2 WAVES ON A ROPE

What is the power that Didier must generate to send these waves having a 3m wavelength, a 1 Hz frequency, and a 50 cm amplitude in these rope with a 5 kg/m linear mass?

www.mensjournal.com/health-fitness/5-combat-rope-moves-to-torch-your-metabolism-w507533/alternating-waves-w507537/

Discover how to solve this problem in this chapter.

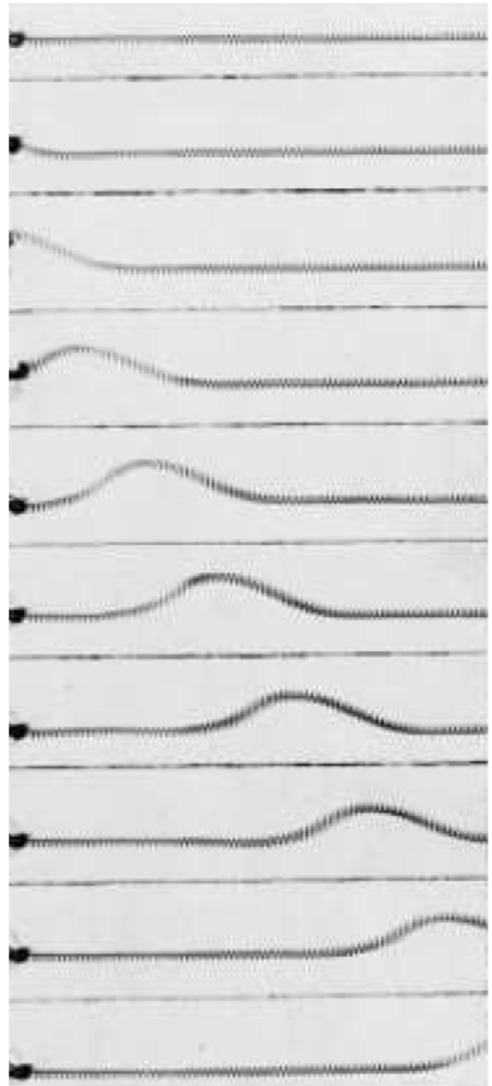
2.1 A TRAVELLING PERTURBATION

Suppose that initially, there is a taut rope at rest, as on the top image of this figure. (It's a stretched spring, but such a spring acts exactly like a rope.) Suddenly, a person slightly moves the end of the spring upwards, creating a disturbance. This disturbance then travels along the spring until it arrives at the other end.

This propagating disturbance is called a ***wave***.

The matter in which the wave travels (here a string) is called the ***medium*** of propagation. (The name *progressive wave* is also used.)

In this video, a wave on two ropes can be seen.
www.youtube.com/watch?v=0DoEpwHYiBE



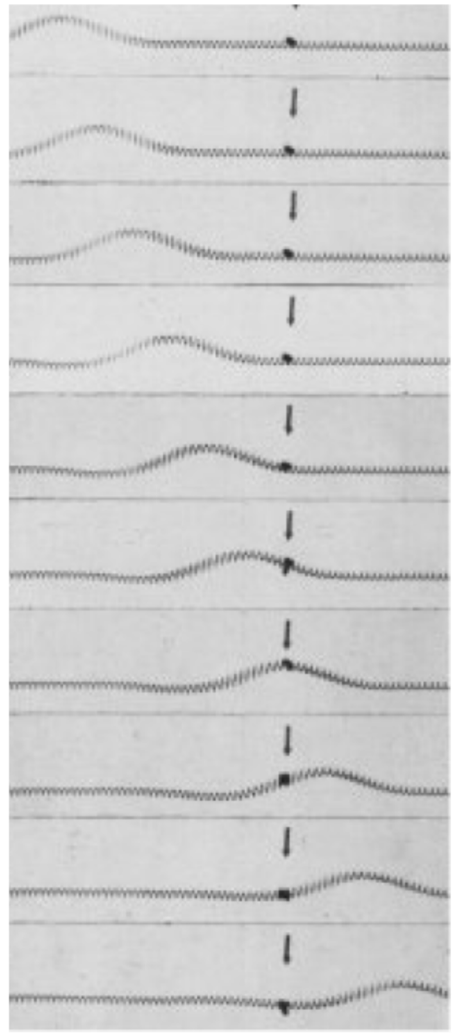
Haber-Schaim, Cross, Dodge, Walter, Tougas, Physique PSSC, éditions CEC, 1974

Note that the motion of the rope is in a direction perpendicular to the direction of propagation of the wave. When the medium is moving in this direction, the wave is called a ***transverse wave***.

Matter Is Not Transported by a Wave

Before the passage of the wave, the rope is at an equilibrium position. When the wave passes, the particles composing the rope moves. However, the wave only temporarily displaces the particles. They return to their equilibrium positions after the passage of the wave.

This phenomenon can be seen in the image to the right. In this image, a wave moves along a stretched spring. A point of the medium has been identified by a black dot. When the wave passes, this point moves away from its equilibrium position and returns to its equilibrium position once the wave has passed. Therefore, the matter composing the medium is not transported by the wave, it only temporarily moves away from the equilibrium position as the wave passes. After the passage of the wave, each particle composing the spring is back to the same position it had before the passage of the wave. The following animation illustrates this phenomenon.
https://www.youtube.com/watch?v=d4grcSK6Igg
We see that the blue dot and the yellow dots oscillate up and down when the wave passes, but they are not carried by the wave.



Haber-Schaim, Cross, Dodge, Walter, Tougas, Physique PSSC, éditions CEC, 1974

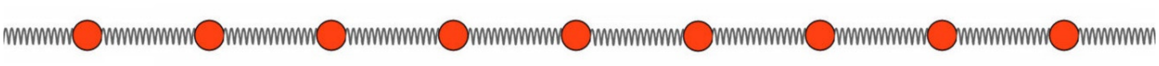
This is true for all types of waves. In this video, a little duck moves around its equilibrium position during the passage of waves. It does not travel with the wave.
http://www.youtube.com/watch?v=-o-VgeabKjI

However, this is not always true since waves on a beach can push objects, such as algae or surfers. These waves are different because they are breaking waves. This is what happens when the height of the wave becomes too large relative to the depth. This is a very different type of wave that will not be considered here. Here, only non-breaking waves will be explored.

Forces Between the Atoms of the Medium

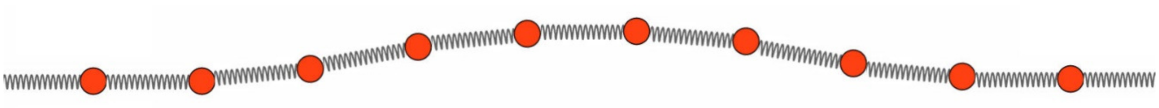
Before the passage of the wave, the matter forming the medium is at its equilibrium position. When the wave passes, each piece of matter in the medium moves for a while and finally returns to its equilibrium position. This means that there are forces acting in the medium that bring each atom back to its equilibrium position.

The forces acting on an atom in the medium are made by the neighbouring atoms. These forces could be represented by springs, as in this image.

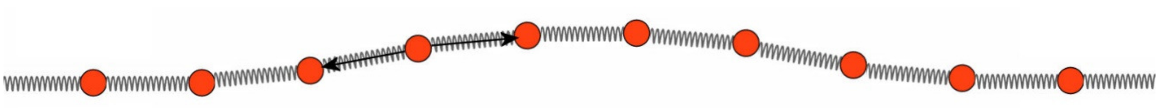


For a string, this force is a force of attraction of electrical origin between the atoms of the string, which is actually the tension of the string. The previous image shows the string at its equilibrium position, and the net force on each atom is zero.

When the wave passes, the medium is distorted, and the following situation could occur.



If we look at the forces acting on a specific atom, say the 4th from the left, we have the following forces.



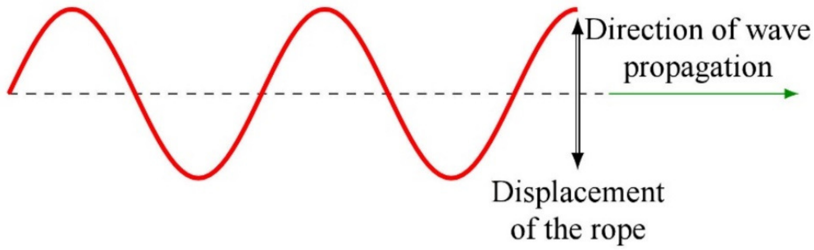
If these forces are added, the net force is not zero. There is a little more downwards force than upwards force, which means that the net force has a downwards component. This downwards force seeks to return the atom to its equilibrium position. Each atom is subjected to a net force that always seeks to bring it back to the equilibrium position.

This force, which opposes the deformation of the medium, is essential for the wave to propagate. If neighbouring atoms exert a force on the 4th atom to bring it back to its equilibrium position, then the 4th atom, according to Newton's 3rd law, also exerts a force on its neighbours. So, if an atom of a medium is displaced, the force that this atom exerts on its neighbour to the right puts that atom in motion, which then puts the next neighbour to the right in motion and so on. The deformation then spreads in the rope. Without forces between the atoms of the medium, the wave does not propagate.

It is therefore necessary that the medium oppose the deformation, so that there are forces on the atoms that seek to bring them back to the position of equilibrium, so that the waves can exist in the medium. In the case of the string, this force is made by the tension of the string.

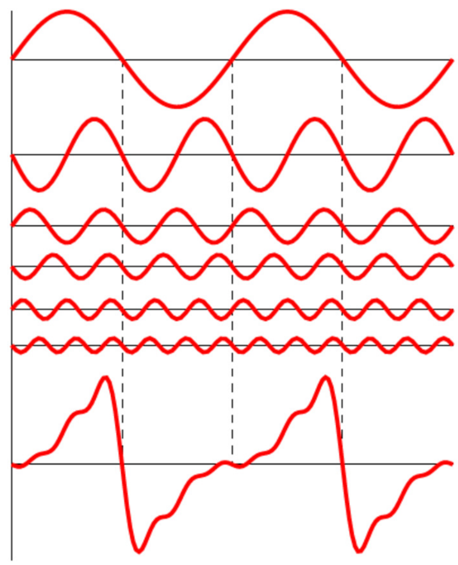
2.2 SINE WAVES

We will now focus on waves whose shape is described by a sine function. This means that if a picture of the wave is taken at a certain time, the form of the string is described by a sine function.



Why Sine Wave?

This seems to be a little restrictive because a wave can have any shape, but it is not since it can be shown that any waveform can be written as a sum of sine waves. This is Fourier's theorem, a theorem that generated a 30-year debate at the beginning of the 19th century.

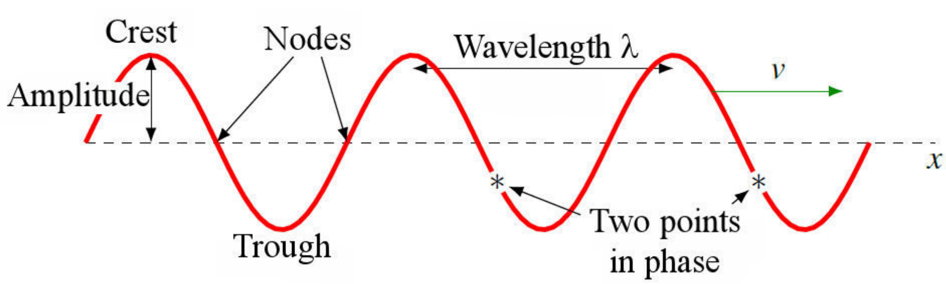


The diagram to the right shows how a sum of sine waves can give a different wave shape. The sum of the 6 sine functions shown on top of the diagram gives the signal at the bottom. This sum is a Fourier series, and it is an important topic of university mathematics.

This next clip shows you how special waveforms can be obtained by adding sine functions (called harmonics here).
http://www.youtube.com/watch?v=Lu2nnvYORec

A Bit of Terminology

For a sine wave in a string, we have the following elements.



The **amplitude** (A) is the value of the maximum displacement of the rope.

The **crests** are the places where the displacement of the rope reaches its largest positive value.

The **troughs** are the places where the displacement of the rope reaches its largest negative value.

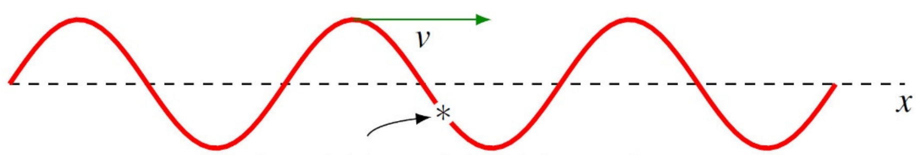
The **nodes** are the places where the displacement of the rope is zero.

Points in phase are points that are in the same position on the cycle of the sine function. There may be one or more complete cycles between these points in phase. Thus, all the peaks are in phase and all the troughs are in phase.

The **wavelength** (λ) is the distance between two adjacent crests. (Properly speaking, the wavelength is the distance between two points in phase that are closest to each other.) Therefore, it is the length of one cycle of the sine function.

The wave travels at a certain **speed** (v) determined by the characteristics of the medium. Whatever the amplitude and the wavelength, this speed is always the same.

Let us now examine the motion of a specific part of the medium to see how it moves. A transverse wave is considered to illustrate.



As the wave passes, the particle of the rope shown in the diagram goes up and down. At the instant shown in the diagram, the particle is going up since a crest is approaching. The particle must, therefore, moves upwards since it will form the crest of the wave in a few moments. It will be proven a little later (in an example) that the motion of this particle is actually a simple harmonic motion. According to what we have seen in Chapter 1, this motion of the particle of the medium is then characterized by a frequency and a period. In the following animation, a sine wave travels through in a medium.
https://www.youtube.com/watch?v=d4grcSK6Igg
It can easily be noted that each piece of the medium seems to make a harmonic oscillation when the wave passes.

Since we have a harmonic motion, we also have the following characteristics.

The **period** of the wave (T) is the time taken by the particles of the medium to make a full oscillation.

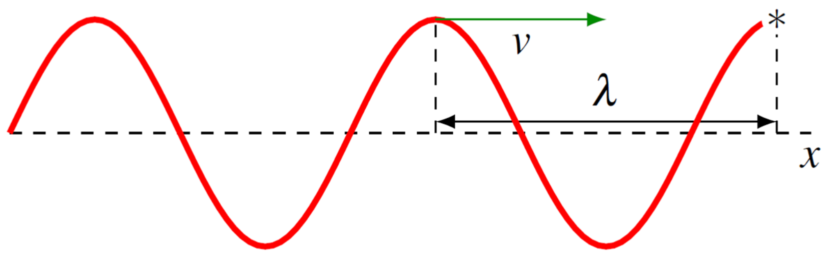
The **frequency** of the wave (f) is the number of oscillations made by the particles of the medium in one second.

The relationship between these two quantities found in Chapter 1 is still valid.

$$f = \frac{1}{T}$$

Relationship between the Frequency and the Wavelength

To find the relationship between the wavelength and the frequency, let's examine the following situation on a rope.



The particle at the end of the rope, identified by *, is, at this time, at the maximum position of its motion. As the wave travels, it will move downwards and then upwards to reach its maximum position again when the next crest arrives at the end of the rope. The particle will then have made a complete cycle. This means that the time it takes to do one oscillation (period T) is equal to the time it takes for the next crest to arrive at the end of the rope. As this crest is initially at a distance λ and is travelling towards the end of the rope at the speed v, the time it takes to arrive is λ/v. Therefore,

$$T = \frac{\lambda}{v}$$

This equation can then be written as

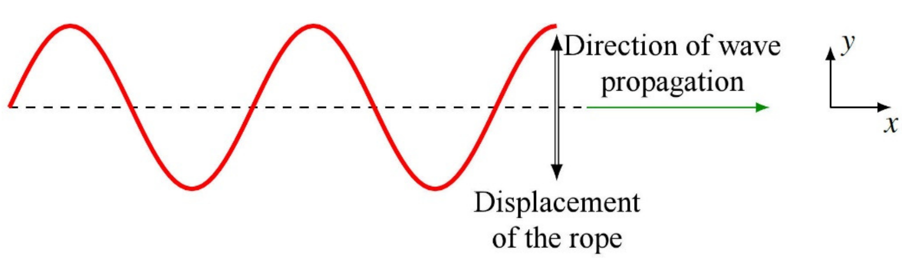
Link between λ and f or Link between λ and T

$$v = \frac{\lambda}{T} = \lambda f$$

2.3 FUNCTION DESCRIBING A SINE WAVE

How to obtain a travelling function

The formula giving the displacement (noted y) of the rope from the equilibrium position (dotted line at $y = 0$) when there is a sine wave on a rope will now be sought.



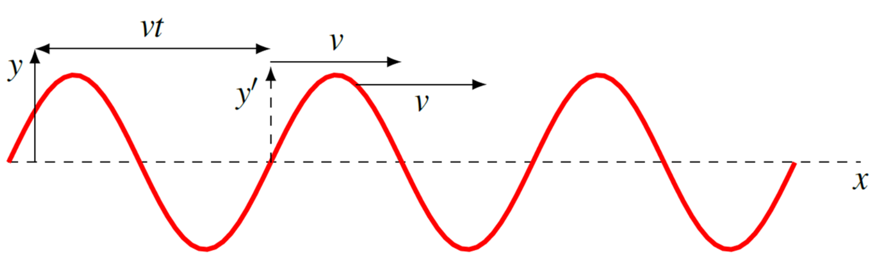
Obviously, the shape of a sine wave is given by a sine function. To have a wave that rises up to height A, the sine function must be multiplied by A because the maximum value of a sine function is 1.

The length of a cycle of a sine function of 2π. The length of the cycle must, therefore, be adjusted to have any wavelength. This can be achieved with the function

$$y = A\sin\left(\frac{2\pi}{\lambda}x\right)$$

If a cycle starts at $x = 0$, the cycle must end at $x = \lambda$. If $x = \lambda$ is put in this equation, then $\sin(2\pi)$ is obtained. This is indeed a full cycle because the period of a sine function is 2π.

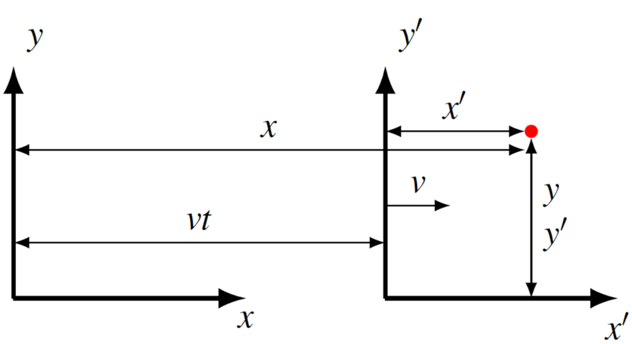
In this form, this function cannot describe a wave, because it is motionless. It must be changed so that this function moves as time passes. To find this moving function, two axes systems will be used. The first is a stationary axis system (x and y) and the second is a moving axes system which follows the sine function in its motion (x' and y').



As the axes x' and y' follow the sine function while it is moving, the sine function is stationary with respect to these axes. Therefore, the equation of the sine with these axes is

$$y' = A\sin\left(\frac{2\pi}{\lambda}x'\right)$$

In order to have this function with the stationary axes, the link between these two axes systems must be found.



The x-value is the distance between a point in the plane and the y-axis, and x'-value is the distance between the point and the y'-axis. According to what is shown in the diagram, we have

$$x' = x - vt$$

The values of y and y' are the distances between the point and the x and x' axes. According to the diagram, this means that

$$y' = y$$

With these changes in the sine function, the following equation is obtained.

$$y' = A\sin\left(\frac{2\pi}{\lambda}x'\right)$$

$$y = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

$$y = A\sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi v}{\lambda}t\right)$$

This wave travels towards the positive x-axis. To have a wave moving towards the negative x-axis, the sign of the speed must be inverted. The equation then becomes

$$y = A\sin\left(\frac{2\pi}{\lambda}x \pm \frac{2\pi v}{\lambda}t\right)$$

where – is used if the wave moves towards the positive x-axis, and + is used if the wave moves towards the negative x-axis.

The equation can be transformed a bit. Since

$$\frac{v}{\lambda} = f$$

the equation can be written as

$$y = A\sin\left(\frac{2\pi}{\lambda}x \pm 2\pi ft\right)$$

Certain combinations of variables are so common in the study of waves that symbols are used to represent them.

Definitions of k and ω

$$k = \frac{2\pi}{\lambda}$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

k (not to be confused with the spring constant seen in Chapter 1) is the ***wave number***. It is in rad/m and represents the number of cycles made by the wave over a distance of 2π metres.

ω, which is identical to the ω used in Chapter 1, is the angular frequency. It is in rad/s and represents the number of oscillation cycles made during 2π seconds.

There is a link between these variables and the speed. This link is

$$v = \lambda f$$

$$v = \frac{\lambda}{2\pi} 2\pi f$$

$$v = \frac{1}{k}\omega$$

Therefore,

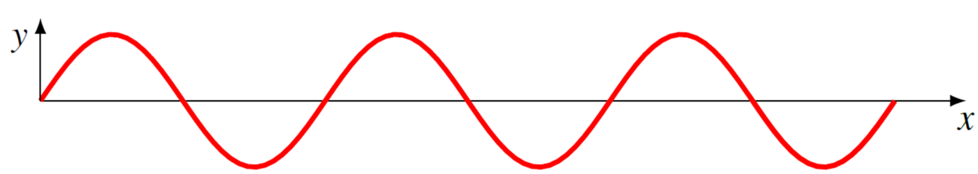
Link between ω and k

$$v = \frac{\omega}{k}$$

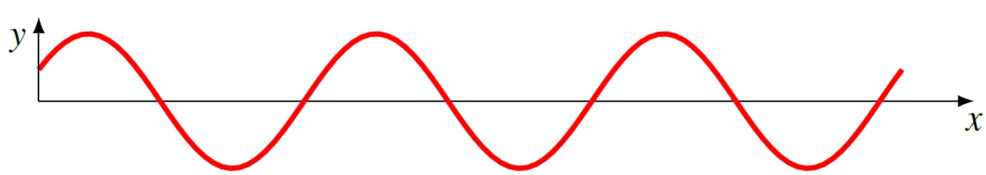
With these new variables, the function that describes the wave becomes

$$y = A\sin\left(kx \pm \omega t\right)$$

Yet, this is not the most general form. For now, the amplitude, the wavelength, and the speed can be modified but the shape of the wave at $t = 0$ cannot. With the current formula, the wave must have the following shape at $t = 0$.



The wave must have $y = 0$ at $x = 0$. But it is possible to have a wave that has a different shape at $t = 0$. For example, the wave could have the following shape.



It is still described by a sine function but shifted towards the left. Fortunately, we know how to shift a sine function: simply add a phase constant ϕ. Depending on the value of the phase constant, the sine function at $t = 0$ will shift to one side or the other along the x-axis. The following final result is thus obtained.

Function Describing a Sine Wave

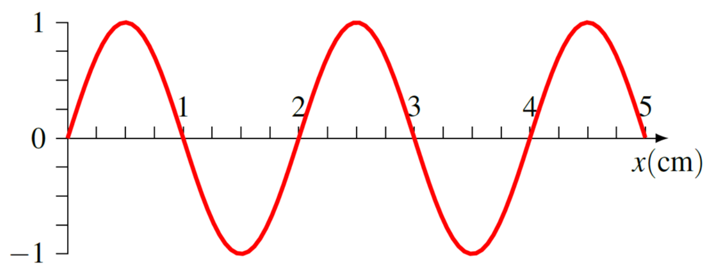
$$y = A\sin\left(kx \pm \omega t + \phi\right)$$

The negative sign is chosen if the wave moves towards the positive x-axis.
The positive sign is chosen if the wave moves towards the negative x-axis.

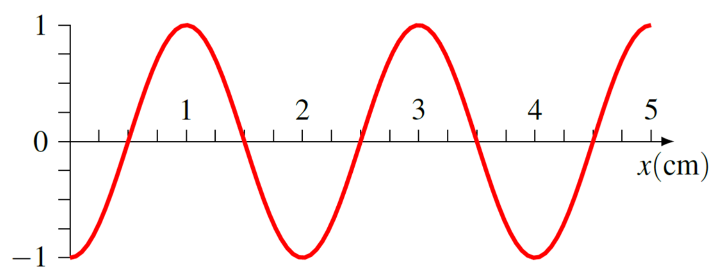
Let's check if a wave travelling at the right speed is obtained. To do this, a wave with a 1 cm amplitude, a 2 cm wavelength and a speed of 1 cm/s towards the positive x-axis is used. This implies that the frequency is 0.5 Hz. A vanishing phase constant is used for this wave. The equation of this wave is thus

$$y = 1cm \cdot \sin\left(100\pi \tfrac{rad}{m} \cdot x - \pi \tfrac{rad}{s} \cdot t\right)$$

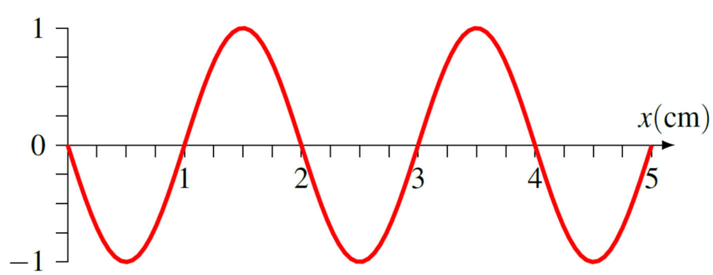
At $t = 0$, the wave equation is $y = 1cm \cdot \sin\left(100\pi \tfrac{rad}{m} \cdot x\right)$. The graph of this function is



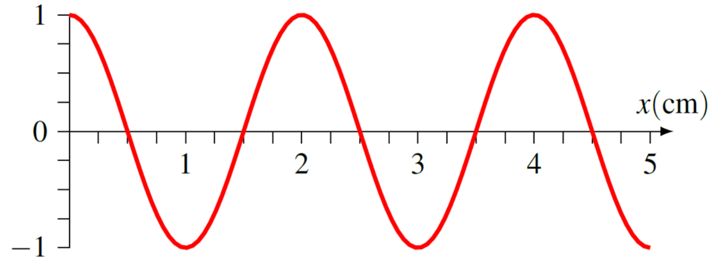
At $t = 0.5$ s, the equation is $y = 1cm \cdot \sin\left(100\pi \tfrac{rad}{m} \cdot x - \tfrac{\pi}{2}\right)$. The graph of this function is



At $t = 1$ s, the equation is $y = 1cm \cdot \sin\left(100\pi \tfrac{rad}{m} \cdot x - \pi\right)$. The graph of this function is



At t = 1.5 s, the equation is $y = 1cm \cdot \sin\left(100\pi \frac{rad}{m} \cdot x - \frac{3\pi}{2}\right)$. The graph of this function is



It is easy to see in this series of graphs that the sine function moves towards the right with a speed of 1 cm/s, as it is supposed to do.

Note that with such a sine wave, each point on the string makes a harmonic motion as the wave passes. Here is the proof.
https://physique.merici.ca/waves/proofshmwave.pdf
The formulas seen in Chapter 1 can therefore be used to describe the movement of each point in the rope.

Speed and Acceleration of the Particles of the Medium

From the formula for the position of the particles of the medium as a function of time, the formulas for the velocity and the acceleration of the particles of the medium can be obtained. Since the velocity is the derivative of the position

$$v_y = \frac{\partial y}{\partial t}$$

the velocity is

Velocity of the Particles of the Medium

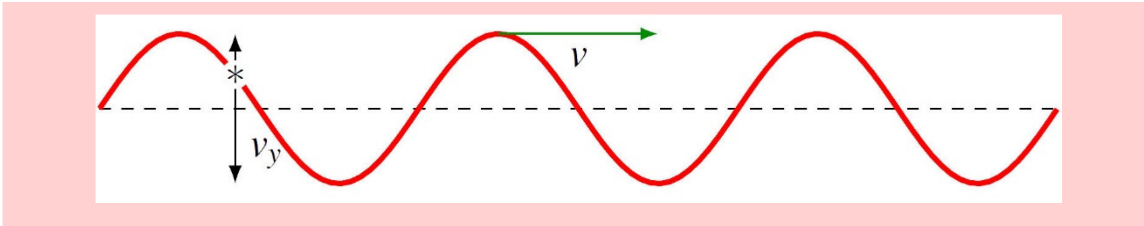
$$v_y = \pm A\omega \cos\left(kx \pm \omega t + \phi\right)$$

The negative sign is chosen if the wave moves towards the positive x-axis.
The positive sign is chosen if the wave moves towards the negative x-axis.

$$v_{y\max} = A\omega$$

Common Mistake: To Confuse v and v_y

Do not confuse the speed of the wave and the speed of a particle composing the medium. For a rope, the speed of the wave (v) is the speed of the crest in the x-direction whereas the speed of the medium (v_y) (here the speed of the rope) is in the y-direction. There's no direct link between these two speeds.



The acceleration is found by differentiating once more.

$$a = \frac{\partial v_y}{\partial t}$$

This derivative gives

Acceleration of the Particles of the Medium

$$a = -A\omega^2 \sin\left(kx \pm \omega t + \phi\right)$$

The negative sign is chosen if the wave moves towards the positive *x*-axis.
The positive sign is chosen if the wave moves towards the negative *x*-axis.

$$a_{\max} = A\omega^2$$

How to Calculate the Value of *A*

The values of *A* can be calculated if the position and the velocity of the string at a certain place and at a certain time are known. This formula is obtained from the formulas for the position and speed of the string.

$$y = A\sin\left(kx \pm \omega t + \phi\right) \qquad \text{and} \qquad v_y = \pm A\omega\cos\left(kx \pm \omega t + \phi\right)$$

If you solve those equation for the sine and cosine functions

$$\sin\left(kx \pm \omega t + \phi\right) = \frac{y}{A} \qquad \cos\left(kx \pm \omega t + \phi\right) = \frac{\pm v_y}{\omega A}$$

and use the following identity

$$\sin^2\left(kx \pm \omega t + \phi\right) + \cos^2\left(kx \pm \omega t + \phi\right) = 1$$

the following result can be obtained

$$\left(\frac{y}{A}\right)^2 + \left(\frac{v_y}{\omega A}\right)^2 = 1$$

By multiplying by A^2, this equation becomes

Calculation of the Amplitude of the Wave

$$y^2 + \left(\frac{v_y}{\omega}\right)^2 = A^2$$

To use this formula, the values of y and v_y at the same time must be used (which is not necessarily $t = 0$).

This is actually the same formula as in Chapter 1, but the motion is in the y-direction rather than x-direction. It's normal to have the same formula since each particle in the string makes a harmonic oscillation as the sine wave passes.

How to Calculate the Values of ϕ

If the displacement of the string is known for a certain value of x and t, the phase constant ϕ can obviously be found with the displacement formula.

$$y = A\sin\left(kx \pm \omega t + \phi\right)$$

Just like in Chapter 1, there will be a small problem if ϕ is calculated with this formula. Since there are two main solutions to the arcsine function, you will have 2 values of ϕ, and you will have to find the right value of ϕ by calculating the speed of the string. All these calculations (calculation of 2 answers of the arcsine and then calculation of the velocities) must be done if only the sign of the velocity is known, but all these complications can be avoided if y and v_y at a certain time are known by using another formula.

This formula is obtained by using the sines and cosines obtained with the formulas of y and v_y

$$\sin\left(kx \pm \omega t + \phi\right) = \frac{y}{A} \qquad \cos\left(kx \pm \omega t + \phi\right) = \frac{\pm v_y}{\omega A}$$

and then by dividing the sine by the cosine

$$\frac{\sin\left(kx \pm \omega t + \phi\right)}{\cos\left(kx \pm \omega t + \phi\right)} = \frac{\left(\frac{y}{A}\right)}{\left(\frac{\pm v_y}{A\omega}\right)}$$

Substituting with tangent on the left and simplifying on the right, the following formula is obtained.

Calculation of ϕ

$$\tan\left(kx \pm \omega t + \phi\right) = \frac{\omega y}{\pm v_y}$$

The negative sign is chosen if the wave moves towards the positive x-axis.
The positive sign is chosen if the wave moves towards the negative x-axis.

To use this formula, the values of y and v_y <u>at the same time</u> must be used (which is not necessarily $t = 0$).

Warnings:

- The value of ϕ is in radians. Put your calculator in radian mode to obtain the correct value.
- If the value of $\pm v_y$ is negative, π radians must be added to the answer given by the calculator.
- If the speed is zero, the inverse tangent of ∞ or -∞ is obtained according to the sign of y. Do not panic, the answers of these inverse tangents are π/2 and -π/ 2.

Examples

Example 2.3.1

A sine wave is travelling towards the positive x-axis on a taut rope. The wavelength is 40 cm, and the frequency is 8 Hz. At $t = 0$ s and $x = 0$ m, the displacement of the rope is $y = 1$ cm, and the velocity of the rope is $v_y = 100$ cm/s.

a) What is the period?

The period is

$$\begin{aligned} T &= \frac{1}{f} \\ &= \frac{1}{8Hz} \\ &= 0.125s \end{aligned}$$

b) What is the speed of the wave?

The speed is

$$\begin{aligned} v &= \lambda f \\ &= 0.4m \cdot 8Hz \\ &= 3.2 \tfrac{m}{s} \end{aligned}$$

c) What is the function describing this wave?

The function is

$$y = A\sin\left(kx - \omega t + \phi\right)$$

The negative sign was chosen since the wave is travelling towards the positive x-axis. k, ω, A, and ϕ must then be found.

The wave number is

$$\begin{aligned} k &= \frac{2\pi}{\lambda} \\ &= \frac{2\pi}{0.4m} \\ &= 5\pi \tfrac{rad}{m} \end{aligned}$$

The angular frequency is

$$\begin{aligned} \omega &= 2\pi f \\ &= 2\pi \cdot 8Hz \\ &= 16\pi \tfrac{rad}{s} \end{aligned}$$

The amplitude is found with

$$A^2 = y^2 + \left(\frac{v_y}{\omega}\right)^2$$

$$A^2 = (0.01m)^2 + \left(\frac{1\frac{m}{s}}{16\pi s^{-1}}\right)^2$$

$$A = 0.02227m$$

and the phase constant is found with

$$\tan(kx - \omega t + \phi) = \frac{\omega y}{-v_y}$$

$$\tan(\phi) = \frac{16\pi s^{-1} \cdot 1cm}{-100\frac{cm}{s}}$$

$$\tan(\phi) = -0.5027$$

$$\phi = 2.676rad$$

(π was added to the value given by the calculator because the divisor is negative.) Therefore, the function describing the wave is

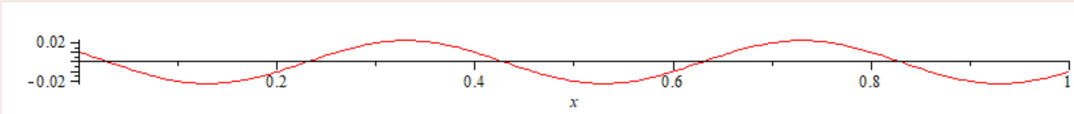
$$y = 0.02227m \cdot \sin\left(5\pi \tfrac{rad}{m} \cdot x - 16\pi \tfrac{rad}{s} \cdot t + 2.676rad\right)$$

d) What is the formula describing the shape of the rope at t = 1s?

To obtain the shape, just put t = 1 s in the equation obtained previously. The result is

$$y = 0.02227m \cdot \sin\left(5\pi \tfrac{rad}{m} \cdot x - 16\pi \tfrac{rad}{s} \cdot 1s + 2.676rad\right)$$
$$= 0.02227m \cdot \sin\left(5\pi \tfrac{rad}{m} \cdot x - 47.590rad\right)$$

Here is the shape of the rope at t = 1 s.



e) What is the position of the piece of rope at x = 20 cm and t = 1 s?

Using x = 20 cm and t = 1 s in the equation of the position obtained previously, the result is

$$y = 0.02227m \cdot \sin\left(5\pi \tfrac{rad}{m} \cdot 0.2m - 16\pi \tfrac{rad}{s} \cdot 1s + 2.676rad\right)$$
$$= 0.02227m \cdot \sin\left(-44.448rad\right)$$
$$= -0.01m$$

It can be seen on the graph that this is indeed the position of the rope at this location.

f) What is the maximum speed of the rope?

The maximum speed of the rope is

$$v_{y\max} = \omega A$$
$$= 16\pi s^{-1} \cdot 0.02227m$$
$$= 1.119 \tfrac{m}{s}$$

g) What is the velocity of the rope at x = 1 m and t = 1 s?

The formula giving the velocity of the rope is

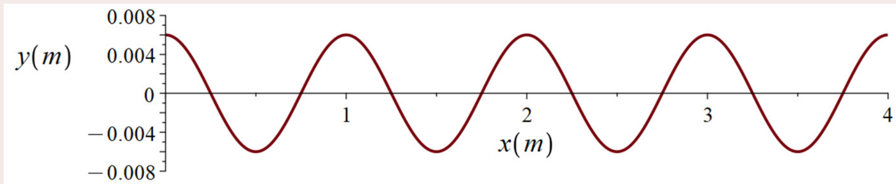
$$v_y = -A\omega\cos\left(kx - \omega t + \phi\right)$$
$$= -0.02227m \cdot 16\pi s^{-1} \cdot \cos\left(5\pi \tfrac{rad}{m} \cdot x - 16\pi \tfrac{rad}{s} \cdot t + 2.676rad\right)$$
$$= -1.119 \tfrac{m}{s} \cdot \cos\left(5\pi \tfrac{rad}{m} \cdot x - 16\pi \tfrac{rad}{s} \cdot t + 2.676rad\right)$$

At t = 1 s and x = 1 s, the velocity is

$$v_y = -1.119 \tfrac{m}{s} \cdot \cos\left(5\pi \tfrac{rad}{m} \cdot 1m - 16\pi \tfrac{rad}{s} \cdot 1s + 2.676rad\right)$$
$$= -1 \tfrac{m}{s}$$

Example 2.3.2

Here is an image that shows the shape of a sine wave on a string at t = 1 s. The speed of the wave is 5.5 m/s and the wave travels towards the left. What is the equation of this wave?



The function is

$$y = A\sin\left(kx + \omega t + \phi\right)$$

We chose the negative sign, because the wave is travelling towards the negative x's. It only remains to find k, ω, A and ϕ.

The graph clearly indicates that the amplitude is 6 mm.

The graph shows that the wavelength is 1 m. So, the wave number k is

$$\begin{aligned} k &= \frac{2\pi}{\lambda} \\ &= \frac{2\pi}{1m} \\ &= 2\pi \tfrac{rad}{m} \end{aligned}$$

The angular frequency can be found with the velocity of the wave.

$$\begin{aligned} v &= \frac{\omega}{k} \\ 5.5 \tfrac{m}{s} &= \frac{\omega}{2\pi \tfrac{rad}{m}} \\ \omega &= 11\pi \tfrac{rad}{s} \end{aligned}$$

The phase constant can be found with the following formula. (The positive signs are used in the formula since the wave is travelling towards the negative x's.)

$$\tan\left(kx + \omega t + \phi\right) = \frac{\omega y}{v_y}$$

$$\tan\left(2\pi \tfrac{rad}{m} \cdot x + 11\pi \tfrac{rad}{s} \cdot t + \phi\right) = \frac{11\pi \tfrac{rad}{s} \cdot y}{v_y}$$

We know that we are at t = 1 s, but we don't have specific values for y and v_y. Luckily, those values can be found with the graph. Any point on the graph can be taken, but it's easier if a point where the string is at $y = 0$ or a point where the string is at $y = A$ or $y = -A$ is taken. Let's say we take the point at $x = 1.25$ m, which is a point where $y = 0$. Since this is the graph at $t = 1$ s, the equation becomes

$$\tan\left(2\pi \tfrac{rad}{m}\cdot 1.25m + 11\pi \tfrac{rad}{s}\cdot 1s + \phi\right) = \frac{11\pi \tfrac{rad}{s}\cdot 0}{v_y}$$
$$\tan\left(2.5\pi rad + 11\pi rad + \phi\right) = 0$$
$$\tan\left(13.5\pi rad + \phi\right) = 0$$

The solution of this equation is

$$13.5\pi + \phi = \pi$$

There's a little subtlety here. Even though division always gives 0 when a point where $y = 0$ is taken, we still need to check if the denominator is negative and add π to the value given by the calculator if this is the case. In our calculation, π was added because the denominator is v_y and this v_y is negative at $x = 1.25$ m (a through of the wave will soon reach $x = 1.25$ m).

Therefore, ϕ is

$$\phi = \pi - 13.5\pi$$
$$= -12.5\pi$$

This value can be used, but 14π can also be added to this answer (since 2π can be added or subtracted as many times as we want to ϕ) to obtain

$$\phi = 1.5\pi = \tfrac{3\pi}{2}$$

So, the wave equation is

$$y = 6mm \cdot \sin\left(2\pi \tfrac{rad}{m} x + 11\pi \tfrac{rad}{s} t + \tfrac{3\pi}{2}\right)$$

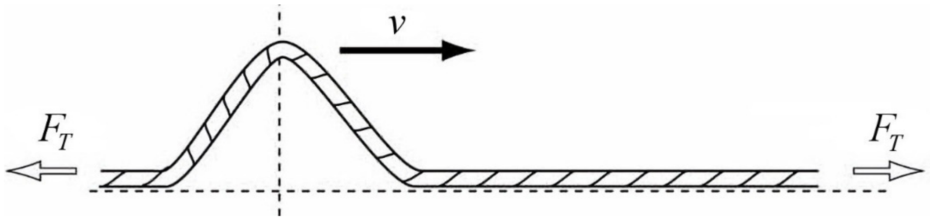
To avoid the subtlety of the sign of the denominator with a tangent function, a point of the graph where a crest of the wave is located could have been used, such as $x = 0$ m for example. In this case, the formula for finding ϕ would have been

$$\tan\left(2\pi \tfrac{rad}{m}\cdot 0m + 11\pi \tfrac{rad}{s}\cdot 1s + \phi\right) = \frac{11\pi \tfrac{rad}{s}\cdot 0.006m}{0}$$
$$\tan\left(11\pi rad + \phi\right) = \infty$$
$$11\pi rad + \phi = \tfrac{\pi}{2}$$
$$\phi = -10.5\pi$$

This answer also leads to $\phi = 3\pi/2$ if 12π is added. For sure, the subtlety of the sign of denominator was avoided, but we must deal now with the subtlety of dividing by 0 with an arctangent...

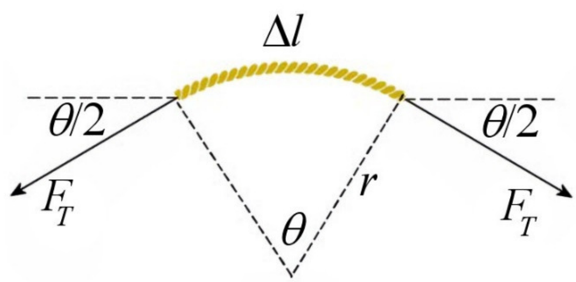
2.4 WAVES SPEED ON A ROPE

To find the speed of waves in a stretched rope, the forces exerted on a small piece of rope will be considered. Suppose that the wave has the following shape.



www.wikipremed.com/image.php?img=010201_68zzzz120700_19602_68.jpg&image_id=120700

The forces acting on a small piece of rope on top of the wave are as follows. This figure clearly shows that the tension forces exerted on the piece of rope seek to bring the piece of rope back to its equilibrium position. This corresponds to what was said previously: there must be a force that seeks to bring the medium back to its equilibrium position. This is an essential condition to have a wave in a medium. For a rope, the tension of the rope is the force that seeks to bring the rope back to its equilibrium position.



The tension forces and the angles being the same on each side of the piece, the sum of the *x*-component of the forces vanishes by symmetry. On the other hand, the *y*-components of the forces do not cancel out. The sum of these components is

$$\sum F_y = -2F_T \sin\left(\frac{\theta}{2}\right)$$

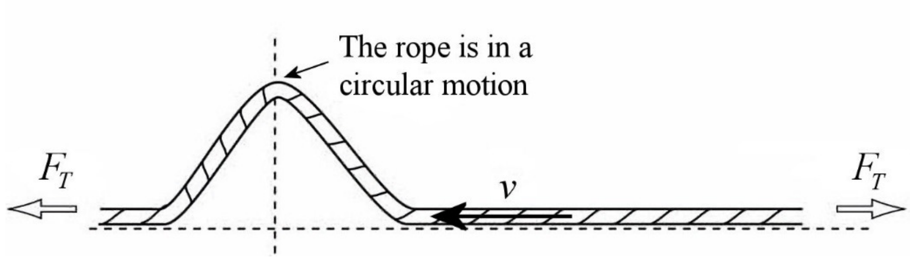
If the piece of rope is very small, the angle is small and the approximation $\sin\theta \approx \theta$ can be used. The sum then becomes

$$\sum F_y = -2F_T\left(\frac{\theta}{2}\right) = -F_t\theta$$

The angle is simply the length of the arc of a circle (the length of the small piece of rope) divided by the radius of curvature. Therefore,

$$\sum F_y = -F_T\frac{\Delta l}{r}$$

Now, the acceleration of the piece of rope must be found. There is a trick to diagram out its value quite easily: using the point of view of an observer moving at the same speed as the wave. For this observer, the wave is stationary and the rope travels towards the left at speed *v*. At the highest point of the wave, the rope moves along an arc and is, therefore, making a circular motion.



As the rope makes part of a circular motion at the top of the wave, its acceleration is

$$a = \frac{v^2}{r}$$

Therefore,

$$\sum F_y = ma_y$$
$$-F_T \frac{\Delta l}{r} = -m\frac{v^2}{r}$$
$$F_T \Delta l = mv^2$$

The mass of the small piece of rope depends on its length. If the linear density of the rope (in kg/m) is

$$\mu = \frac{mass}{length}$$

then, the mass of the piece is

$$mass = \mu \cdot length$$
$$m = \mu \Delta l$$

The equation then becomes

$$F_T \Delta l = mv^2$$
$$F_T \Delta l = \mu \Delta l v^2$$
$$F_T = \mu v^2$$

If this equation is solved for the speed, the following result is obtained.

Wave Speed in a Rope

$$v = \sqrt{\frac{F_T}{\mu}}$$

It can be seen that this speed does not depend on the shape of the wave. It depends only on the characteristics of the medium such as the tension force and the linear density of the rope. This means that the waves on a rope, regardless of their amplitude and their wavelength, all travel at the same speed.

This conclusion that the speed is the same for all waves is also true for several other types of waves propagating in other media (such as sound for example). However, this is not always true.

When all the waves have the same speed in a medium, the medium is called a ***non-dispersive*** medium**.** When the waves do not have the same speed in a medium (speed depending on the wavelength or the amplitude), the medium is called a ***dispersive*** medium.

The waves are all travelling at the same speed in a non-dispersive medium.

Almost everywhere in these notes, the media will be non-dispersive.

To change the speed of a wave, the medium must be changed. For a wave travelling in a rope, the tension of the rope or the linear density of the rope has to be changed to change the speed of the wave.

A non-dispersive medium also implies that a wave will always keep the same shape while propagating. In a dispersive medium, the wave shape changes as the wave travels.

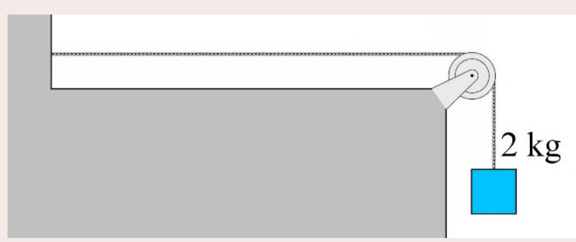
Common Mistake: Thinking that the Speed Changes if λ or f Changes

Sometimes, students use the formula $v = \lambda f$ to deduce that the speed of the wave changes if the wavelength or the frequency is changed. This is wrong. The speed of the wave is the same regardless of the wavelength. This formula rather indicates that the frequency must change if the wavelength is changed since the speed must remain the same.

A demonstration of this can be seen in the following video. In the top part of the clip, there is a wave with a smaller frequency than the wave in the bottom part. It is clear that a lower frequency generates a wave with a longer wavelength. It is also possible to note that the speed of the wave is the same even if the wave is different.
http://www.youtube.com/watch?v=7Iv4GmyXsCQ

Example 2.4.1

A 2 kg mass is suspended at the end of a 6 m long rope as shown in the diagram. The mass of the rope is 300 g. What is the speed of the waves travelling on the rope?



The speed of waves in a rope is given by

$$v = \sqrt{\frac{F_T}{\mu}}$$

To find the speed, the tension and the linear density must be found first.

The tension of the rope is found with the sum of the forces acting on the mass.

$$\sum F_y = -mg + F_T = 0$$
$$F_T = mg$$
$$F_T = 19.6N$$

The linear density of the string is

$$\mu = \frac{mass}{length} = \frac{0.3kg}{6m} = 0.05\tfrac{kg}{m}$$

Therefore, the speed is

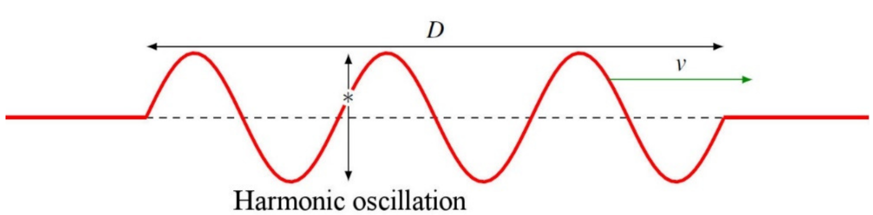
$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{19.6N}{0.05\tfrac{kg}{m}}} = 19.8\tfrac{m}{s}$$

2.5 ENERGY AND POWER

A wave does not transport matters, but it carries energy. The amount of energy in a wave will now be determined.

Energy in a Sine Wave

Suppose that there is a sine wave with a certain length on a string, as shown in this diagram.



Each particle (of mass *m*) composing the string is undergoing a harmonic oscillation. The energy of a mass in a harmonic oscillation being

$$E = \frac{1}{2} m\omega^2 A^2$$

the energy of all the oscillating masses is

$$\begin{aligned} E &= \sum \frac{1}{2} m\omega^2 A^2 \\ &= \frac{1}{2}\left(\sum m\right)\omega^2 A^2 \\ &= \frac{1}{2} M\,\omega^2 A^2 \end{aligned}$$

where *M* is the mass of the rope in the area where there is the wave. This mass is

$$\begin{aligned} M &= \mu \cdot length \\ &= \mu D \end{aligned}$$

The energy is then

Energy of a Sine Wave of Length *D* (in joules)

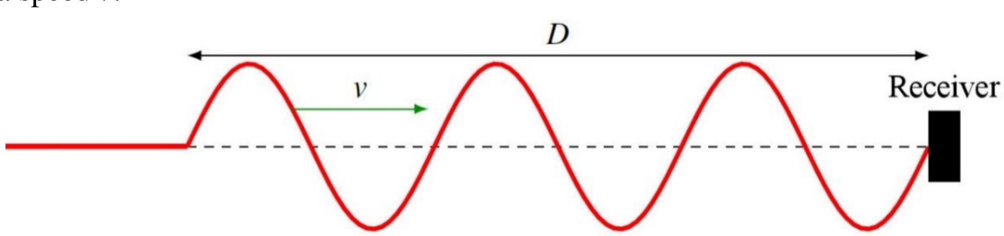
$$E = \frac{1}{2} \mu D\omega^2 A^2$$

Power of a Sine Wave

The rate at which energy arrives at the end of the rope will now be calculated. This power is

$$Power = \frac{Energy}{time}$$

The total amount of energy in the wave is already known but not the time taken for this energy to arrive at the end of the rope. The time starts when the front of the wave arrives, and it stops when the back of the wave arrives. Therefore, the time is the time taken by the end of the wave to arrive when the front of the wave has just arrived at the end of the rope. At this instant, the end of the wave is at a distance *D* and is coming towards the receiver with a speed *v*.



Thus, the time is

$$time = \frac{D}{v}$$

and the power is

$$Power = \frac{\frac{1}{2}\mu D\omega^2 A^2}{\frac{D}{v}}$$

Simplifying, the result is

$$P = \frac{1}{2}\mu v\omega^2 A^2$$

Impedance

In the power formula, there are elements that depend only on the medium (μ and v), so the characteristics of the rope. All the elements that depend only on the medium in the formula of power are called the ***impedance of the medium***. Therefore, for a wave propagating on a rope, the impedance of the medium is

$$Z = \mu v$$

Actually, several versions of the formula of the impedance of a rope can be obtained by using the formula for the speed of the wave. The results are

Impedance of the Medium for Waves on a Rope

$$Z = \mu v = \sqrt{F_T \mu} = \frac{F_T}{v}$$

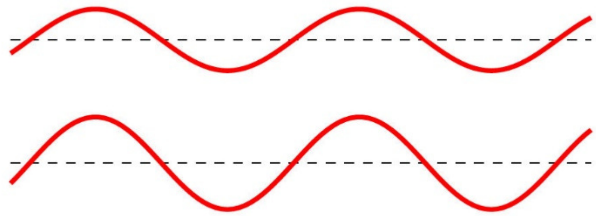
Thus, the formula of power can be written as

Power of a Sine Wave (in Watts) on a Rope

$$P = \frac{1}{2}\mu v\omega^2 A^2 = \frac{1}{2}Z\omega^2 A^2$$

To understand what impedance means, imagine that two waves with the same power and the same frequency are sent in ropes with different impedances. The power equation indicates that the amplitude must be smaller in the rope with the biggest impedance. (A must decrease if Z increases if all other variables remain the same.)

Thus, the greater the impedance, the lower the amplitude, as shown on the diagram.



Therefore, the impedance can be interpreted as the response of the medium to a disturbance. The more impedance there is, the smaller the response of the medium will be for a similar disturbance.

Example 2.5.1

A sine wave travelling in a string has a length of 3 m, a wavelength of 25 cm and an amplitude of 1 cm. The tension of the rope is 500 N, and its linear density is 50 g/m.

a) What is the total energy in the wave?

The energy is

$$E = \frac{1}{2}\mu D\omega^2 A^2$$

The speed of the wave and frequency are needed to solve this problem.

The speed is

$$\begin{aligned} v &= \sqrt{\frac{F_T}{\mu}} \\ &= \sqrt{\frac{500N}{0.05\frac{kg}{m}}} \\ &= 100\tfrac{m}{s} \end{aligned}$$

The frequency is

$$\begin{aligned} v &= \lambda f \\ 100\tfrac{m}{s} &= 0.25m \cdot f \\ f &= 400Hz \end{aligned}$$

Therefore, the energy is

$$E = \frac{1}{2}\mu D\omega^2 A^2$$

$$= \frac{1}{2} \cdot 0.05 \tfrac{kg}{m} \cdot 3m \cdot (2\pi \cdot 400Hz)^2 \cdot (0.01m)^2$$
$$= 47.37J$$

b) What is the power of this wave?

The power is

$$P = \frac{1}{2}\mu v \omega^2 A^2$$
$$= \frac{1}{2} \cdot 0.05 \tfrac{kg}{m} \cdot 100 \tfrac{m}{s} \cdot (2\pi \cdot 400Hz)^2 \cdot (0.01m)^2$$
$$= 1579W$$

Example 2.5.2

What is the power that Didier must generate to send these waves having a 3m wavelength, a 1 Hz frequency, and a 50 cm amplitude in these rope with a 5 kg/m linear mass?

www.mensjournal.com/health-fitness/5-combat-rope-moves-to-torch-your-metabolism-w507533/

The power is found with this formula.

$$P = \frac{1}{2}\mu v \omega^2 A^2$$

To calculate the power, the speed of waves is needed. This speed is

$$v = \lambda f$$
$$= 3m \cdot 1s^{-1}$$
$$= 3 \tfrac{m}{s}$$

Thus, the power needed for one wave is

$$P = \frac{1}{2}\mu v \omega^2 A^2$$
$$= \frac{1}{2} 5 \tfrac{kg}{m} \cdot 3 \tfrac{m}{s} \cdot (2\pi \cdot 1Hz)^2 \cdot (0.5m)^2$$
$$= 74.02W$$

For 2 ropes, Didier must generate 148.04 W.

SUMMARY OF EQUATIONS

Link between λ and f or Link between λ and T

$$v = \frac{\lambda}{T} = \lambda f$$

Definition of k and ω

$$k = \frac{2\pi}{\lambda}$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Link between ω and k

$$v = \frac{\omega}{k}$$

Function Describing a Sine Wave

$$y = A\sin\left(kx \pm \omega t + \phi\right)$$

The negative sign is chosen if the wave moves towards the positive x-axis.
The positive sign is chosen if the wave moves towards the negative x-axis.

Velocity of the Particles of the Medium

$$v_y = \pm A\omega\cos\left(kx \pm \omega t + \phi\right)$$

The negative sign is chosen if the wave moves towards the positive x-axis.
The positive sign is chosen if the wave moves towards the negative x-axis.

$$v_{y\max} = A\omega$$

Acceleration of the Particles of the Medium

$$a = -A\omega^2 \sin\left(kx \pm \omega t + \phi\right)$$

The negative sign is chosen if the wave moves towards the positive x-axis.
The positive sign is chosen if the wave moves towards the negative x-axis.

$$a_{\max} = A\omega^2$$

Calculation of the Amplitude of the Wave

$$y^2 + \left(\frac{v_y}{\omega}\right)^2 = A^2$$

Calculation of ϕ

$$\tan\left(kx \pm \omega t + \phi\right) = \frac{\omega y}{\pm v_y}$$

The negative sign is chosen if the wave moves towards the positive x-axis.
The positive sign is chosen if the wave moves towards the negative x-axis

Wave Speed in a Rope

$$v = \sqrt{\frac{F_T}{\mu}}$$

Energy of a Sine Wave of Length *D* (in Joules) on a Rope

$$E = \frac{1}{2}\mu D\omega^2 A^2$$

Power of a Sine Wave (in Watts) on a Rope

$$P = \frac{1}{2}\mu v\omega^2 A^2 = \frac{1}{2}Z\omega^2 A^2$$

Impedance of the Medium for a Wave on a Rope

$$Z = \frac{F_T}{v} = \sqrt{F_T\mu} = \mu v$$

EXERCISES

2.2 Sine Wave

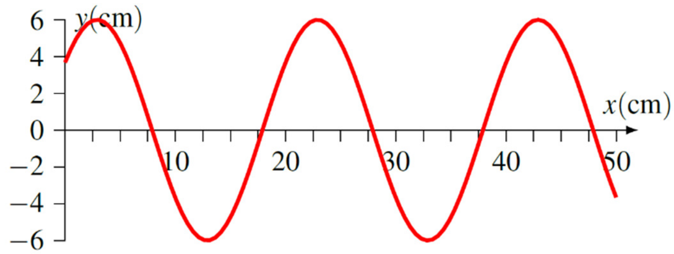
1. A wave on a rope has a speed of 350 m/s and a frequency of 400 Hz.

   a) What is the period of the wave?
   b) What is the wavelength of the wave?

2. In 4 seconds, a wave on a rope travelled 30 m while a piece of the rope has made 20 complete oscillations. What is the wavelength of the wave?

2.3 Function Describing a Sine Wave

3. This wave moves at 40 m/s towards the right on a rope. (Distances are in centimetres on the graph.)



a) What is the frequency of the wave (approximately)?
b) What is the maximum speed of the rope (approximately)?

4. The equation of a wave on a rope is

$$y = 0.2m \cdot \sin\left(10\tfrac{rad}{m} \cdot x + 200\tfrac{rad}{s} \cdot t + \frac{\pi}{4}\right)$$

a) In which direction is this wave travelling?
b) What is the wavelength?
c) What is the speed of the wave?
d) What is the velocity of the rope at x = 1 m and t = 1 s?

5. At a specific position on a rope, the maximum speed is 2 m/s and the maximum acceleration is 200 m/s². What is the amplitude of the wave?

6. The equation of a wave on a rope is

$$y = 0.1m \cdot \sin\left(5\tfrac{rad}{m} \cdot x - 50\tfrac{rad}{s} \cdot t\right)$$

a) What is the maximum speed of the rope?
b) What is the maximum acceleration of the rope?
c) What are the three first instants (t > 0) at which the crest of the wave passes at the point x = 1 m?

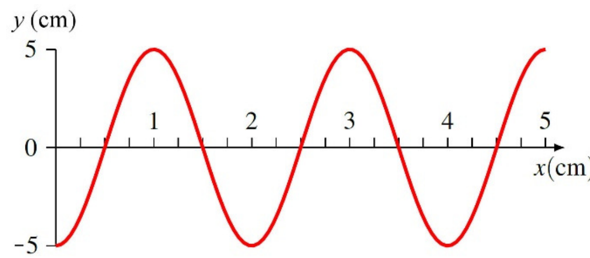
7. A transverse wave is moving towards the positive x-axis with a speed of 50 m/s. At t = 0 s, the string at the position x = 0 m is at y = 2 cm and has a speed of -100 cm/s. Knowing that the wave period is 0.1 s, write the equation of the wave.

$$y = A\sin\left(kx \pm \omega t + \phi\right)$$

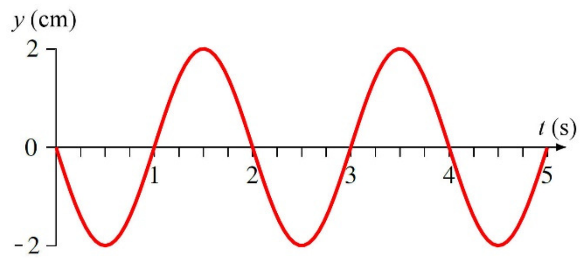
8. At a specific position and at some instant during its oscillation motion, the rope on which a wave is travelling is at y = 1 cm, has a speed of 1.2 m/s and has an acceleration of -100 m/s².

a) What is the period of the wave?
b) What is the amplitude of the wave?

9. The graph shows the shape of a wave at t = 5 s. The wave travels towards the left at 2 cm/s. What is the equation for this wave (with ϕ between 0 and 2π)?



10. The graph shows the oscillation as a function of time of a piece of rope at x = 3 cm. The wave travels towards the right at 3 cm/s. What is the equation of this wave (with ϕ between 0 and 2π)?



2.4 Waves Speed on a Rope

11. A wave with a 50 cm wavelength is moving at 30 m/s on a rope. How fast will a wave with a wavelength of 20 cm travel if it moves on the same string with the same tension?

12. A 2 m long rope has a 200 N tension. A wave passes from one end of the rope to the other in 0.05 s. What is the mass of the rope?

13. A 2 m long rope has a 50 g mass. The speed of the waves in this rope is 50 m/s. What is the tension of the rope?

14. When the tension of a rope is 50 N, the speed of the wave on the rope is 40 m/s. What will the speed of the waves be if the tension is increased to 80 N?

15. On a rope with a tension of 50 N, there is a wave described by the equation

$$y = 0.2m \cdot \sin\left(10\tfrac{rad}{m} \cdot x + 200\tfrac{rad}{s} \cdot t + \frac{\pi}{4}\right)$$

a) What is the linear density of the string?
b) What is the maximum speed of the rope?

16. A rope is made of nylon, whose density is 1150 kg/m³. What is the speed of the waves in this rope if it has a 4 mm diameter and a 300 N tension?

17. In a specific position on a rope, the amplitude of the wave is 2 cm and the maximum speed of the rope is 1 m/s. What is the distance between the nodes of the wave if the tension of the string is 500 N and the linear mass is 50 g/m?

2.5 Energy and Power

18. A wave has the following characteristics:

Amplitude: 2 mm
Wavelength: 40 cm
Wave speed: 50 m/s
Length of the wave on the rope: 10 m

a) How much energy is in this wave if the linear density of the string is 25 g/m?
b) What is the power of the wave if the linear density of the string is 25 g/m?

19. A source with a power of 20 W generates a wave with a wavelength of 12.5 cm and a frequency of 200 Hz. What is the amplitude of the wave if the tension of the rope is 80 N?

20. The wave described by this equation

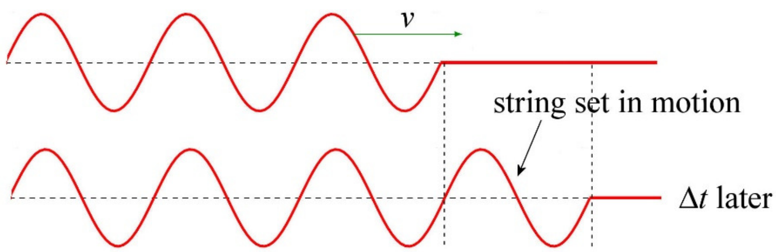
$$y = 2cm \cdot \sin\left(10\pi\tfrac{rad}{m} \cdot x - 50\pi\tfrac{rad}{s} \cdot t\right)$$

is travelling along a rope having a linear density of 50 g/m. What is the power of this wave?

21. The impedance of a rope is 4 kg/s and the speed of the waves in this rope is 80 m/s. What are the values tension and the linear density of this rope?

22. A source with a power of 100 W generates a wave in a rope. The amplitude of this wave is 2 cm. The rope is then changed for another rope made of the same material but with a radius 2 times larger while keeping the same tension in the rope. What is the amplitude of the waves in this new rope if the source still has the same frequency and the same 100 W power?

23. A wave propagates on a string. As it moves forward on the string, the wave sets in motion a part of the string that was not in motion before.

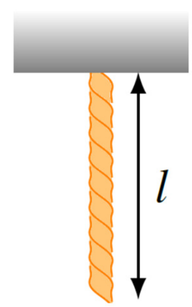


Show that the mass of the string set in motion per unit of time ($m/\Delta t$) is equal to the impedance of the string (Z).

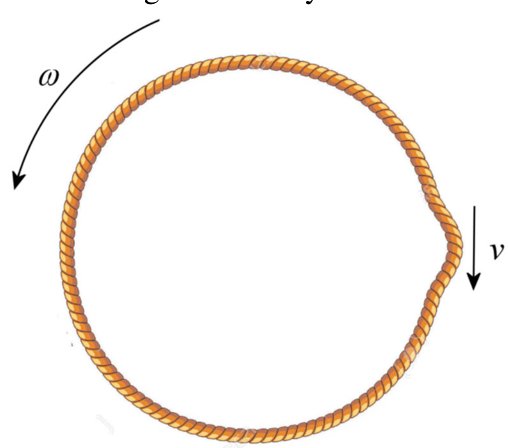
Challenges

(Questions more difficult than the exam questions.)

24. One end of a rope having a linear density of 0.2 kg/m is attached to the ceiling, as shown in the diagram. In this position, the tension of the string varies from one point to another in the rope. The length of the rope is 2 m. How long will take a wave to travel from the top of the rope to the bottom of the rope?



25. A loop of rope with a 25 cm radius rotates with an angular velocity of $\omega = 2$ rad/s. This rotation puts the rope under tension. A wave is present on the rope. What is the speed of this wave? (Neglect the effects of gravitation.)



www.123rf.com/photo_7684927_ropes-set--isolated-design-elements-gibbet-knot-loop-spiral.html

ANSWERS

2.2 Sine Wave

1. a) 0.0025 s b) 0.875 m
2. 1.5 m

2.3 Function Describing a Sine Wave

3. a) 200 Hz b) 75.4 m/s
4. a) towards the negative x-axis b) 0.6283 m c) 20 m/s d) -38.23 m/s
5. 2 cm
6. a) 5 m/s b) 250 m/s² c) 0.06858 s 0.19425 s 0.31991 s
7. $y = 0.02556m \cdot \sin\left(\frac{2\pi}{5}\frac{rad}{m} \cdot x - 20\pi\frac{rad}{s} \cdot t + 0.8986\right)$
8. a) 0.06283 s b) 1.562 cm
9. $y = 5cm \cdot \sin\left(\pi\frac{rad}{cm} \cdot x + 2\pi\frac{rad}{s} \cdot t + \frac{3\pi}{2} rad\right)$
10. $y = 2cm \cdot \sin\left(\frac{\pi}{3}\frac{rad}{cm} \cdot x - \pi\frac{rad}{s} \cdot t + \pi rad\right)$

2.4 Wave Speed on a Rope

11. 30 m/s
12. 0.25 kg
13. 62.5 N
14. 50.6 m/s
15. a) 0.125 kg/m b) 40 m/s
16. 144.1 m/s
17. 6.283 m

2.5 Energy in Sine Waves

18. a) 0.3084 J b) 1.542 W
19. 2.813 mm
20. 1.234 W
21. The tension is 320 N, and the linear density is 50 g/m
22. 1.414 cm

Challenges

24. 0.9035 s
25. 0.5 m/s (The same speed as the rope!)