1 HARMONIC OSCILLATIONS

A 1 kg bob is at the end of 20 m long rope to form a pendulum. It has a speed of 5 m/s when it is at the equilibrium position. What is the speed of the bob when the angle is 10°?

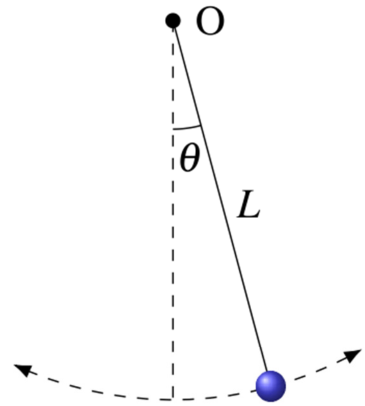
www.michaelfreemanphoto.com/media/1714951c-8d53-11e1-b996-f18e61a19d8d-foucault-pendulum

Discover how to solve this problem in this chapter.

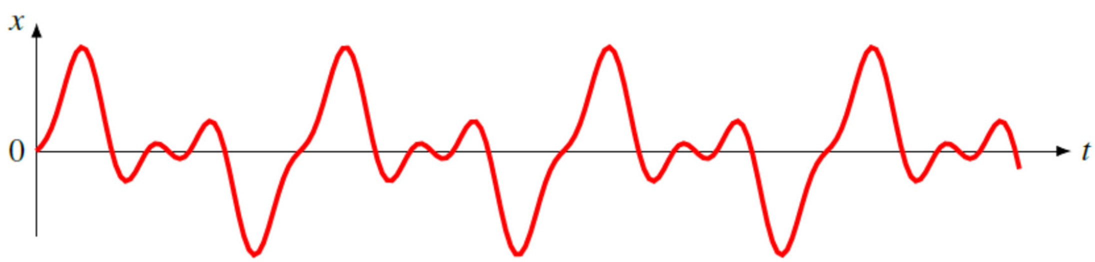
1.1 HARMONIC OSCILLATIONS

Description of the Motion with a Sine Function

When an object is oscillating, it passes continuously from one side to the other of a position called the *equilibrium position*. This could be, for example, a pendulum. In this case, the mass passes alternately from one side of the equilibrium position, which is at the lowest point of the motion of the pendulum, to the other.



If the equilibrium position is set at $x = 0$, then the position takes alternately positive and negative values. The graph of the position of the object over time could look like the following graph.



The graph can take a variety of forms. The only thing pointing to the fact that there is an oscillation is the repetition of an identical motion at each cycle.

There is, however, an important case: the oscillating motion described by a sinusoidal function. This motion is called *harmonic oscillation*. Then, the position is given by the function

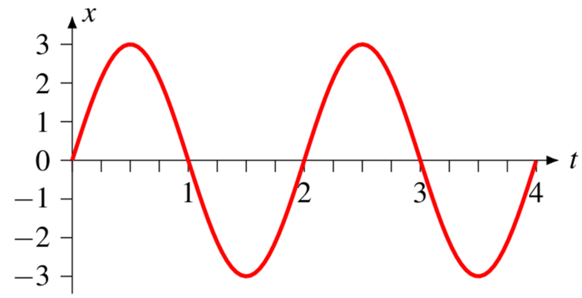
$$x = A\sin\left(\frac{2\pi}{T}t\right)$$

(Correctly, this is a simple harmonic motion. In a complex harmonic motion, the motion is described by adding several sine functions.)

Here is an example of a harmonic oscillation motion. The formula

$$x = 3cm \cdot \sin\left(\frac{2\pi}{2s}t\right)$$

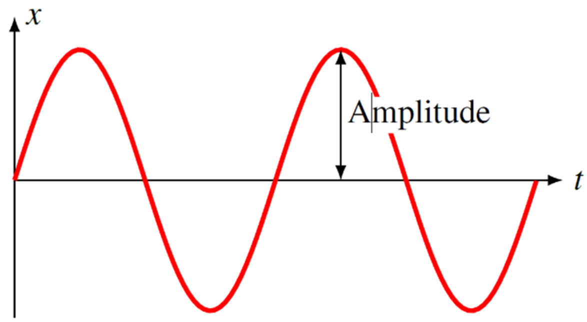
The graph on the right describes this motion.



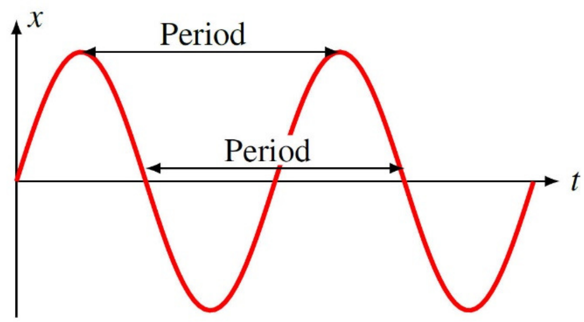
In this clip, the graph of the position as a function of time for a mass-spring system (which is a system making a harmonic oscillation) can be seen. It is similar to the graph showed.

http://www.youtube.com/watch?v=T7fRGXc9SBI

In the formula, A is the amplitude of the motion. The height of the sine function can be adjusted with this parameter. Normally, a sine has a maximum value of 1 and a minimum value of -1. Multiplying by A allows the sine to have a maximum value of A and a minimum value of -A. This amplitude indicates the greatest possible distance between the object and the equilibrium position. In the example at the top of the page, the amplitude is 3 cm.



T is the period of the motion. It indicates the time it takes for the object to make one complete cycle of the oscillation. Normally, a sine has a period of 2π (in radians). By multiplying the time by $2\pi/T$, the sine will then have a period of T. In the example at the top of the page, the period is 2 seconds.



Common Mistake: Using a Calculator in Degree Mode

The values in the trigonometric functions are in radians in this chapter. It is very common to see people forgetting to put their calculator in radian mode and getting incorrect values.

f is the frequency. It indicates the number of oscillations made by the object in one second. It is measured in hertz (Hz), which are s^{-1}.

Obviously, there is a link between the period and the frequency since the more oscillations the objects make per second, the smaller the period is. If the object makes 10 oscillations per second, then each oscillation lasts 0.1 s (1 second/10). If the object makes 50 oscillations per second, then each oscillation lasts 0.02 s (1 second / 50). Therefore,

Link between *T* and *f*

$$T = \frac{1}{f}$$

The sine function can, therefore, be written

$$x = A\sin\left(2\pi ft\right)$$

The quantity $2\pi f$ comes back continually in the study of harmonic oscillations. Physicists got tired of writing it and decided to use a symbol to represent it. It is the angular frequency.

Definition of Angular Frequency

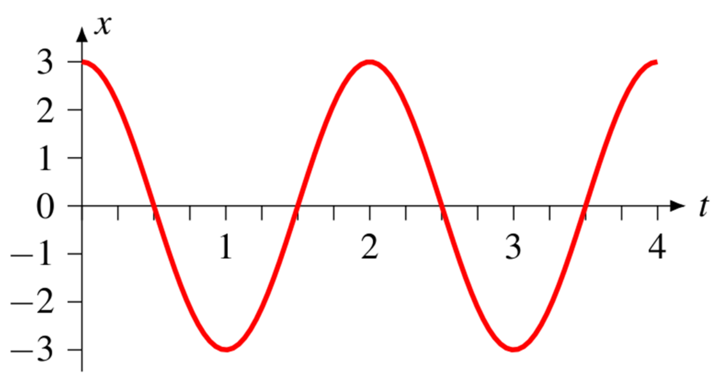
$$\omega = 2\pi f = \frac{2\pi}{T}$$

The angular frequency is in rad/s, and it represents the number of cycles of the oscillation made during a time of 2π seconds.

The sine function can now be written

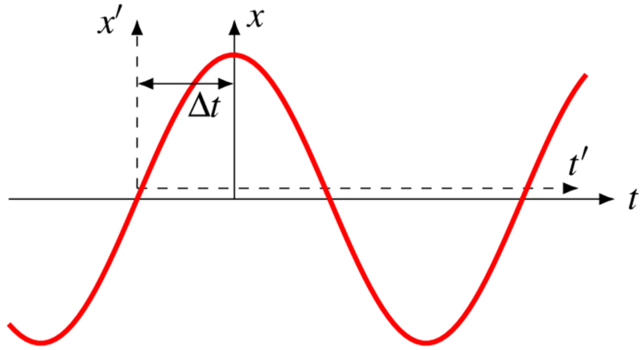
$$x = A\sin\left(\omega t\right)$$

Still, this is not the most general formula to describe harmonic oscillations, even if the amplitude and the period can be adjusted. The beginning of the motion must also be adjusted. The oscillation motion does not necessarily start at $x = 0$ as it is required with a sine function. For example, the oscillation motion could also begin at the maximum position as shown in this graph.



This graph is not the graph of a sine function. Something must be changed in the sine function to represent this function. In fact, the shape of the function has not changed; the sine function has only moved along the time axis. This is a simple translation of the function.

On this graph, the sine function is shifted by Δt towards the left.



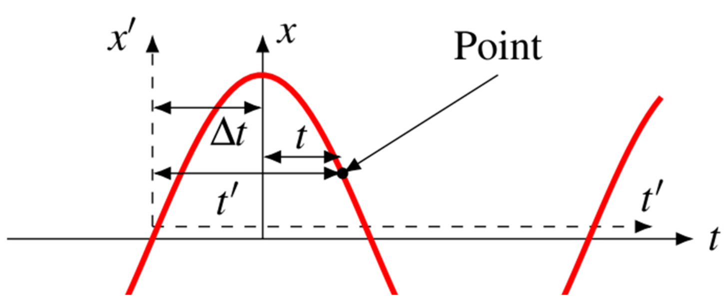
The dotted axes represent the original axes of the function, before the shift. With these dotted line axes, we still have the graph of a sine function. The equation is, therefore,

$$x' = A\sin(\omega t')$$

The continuous line axes are the new axes. With these axes, the sine function does not begin at zero. The equation of this new sine function can be determined by finding the laws of transformation between coordinates x' and t' and the coordinates x and t. As the height above the horizontal axes was not changed by the translation, the values of x were not changed.

$$x' = x$$

The points on the graph are, however, not at the same distance from the vertical axis since the x'-axis has been moved by Δt towards the left.



The coordinate transformations are then

$$t' = t + \Delta t$$

Using these transformations law in the sine function, the equation of this graph with the axes x and t is obtained.

$$x' = A\sin(\omega t')$$
$$x = A\sin(\omega(t + \Delta t))$$
$$x = A\sin(\omega t + \omega \Delta t)$$

since ω and Δt are constants, their product is also a constant. This new constant is denoted ϕ, and is called the *phase constant*.

$$\phi = \omega \Delta t$$

Therefore, the most general equation for a harmonic oscillation motion is

Position as a Function of Time for a Harmonic Oscillation

$$x = A\sin\left(\omega t + \phi\right)$$

Note that the expression inside the parenthesis is called the *phase of the sine function*. (This is a general term in mathematics, and it does not only apply to harmonic oscillations.)

Let's take a closer look at this phase constant that allows the sine to be shifted. Since $\omega = 2\pi/T$, we can write

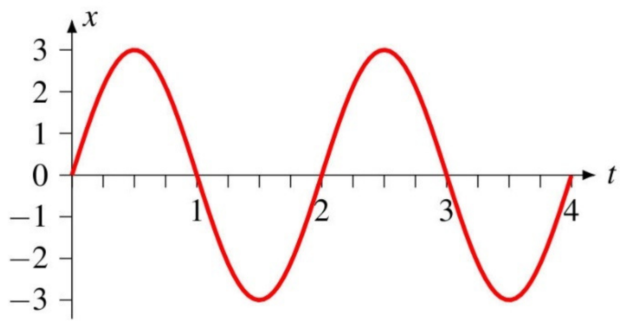
$$\begin{aligned}\phi &= \omega \Delta t \\ &= \frac{2\pi}{T}\Delta t\end{aligned}$$

The result is

Phase Constant

$$\phi = \frac{\Delta t}{T} 2\pi$$

Let's illustrate this with a few simple examples. We will start with the graph of a sine function with a zero phase constant. In this case, the sine is not shifted, and it starts at 0 with a positive slope.



$3\sin(\pi t)$

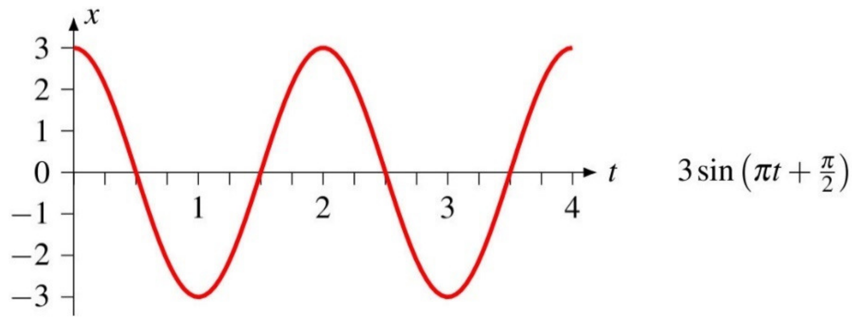
If the sine is shifted by one-quarter of a cycle, we have

$$\frac{\Delta t}{T} = \frac{1}{4}$$

and the phase constant is

$$\phi = \frac{1}{4}\cdot 2\pi = \frac{\pi}{2}$$

Here is the graph of this sine function shifted by one-quarter cycle to the left.



With ϕ = π/2, the sine function always starts at a maximum with zero slope.

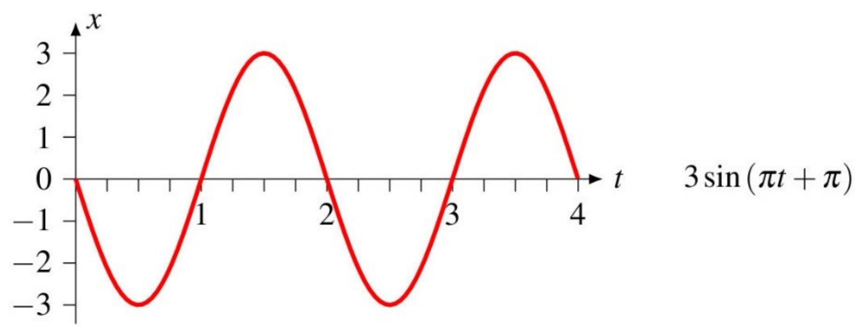
If the sine is shifted by one-half of a cycle, we have

$$\frac{\Delta t}{T} = \frac{1}{2}$$

and the phase constant is

$$\phi = \frac{1}{2} \cdot 2\pi = \pi$$

Here's the graph of this sine function shifted by one-half a cycle to the left.



With ϕ = π, the sine function always starts at 0 with a negative slope.

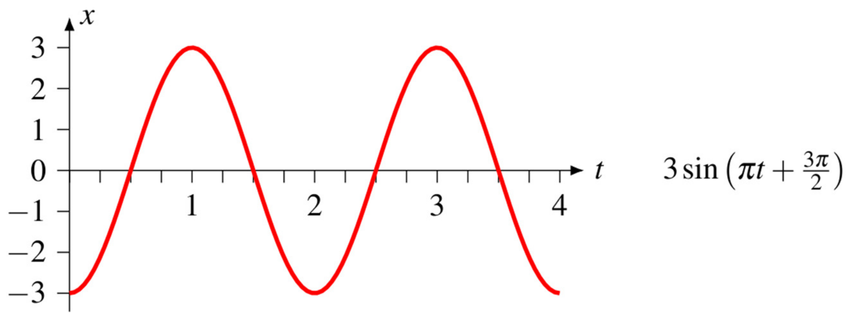
If the sine is shifted by three-quarters of a cycle, we have

$$\frac{\Delta t}{T} = \frac{3}{4}$$

and the phase constant is

$$\phi = \frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}$$

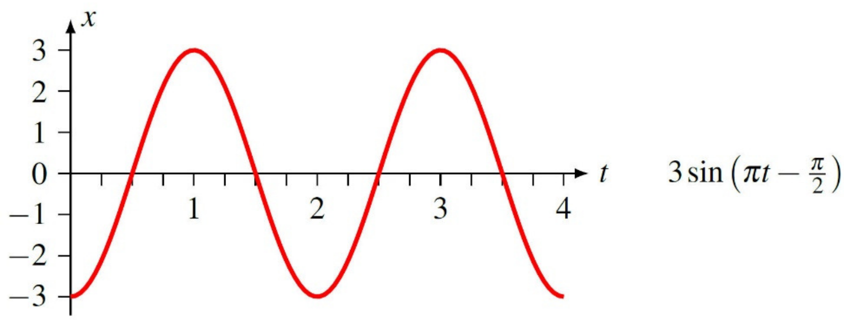
Here is the graph of this sine function shifted by three-quarters of a cycle to the left.



With ϕ =3π/2, the sine function always starts at a minimum with zero slope.

There is no need to use a phase constant of 2π since the function then shifts a full cycle and we are back to the same function.

Positive phase constants cause the sine function to shift towards negative *x*'s. To shift the sine function towards the positive *x*'s, simply use a negative phase constant. For example, here's a graph of a sine function shifted by one quarter of a cycle to the right (which means that ϕ = –π/2).



You may notice that this graph is identical to the graph for ϕ =3π/2. This is normal since the difference between –π/2 and 3π/2 is equal to 2π. When 2π is added or subtracted to the phase constant, the exact same graph is obtained. Therefore, you can add or subtract 2π to the phase constant as much as you want without changing the graph.

You may have already noticed that the graph of a cosine function is obtained when the phase constant is π/2. This means that it is also possible to describe harmonic oscillations with a cosine function. It is correct to do so, and some books on oscillations do indeed use a cosine function. You can quickly switch from one to the other with the following trigonometric identity.

$$\sin\left(\omega t+\phi\right)=\cos\left(\omega t+\phi-\tfrac{\pi}{2}\right)$$

Here a some of these transformations.

$$\sin\left(\omega t+\pi\right)=\cos\left(\omega t+\tfrac{\pi}{2}\right)$$
$$\sin\left(\omega t+4.206\right)=\cos\left(\omega t+2.635\right)$$

$$\sin(\omega t + 0.902) = \cos(\omega t - 0.669)$$

The phase constant is always smaller by π/ 2 with the cosine function than with the sine function.

The following identities can also be useful sometimes.

$$\sin\left(\omega t + \frac{\pi}{2}\right) = \cos(\omega t)$$
$$\sin(\omega t + \pi) = -\sin(\omega t)$$
$$\sin\left(\omega t + \frac{3\pi}{2}\right) = -\cos(\omega t)$$
$$\sin(\omega t + 2\pi) = \sin(\omega t)$$

The amplitude of the oscillation is never negative. A minus sign in front of the function is a hidden phase constant of π, as indicated in the second identity. Thus, the equation $x = -3 \sin(2t)$, is actually the same equation as $x = 3 \sin(2t + \pi)$, which clearly shows that the amplitude is 3 (and not - 3).

Velocity and Acceleration

With the position as a function of time, the velocity and the acceleration of the object as functions of time can be found. Since the velocity is

$$v = \frac{dx}{dt}$$

the velocity in a harmonic motion is

Velocity as a Function of Time for a Harmonic Oscillation

$$v = A\omega\cos(\omega t + \phi)$$

As the cosine can only take values between -1 and 1, the largest value for the speed is

Maximum Speed

$$v_{max} = A\omega$$

Since the acceleration is

$$a = \frac{dv}{dt}$$

the acceleration in a harmonic motion is

Acceleration as a Function of Time for a Harmonic Oscillation

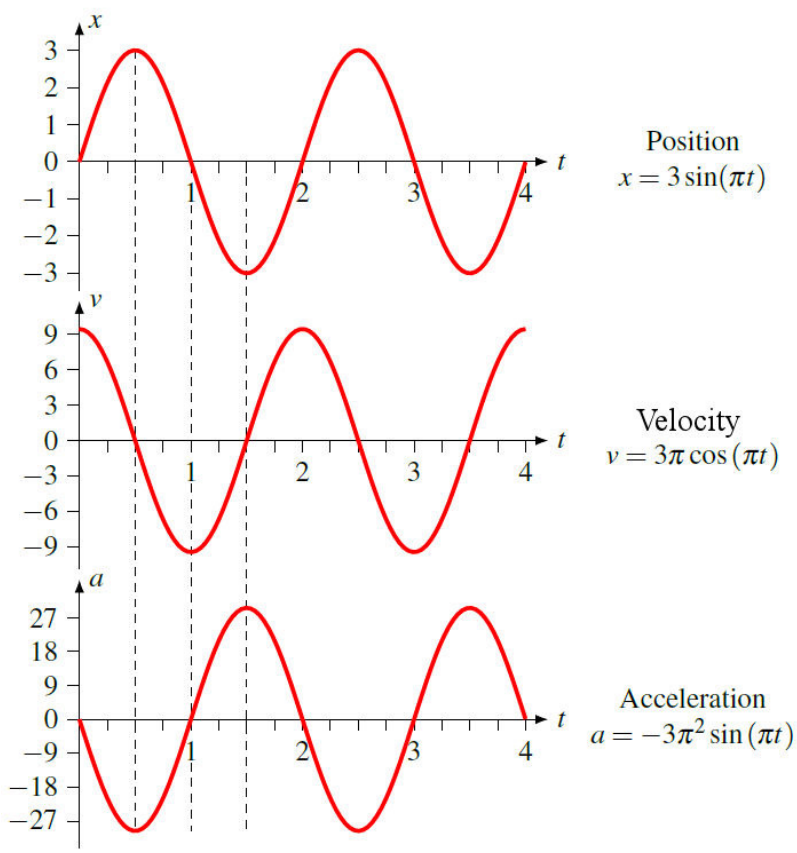
$$a = -A\omega^2 \sin(\omega t + \phi)$$

As a sine function can only take values between -1 and 1, the largest value of the acceleration (in absolute value) is

Maximum Acceleration

$$a_{max} = A\omega^2$$

Here are the graphs of the position, velocity, and acceleration for a harmonic motion.



It can be noted that the speed is zero when the position and the acceleration are maximum (positive or negative) and that the position and acceleration are both zero when the speed is maximum. This means that the object reaches its maximum speed at the equilibrium position.

You can marvel at these graphics in action in the next clip.
http://www.youtube.com/watch?v=eeYRkW8V7Vg

Link between Position and Velocity

A very useful link between the position and the velocity for an object performing a harmonic oscillation can be found by using the following trigonometric identity. This formula is obtained from the formulas for position and velocity.

$$x = A\sin(\omega t + \phi) \quad \text{and} \quad v = A\omega\cos(\omega t + \phi)$$

First, solve these equations for the sine and cosine functions

$$\sin(\omega t + \phi) = \frac{x}{A} \qquad \cos(\omega t + \phi) = \frac{v}{A\omega}$$

and then use this property of sine and cosine functions

$$\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi) = 1$$

To obtain

$$\left(\frac{x}{A}\right)^2 + \left(\frac{v}{A\omega}\right)^2 = 1$$

Multiplying by A^2, the following link is obtained.

Link between *x* and *v* at Some Instant

$$x^2 + \left(\frac{v}{\omega}\right)^2 = A^2$$

Link between Position and Acceleration

The position and the acceleration are given by

$$x = A\sin(\omega t + \phi) \quad \text{and} \quad a = -A\omega^2\sin(\omega t + \phi)$$

Then, it can be noted that

$$a = -A\omega^2\sin(\omega t + \phi)$$
$$a = -\omega^2\left[A\sin(\omega t + \phi)\right]$$

and, therefore, that

Link between *x* and *a* at Some Instant

$$a = -\omega^2 x$$

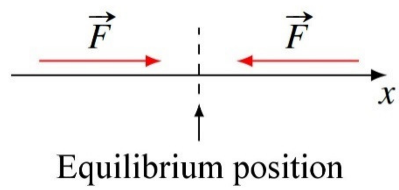
Condition to Have a Harmonic Oscillation

This last equation is also the sufficient condition to have a harmonic oscillation. When a system is analyzed by finding the forces acting on an object in order to determine its acceleration, it can be concluded that the motion of the object is a harmonic oscillation if the following result is obtained.

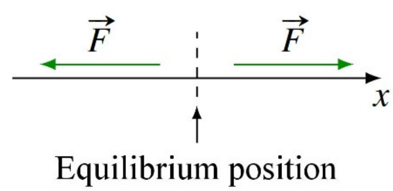
Condition for a Harmonic Oscillation

$$a = -(\text{constant})\, x$$

Moreover, the value of the constant gives ω^2, and this can be used to calculate the frequency and the oscillation period. This idea will be used later to determine whether some systems are doing a harmonic oscillation or not.



The minus sign in front of the constant is very important. It indicates that the acceleration, and hence the force, is opposed to the displacement. Thus, if the object is to the right of the equilibrium position, the force is towards the left. If the object is to the left of the equilibrium position, the force is towards the right. In this case, the force always brings the object back to the equilibrium position, which is necessary to obtain an oscillation.



If there were a positive sign, there would be no oscillations. To the right of the equilibrium position, the force would be towards the right and to the left of the equilibrium position, the force would be towards the left. These forces would push the object away from the equilibrium position rather than bring it back. The object then moves away from the equilibrium position and never returns to the equilibrium position. There is no oscillation in this case.

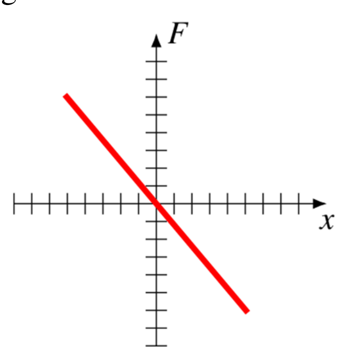
Graphical Interpretation

The condition to get a harmonic oscillation can be written, using

$$F_{net} = ma$$

as

$$F_{net} = -m\omega^2 x$$



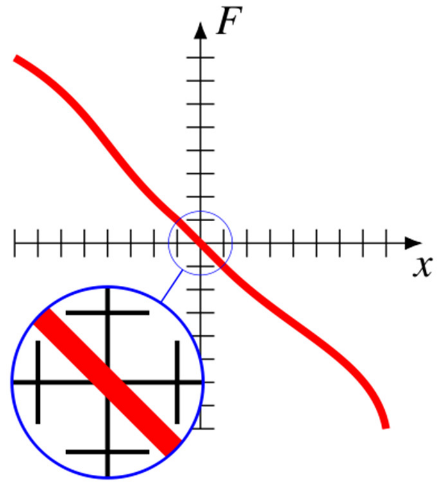
Since $m\omega^2$ is also a constant, the graph of the force must be a straight line with a negative slope to have a harmonic oscillation. In addition, the slope of this graph allows finding ω since the slope must be

$$Slope = -m\omega^2$$

The equilibrium position is at $F = 0$, so at the place where the F line crosses the x-axis. The slope of this graph determines the frequency of the oscillations.

Importance of Harmonic Oscillations

The equations for harmonic oscillations are important even if the graph of the net force acting on a system is not a straight line. For example, the graph of the force on the top part of this diagram is not a straight line and the oscillation of the system on which this force acts is not harmonic.



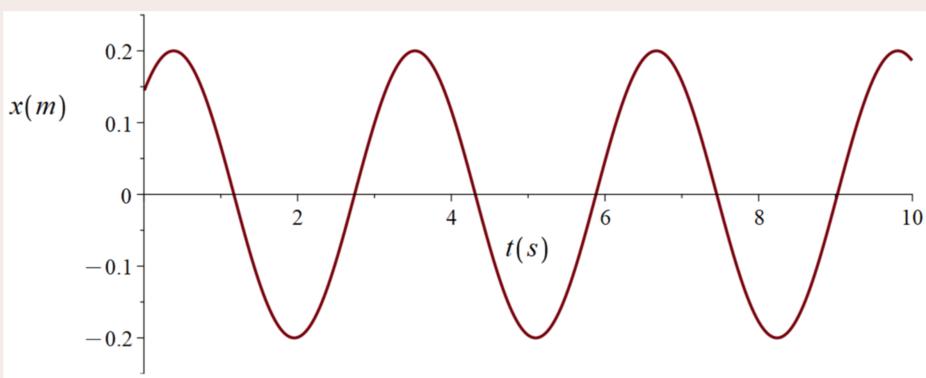
However, for low-amplitude oscillations, the object stays near the point of equilibrium. Close enough to the point of equilibrium, any function will be a straight line. This means that with small amplitude oscillations, the oscillation is always harmonic, regardless of the system.

Example 1.1.1

An object is performing a harmonic oscillation along the x-axis. Its position is given by

$$x = 0.2m \cdot \sin\left(2\tfrac{rad}{s} \cdot t + 0.8rad\right)$$

Here is the graph of this motion.



a) What is the period of this motion?

The period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2s^{-1}} = 3.142s$$

b) What is the velocity of the object at t = 1 s?

The velocity is the derivative of the position. Therefore, it is

$$\begin{aligned} v &= A\omega\cos(\omega t + \phi) \\ &= 0.2m \cdot 2s^{-1} \cdot \cos\left(2\tfrac{rad}{s} \cdot t + 0.8rad\right) \\ &= 0.4\tfrac{m}{s} \cdot \cos\left(2\tfrac{rad}{s} \cdot t + 0.8rad\right) \end{aligned}$$

At t = 1 s, the velocity is

$$\begin{aligned} v &= 0.4\tfrac{m}{s} \cdot \cos\left(2\tfrac{rad}{s} \cdot 1s + 0.8rad\right) \\ &= -0.377\tfrac{m}{s} \end{aligned}$$

c) What is the velocity of the object at x = 0.1 m?

The velocity is found with

$$x^2 + \left(\frac{v}{\omega}\right)^2 = A^2$$

$$(0.1m)^2 + \left(\frac{v}{2s^{-1}}\right)^2 = (0.2m)^2$$

$$v^2 = 0.12\tfrac{m^2}{s^2}$$

$$v = \pm 0.3464\tfrac{m}{s}$$

This velocity can be positive or negative because the object passes at a specific position twice in a cycle. Once while moving away from the equilibrium position and once while going towards the equilibrium position.

d) When will the object be at x = 0.1 m and have a positive velocity for the first time? (If the motion begins at t = 0 s (which is what is assumed most of the time)).

If the object is at x = 0.1 m, then

$$0.1m = 0.2m \cdot \sin\left(2\tfrac{rad}{s} \cdot t + 0.8rad\right)$$

If this equation is solved for t, the result is

$$0.1m = 0.2m \cdot \sin\left(2\tfrac{rad}{s} \cdot t + 0.8rad\right)$$

$$\frac{1}{2} = \sin\left(2\tfrac{rad}{s} \cdot t + 0.8rad\right)$$

$$\frac{\pi}{6} = 2s^{-1} \cdot t + 0.8 \qquad \text{or} \qquad \frac{5\pi}{6} = 2s^{-1} \cdot t + 0.8$$

$$t = -0.1382s \qquad \text{or} \qquad t = 0.9090s$$

(See the note on the inverse sine functions following this example to understand why there are two answers.)

To determine which of these two answers is good, the velocity at these two instants must be calculated.

$$v = 0.4\tfrac{m}{s} \cdot \cos\left(2\tfrac{rad}{s} \cdot t + 0.8rad\right)$$

$$v = 0.4\tfrac{m}{s} \cdot \cos\left(2\tfrac{rad}{s} \cdot -0.1382s + 0.8rad\right) \quad \text{or} \quad v = 0.4\tfrac{m}{s} \cdot \cos\left(2\tfrac{rad}{s} \cdot 0.9090s + 0.8rad\right)$$

$$v = 0.3464\tfrac{m}{s} \qquad \text{or} \qquad v = -0.3464\tfrac{m}{s}$$

As it was specified that the velocity is positive, the correct answer must be t = - 0.1382 s.

However, as the motion begins at $t = 0$, a negative value of t is unacceptable. It is possible to obtain a negative value for t because, mathematically, the sine function extends all the way from $-\infty$ to ∞. A correction to this answer must, therefore, be made to obtain a positive answer.

Actually, there are several other possible values of t because the motion is always repeating itself. If the object is at $x = 0.1$ m at some instant, it will also be there a period later. It will again return to this position another period later and so on. By adding the period to the obtained t values, many other possible answers are obtained. Therefore, the other possible answers are

$$-0.1382s \xrightarrow{+3.142s} 3.003s \xrightarrow{+3.142s} 6.145s$$

The smallest positive value is thus 3.003 s. This is the answer.

Common Mistake: Not finding all the solutions to the inverse sine or inverse cosine functions.

Be aware that there are several answers to arcsin and arccos functions. The two primary solutions are

Arcsin: answer given by the calculator and π – answer given by the calculator
Arccos: answer given by the calculator and – answer given by the calculator

Actually, there is an infinite number of solutions to an arcsin and an arccos function. They can all be found by adding or subtracting as many 2π as we want to these two primary solutions. Actually, something simpler can be done to find the other solutions because the only thing that is calculated here with arcsin and arccos functions are times. Once the values of t corresponding to the two primary solutions are obtained, all the other solutions can be found by adding or subtracting the period as many times as needed to each of these solutions. Remember that negative values of t cannot be kept for a harmonic oscillation since it is assumed that the motion begins at $t = 0$.

How to Find A if x and v at Some Instant Are Known

A can easily be found if x and v are known at any given time using the formula linking x to v, because the amplitude is present in this equation.

$$A^2 = x^2 + \left(\frac{v}{\omega}\right)^2$$

To calculate A with this formula, be sure to use the values of x and v at the same time (which is not necessarily $t = 0$).

How to Find ϕ if x and v at Some Instant Are Known

With graphs, we have seen that it is relatively easy to find ϕ when the shift of the sine function is ¼, ½ or ¾ of a cycle. It's more difficult for the other values of ϕ.

If the position for a certain value of t is known, we can obviously try to find ϕ with the following formula.

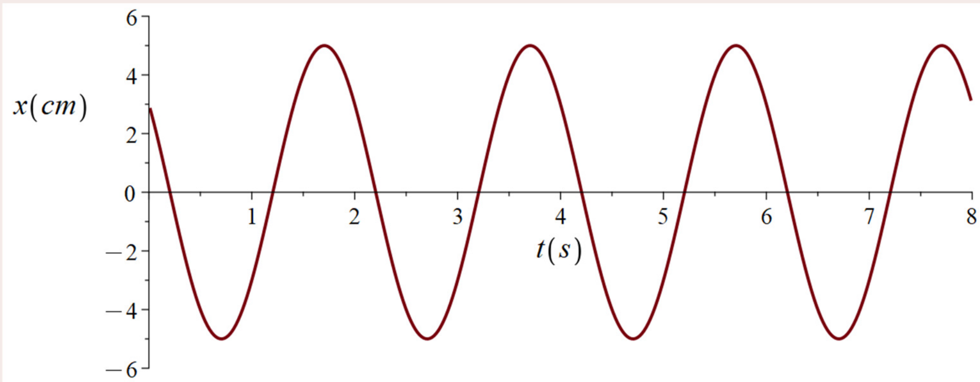
$$x = A\sin\left(\omega t + \phi\right)$$

Using the values of x and t, ϕ will be the only remaining unknown provided that ω and A are known. However, there will be a small problem. Since there are two main solutions to an arcsine function, we're going to obtain 2 values of ϕ. How can we know which of the 2 values is the right answer?

To choose between the 2 values of ϕ, you need to have information about the speed at time t. The sign of the speed may be enough. Just use the two values of ϕ obtained with the arcsine in the velocity formula. One of the values of ϕ is going to give a positive velocity, and the other value of ϕ is going to give a negative velocity. Keep the value of ϕ which gives the right sign for the speed.

Example 1.1.2

A system oscillates with a period of 2 s and an amplitude of 5 cm. At t = 0 s, the object is at x = 3 cm, and its velocity is negative. What is the value of the phase constant?



It is difficult to accurately assess the value of the phase constant from the graph. So, we're going to calculate it.

We know that the amplitude is 5 cm, and we could easily find the value of ω from T (but we'll see that we don't even need to know the value of ω here). So, the equation of motion is

$$x = 5cm \cdot \sin(\omega t + \phi)$$

Since it is known that the object is at x = 3 cm at t = 0, the equation becomes

$$3cm = 5cm \cdot \sin(\omega \cdot 0s + \phi)$$

$$0.6 = \sin\phi$$

$$0.6435 = \phi \qquad \text{or} \qquad 2.4981 = \phi$$

Now we need to find out which of these 2 answers is correct. We're going to determine this with the sign of the velocity at t = 0. (Note that we could also do this with the graph since it indicates that the phase constant must be between π/2 (a shift of ¼ of a cycle towards the left) and π (a shift of ½ of a cycle towards the left). Therefore, 2.4981 must be the correct value of ϕ.)

The formula for velocity is

$$v = A\omega\cos(\omega t + \phi)$$

At t = 0, the speed is

$$v = A\omega\cos(\phi)$$

As A and ω are positive, the velocity will be negative if cos(ϕ) is negative. The values of cos(ϕ) with our two values of ϕ are

$$\cos(0.6435) = 0.8 \qquad \cos(2.4981) = -0.8$$

We then understand that $\phi = 2.4981$ is the correct phase constant.

If the value of v is known, all these complications (calculation of 2 responses of the arcsine and then calculation of the velocities) can be avoided by using another formula. This formula is obtained from the position and velocity formulas.

$$x = A\sin(\omega t + \phi) \quad \text{and} \quad v = A\omega\cos(\omega t + \phi)$$

First, solve these equations for the sine and cosine functions

$$\sin(\omega t + \phi) = \frac{x}{A} \qquad \cos(\omega t + \phi) = \frac{v}{A\omega}$$

and then divide the sine by the cosine

$$\frac{\sin(\omega t + \phi)}{\cos(\omega t + \phi)} = \frac{x/A}{v/A\omega}$$

By substituting with tangent on the left and simplifying on the right, the following formula that allows to find the phase constant is obtained.

ϕ calculation

$$\tan(\omega t + \phi) = \frac{\omega x}{v}$$

To calculate ϕ with this formula, be sure to use the values of x and v at the same time (which is not necessarily $t = 0$).

Be careful:

- The value of ϕ is in radians. Set your calculator in radian mode to obtain the correct value.
- If the value of v is negative, remember to add π radians to the answer given by the calculator.
- If the speed is zero, the inverse tangent of ∞ or –∞ (the sign depends on the sign of x) must be calculated. Do not panic, the answer to these inverse tangents are π/2 and –π/ 2 respectively.

An arctangent function also has 2 main solutions (answer given by the calculator and π + answer given by the calculator). However, it is easy to know which of these two answers to take by following the 2nd rule: take the answer given by the calculator if the speed is positive and take π + the answer given by the calculator if the speed is negative.

Example 1.1.3

An object is performing a harmonic oscillation along the x-axis with a period of 0.5 seconds. At t = 0 s, the object is at x = -3 cm and has a velocity of -40 cm/s. What is the equation giving the position of the object as a function of time?

The value of the parameters in the following formula must be calculated.

$$x = A\sin\left(\omega t + \phi\right)$$

First, the angular frequency is calculated from the period.

$$\begin{aligned}\omega &= \frac{2\pi}{T} \\ &= \frac{2\pi}{0.5s} \\ &= 4\pi \tfrac{rad}{s}\end{aligned}$$

The amplitude is then found with

$$x^2 + \left(\frac{v}{\omega}\right)^2 = A^2$$

$$\left(-0.03m\right)^2 + \left(\frac{-0.4\frac{m}{s}}{4\pi s^{-1}}\right)^2 = A^2$$

$$A = 0.04374m$$

Finally, the phase constant is calculated with

$$\tan\left(\omega t + \phi\right) = \frac{\omega x}{v}$$

$$\tan\left(\omega \cdot 0 + \phi\right) = \frac{4\pi s^{-1} \cdot \left(-0.03m\right)}{-0.4\frac{m}{s}}$$

$$\tan\phi = \frac{3\pi}{10}$$

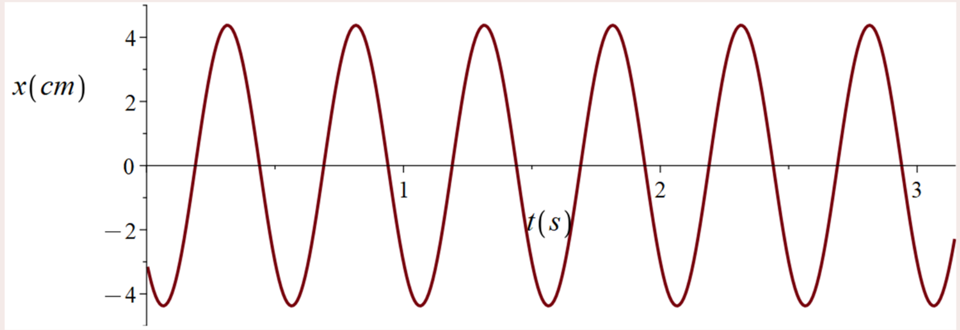
$$\phi = 3.8974 rad$$

π was added to the answer given by the calculator because the velocity is negative.

Therefore, the equation of motion is

$$x = 0.04374m \cdot \sin\left(4\pi \tfrac{rad}{s} \cdot t + 3.8974 rad\right)$$

This equation corresponds to the following graph.



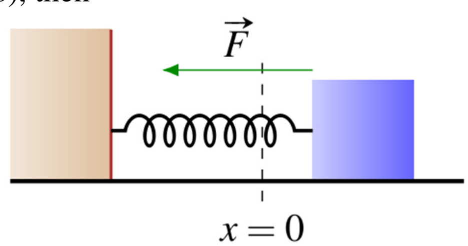
1.2 MASS-SPRING SYSTEMS

In order to determine if a mass-spring system on a frictionless surface performs a harmonic motion, the relation between the acceleration and the position must be found to see if it meets the condition for a harmonic oscillation. Remember that this condition is

$$a = -(\text{constant})\, x$$

Three forces act on the block: the weight (downwards), a normal force (upwards) and the force exerted by the spring (horizontal). The weight and the normal force cancel each other, and the only remaining force is the spring force. If the object is not at the equilibrium position (where the force exerted by the spring is zero), then

$$\sum F_x = ma$$
$$-kx = ma$$
$$a = -\left(\frac{k}{m}\right)x$$



This equation has the same form as the condition. Thus, it can be concluded that

The oscillations of a mass-spring system are harmonic oscillations.

As the value of the constant (in parentheses) must also be equal to the square of the angular frequency, the angular frequency is

Angular Frequency for a Mass-Spring System

$$\omega = \sqrt{\frac{k}{m}}$$

From this equation, the period of oscillation of a mass-spring system can be found. It is

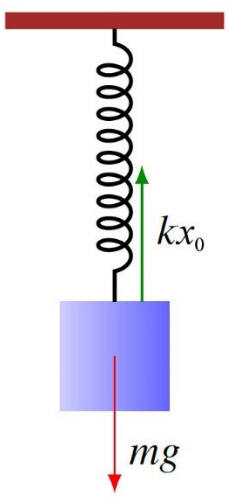
$$T = \frac{2\pi}{\omega}$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Here, a crucial and somewhat surprising fact is clear: the period of oscillation of a mass-spring system does not depend on the amplitude. Whatever the amplitude of oscillations, the period is always the same! The greatest distance to travel with a larger amplitude is exactly offset by larger speeds. This is most surprising and occurs only for harmonic oscillations. For any other types of oscillations, the period depends on the amplitude. This gives a way to know if you are dealing with a harmonic oscillation: measure the period of oscillation for different amplitude. If the period does not depend on the amplitude, the system is performing a harmonic oscillation.

The period of a harmonic motion does not depend on the amplitude of the motion.

Galileo was the first to notice this by examining the oscillation of a chandelier hanging from the roof of a church during mass in 1583. As the amplitude decreased due to friction, he noticed that the period of oscillation remained the same.

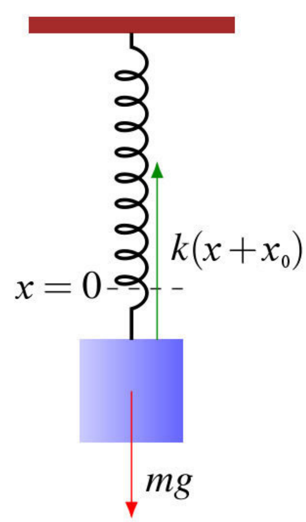


Will the system still perform harmonic oscillations if the system is vertical? In this case, is the force of gravity destroying the harmonic motion? To answer this question, the position of equilibrium must be found first. It is situated at the position where the sum of the forces acting on the object is zero. There, the spring is stretched a distance x_0 to cancel the force of gravity.

Stretching of the Spring at the Equilibrium Position

$$kx_0 = mg$$

We will now check if we still have harmonic oscillations around this equilibrium position. As specified previously, the $x = 0$ must be at the equilibrium position (we will continue to use x for the position even if the motion is vertical). If the mass is a little lower than the equilibrium position, Newton's second law gives (with an axis directed downwards)



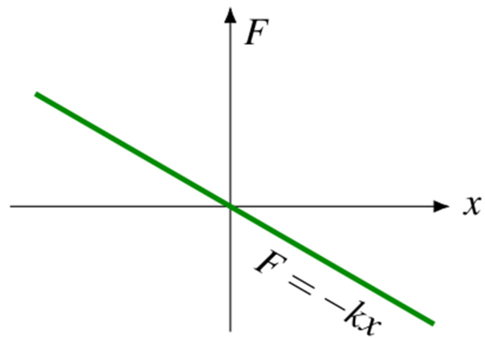
$$\sum F_x = ma$$
$$mg - k\left(x + x_0\right) = ma$$
$$mg - kx - kx_0 = ma$$

However, as $mg = kx_0$ (position of equilibrium formula), the equation becomes

$$\cancel{mg} - kx - \cancel{kx_0} = ma$$

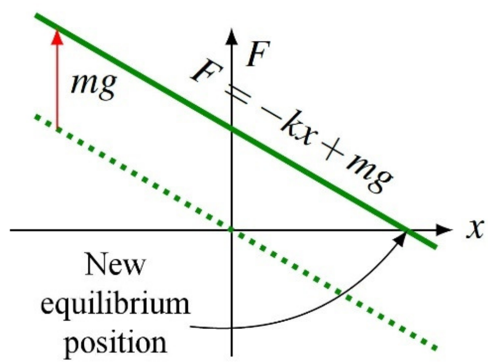
$$a = -\left(\frac{k}{m}\right)x$$

This still meets the requirement for a harmonic motion, and the angular frequency is the same as it was for a horizontal motion. The period of oscillation of a vertical mass-spring system is thus identical to the period of a system oscillating on a frictionless horizontal surface.



It is also possible to conclude, with a simple graph, that the frequency does not change. Without gravity, the graph is the graph of the force $F = -kx$. This graph (a straight line with a negative slope) is exactly the graph required to have a harmonic oscillation.

By adding the gravity mg to this force, the curve moves upwards by mg and the graph on the right is now obtained.



This graph now indicates that the equilibrium position (the place where the F line intersects the axis) has changed, but the oscillation frequency has not changed since the slope of the line remained the same.

The following convention in a mass-spring system will be used: the x-value is positive when the spring is stretched.

Example 1.2.1

A 200 g object is connected to a spring having a constant k = 5 N/m. The spring is stretched 10 cm, and the mass is released (no initial speed is given to the mass).

a) What is the period of the motion?

Let start by calculating the angular frequency.

$$\begin{aligned}\omega &= \sqrt{\frac{k}{m}} \\ &= \sqrt{\frac{5\frac{N}{m}}{0.2kg}} \\ &= 5\tfrac{rad}{s}\end{aligned}$$

With ω, the period can then be found.

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= 1.2566s \end{aligned}$$

b) What is the equation of the position of the object as a function of time?

The equation of the position is $x = A\sin(\omega t + \phi)$, ω is known. A and ϕ must then be found.

A is calculated with

$$x^2 + \left(\frac{v}{\omega}\right)^2 = A^2$$

$$(0.10m)^2 + \left(\frac{0\frac{m}{s}}{5s^{-1}}\right)^2 = A^2$$

$$A = 0.10m$$

ϕ is calculated with

$$\tan(\omega t + \phi) = \frac{\omega x}{v}$$

$$\tan(\omega \cdot 0 + \phi) = \frac{5s^{-1} \cdot 0.1m}{0\frac{m}{s}}$$

$$\tan\phi = \infty$$

$$\phi = \frac{\pi}{2} rad$$

The equation is thus

$$x = 0.1m \cdot \sin\left(5\tfrac{rad}{s} \cdot t + \frac{\pi}{2} rad\right)$$

c) What is the maximum speed of the object?

The maximum speed is

$$\begin{aligned} v_{\max} &= A\omega \\ &= 0.1m \cdot 5s^{-1} \\ &= 0.5\tfrac{m}{s} \end{aligned}$$

d) What is the maximum acceleration of the object?

The maximum acceleration is

$$\begin{aligned} a_{\max} &= A\omega^2 \\ &= 0.10m \cdot \left(5\tfrac{rad}{s}\right)^2 \\ &= 2.5\tfrac{m}{s^2} \end{aligned}$$

e) What are the first three instants at which the velocity of the object is 0.3 m/s?

The velocity formula

$$\begin{aligned} v &= A\omega\cos\left(\omega t+\phi\right) \\ &= 0.1m \cdot 5s^{-1} \cdot \cos\left(5\tfrac{rad}{s} \cdot t + \frac{\pi}{2} rad\right) \\ &= 0.5\tfrac{m}{s} \cdot \cos\left(5\tfrac{rad}{s} \cdot t + \frac{\pi}{2} rad\right) \end{aligned}$$

If the velocity is 0.3 m/s, then

$$0.3\tfrac{m}{s} = 0.5\tfrac{m}{s}\cos\left(5\tfrac{rad}{s} \cdot t + \frac{\pi}{2} rad\right)$$

$$0.6 = \cos\left(5\tfrac{rad}{s} \cdot t + \frac{\pi}{2} rad\right)$$

$$_{calculator}\swarrow \qquad \searrow_{-calculator}$$

$$0.9273 = 5s^{-1} \cdot t + \frac{\pi}{2} \qquad -0.9273 = 5s^{-1} \cdot t + \frac{\pi}{2}$$

$$t = -0.1287s \qquad t = -0.4996s$$

None of these two answers is good because they are both negative. However, there is an infinite number of solutions to an inverse cosine function, and these other solutions are found by adding the period as many times as needed to these two answers.

$$t = -0.1287s \qquad t = -0.4996s$$

$$\downarrow +1.2566s$$

$$t = 1.1279s \qquad t = 0.7570s$$

$$\downarrow +1.2566s$$

$$t = 2.3845s \qquad t = 2.0136s$$

$$\downarrow +1.2566s$$

$$t = 3.6411s \qquad t = 3.2702s$$

Therefore, the first three moments are

t = 0.7570 s, t = 1.1279 s, and t = 2.0136 s.

1.3 MECHANICAL ENERGY IN HARMONIC MOTION

Energy Formula

Energy can take two forms in a harmonic oscillation: kinetic energy (E_k) and potential energy (U). For example, in a mass-spring system, there are kinetic energy and spring potential energy. Mechanical energy is the sum of these two forms of energy.

Mechanical Energy

$$E_{mec} = E_k + U$$

Kinetic Energy

As usual, the kinetic energy is

Kinetic Energy

$$E_k = \frac{1}{2}mv^2$$

Potential Energy

Since

$$F = -\frac{dU}{dx}$$

the potential energy can be found with

$$\begin{aligned} a &= -\omega^2 x \\ F &= -m\omega^2 x \\ -\frac{dU}{dx} &= -m\omega^2 x \\ U &= \frac{1}{2}m\omega^2 x^2 + Cst \end{aligned}$$

Any value can be chosen for the integration constant. The simplest choice, of course, is to use zero.

Potential Energy (always good)

$$U = \frac{1}{2} m\omega^2 x^2$$

It is important to point out the following fact.

The potential energy must be proportional to x^2 to have a harmonic oscillation.

For a mass-spring system, $\omega^2 = k/m$ can be used to obtain

Potential Energy (good only for the mass-spring system)

$$U_S = \frac{1}{2} kx^2$$

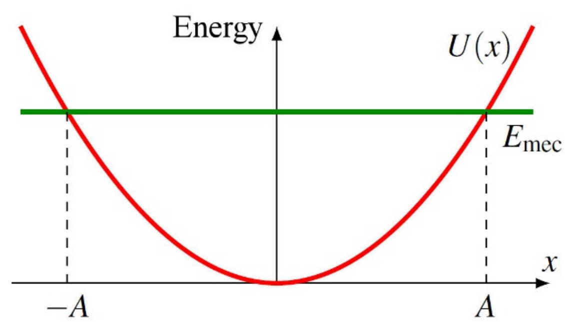
Maybe you recognize (first equation) the spring energy formula obtained in mechanics.

Potential Energy Graph

The potential energy formula is

$$U = \frac{1}{2} m\omega^2 x^2$$

The graph of this potential energy is shown to the right.



This graph shows that the object must stay between $-A$ and A (since the object cannot be at locations where U is greater than E_{mec}) and that its maximum speed is reached at the equilibrium position (where the gap between U and E_{mec} is the largest).

Proof of Mechanical Energy Conservation

It can now be shown that mechanical energy is conserved in a harmonic motion. This is achieved by adding the kinetic and potential energies and by using the formulas for the velocity and position as a function of time. The end result shows that the energy does not depend on time.

$$\begin{aligned} E_{mec} &= \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2 x^2 \\ &= \frac{1}{2}mA^2\omega^2\cos^2(\omega t+\phi) + \frac{1}{2}m\omega^2 A^2\sin^2(\omega t+\phi) \\ &= \frac{1}{2}m\omega^2 A^2\left(\cos^2(\omega t+\phi)+\sin^2(\omega t+\phi)\right) \\ &= \frac{1}{2}m\omega^2 A^2 \end{aligned}$$

As the result is a constant, mechanical energy is conserved.

Mechanical Energy

With this proof, the formula for the value of the mechanical energy was also obtained.

Mechanical Energy

$$E_{mec} = \frac{1}{2}m\omega^2 A^2$$

For a mass-spring system, $\omega^2 = k/m$ can be used to obtain

Mechanical Energy (good only for a mass-spring system)

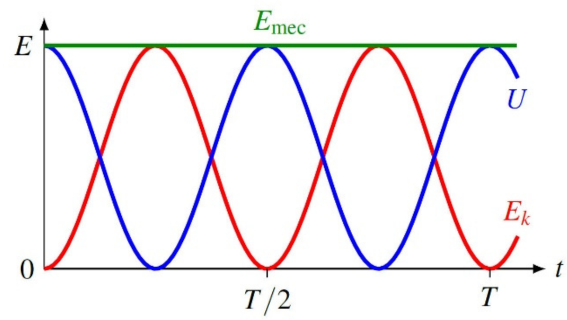
$$E_{mec} = \frac{1}{2}kA^2$$

In both cases, it can be seen that the mechanical energy is equal to the maximum value of the potential energy since the maximum value of x is A. Moreover, as ωA is the maximum speed, the mechanical energy is also

Mechanical Energy

$$E_{mec} = \frac{1}{2}mv_{\max}^2$$

This formula also shows that the mechanical energy is equal to the maximum value of the kinetic energy. You can admire here the graph of the energies as functions of time.



Suppose that this is a mass-spring system. It is clear that the energy passes alternatively from kinetic energy to spring energy. Initially here, the object is at its maximum position, and its velocity is zero. The mechanical energy is then entirely in the form of spring energy since the spring is stretched to its maximum. As the object moves towards the equilibrium position, the spring energy decreases and the kinetic energy increases so that at the equilibrium position, all the mechanical energy is in the form of kinetic energy. Then the object moves away from its equilibrium position. This increases the spring energy and decreases the kinetic energy. When the object reaches its greatest distance from the equilibrium position, the kinetic energy is again zero and all the mechanical energy is back in the form of the energy of the spring. Then the process starts all over again.

You can see in this video how the mechanical energy passes from one form to the other in a mass-spring system.
http://www.youtube.com/watch?v=PL5g_IwrC5U

Example 1.3.1

A 200 g object is connected to a spring having a 5 N/m constant. The position of the object is given by

$$x = 0.1m \cdot \cos\left(5 \tfrac{rad}{s} \cdot t\right)$$

(For those who are wondering why the position is given by a cosine function rather than a sine function, this is just a sine function with a hidden π/ 2 phase constant.)

a) What is the spring energy at t = 3 s?

The spring energy (or potential energy) is

$$U_S = \frac{1}{2}kx^2$$

The position at t = 3 s is needed to calculate the energy. This position is

$$\begin{aligned} x &= 0.1m \cdot \cos\left(5 \tfrac{rad}{s} \cdot 3s\right) \\ &= -0.07597m \end{aligned}$$

Therefore, the spring energy is

$$\begin{aligned} U_S &= \frac{1}{2}kx^2 \\ &= \frac{1}{2} \cdot 5 \tfrac{N}{m} \cdot \left(-0.07597m\right)^2 \\ &= 0.01443J \end{aligned}$$

b) What is the kinetic to energy at t = 3 s?

Kinetic energy is

$$E_k = \frac{1}{2}mv^2$$

The velocity is needed to calculate the kinetic energy. The formula $v = A\omega\cos(\omega t+\phi)$ cannot be used since the equation of the position is written with a cosine function and not a sine function. To find the velocity, there are two options.

1) Change the position formula so that it is written with a sine function.

When changing from a cosine function to a sine function, the phase constant must be increased by π/2. Thus, the position is

$$\begin{aligned} x &= 0.1m \cdot \cos\left(5\tfrac{rad}{s} \cdot t\right) \\ &= 0.1m \cdot \sin\left(5\tfrac{rad}{s} \cdot t + \frac{\pi}{2}\right) \end{aligned}$$

Then the formula $v = A\omega\cos(\omega t+\phi)$ can be used to find the velocity.

$$\begin{aligned} v &= A\omega\cos(\omega t + \phi) \\ &= 0.1m \cdot 5s^{-1} \cdot \cos\left(5\tfrac{rad}{s} \cdot t + \frac{\pi}{2}\right) \\ &= 0.5\tfrac{m}{s} \cdot \cos\left(5\tfrac{rad}{s} \cdot t + \frac{\pi}{2}\right) \end{aligned}$$

At t = 3 s, the velocity is

$$\begin{aligned} v &= 0.5\tfrac{m}{s} \cdot \cos\left(5\tfrac{rad}{s} \cdot 3s + \frac{\pi}{2}\right) \\ &= -0.3251\tfrac{m}{s} \end{aligned}$$

2) Calculate the derivative of the position formula.

$$\begin{aligned} v &= \frac{dx}{dt} \\ &= \frac{d\left(0.1m \cdot \cos\left(5\tfrac{rad}{s} \cdot t\right)\right)}{dt} \\ &= -0.1m \cdot 5s^{-1} \cdot \sin\left(5\tfrac{rad}{s} \cdot t\right) \\ &= -0.5\tfrac{m}{s} \cdot \sin\left(5\tfrac{rad}{s} t\right) \end{aligned}$$

At t = 3 s, the velocity is

$$\begin{aligned} v &= -0.5\tfrac{m}{s}\cdot\sin\left(5\tfrac{rad}{s}\cdot 3s\right) \\ &= -0.3251\tfrac{m}{s} \end{aligned}$$

The kinetic energy is thus

$$\begin{aligned} E_k &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}\cdot 0.2kg\cdot\left(-0.3251\tfrac{m}{s}\right)^2 \\ &= 0.01057J \end{aligned}$$

c) What is the mechanical energy at t = 3 s?

Here are 3 ways to calculate the mechanical energy.

1) Add the kinetic and potential energy at t = 3 s.

$$\begin{aligned} E_{mec} &= E_k + U \\ &= 0.01057J + 0.01443J \\ &= 0.025J \end{aligned}$$

2) Calculate the mechanical energy from the amplitude.

$$\begin{aligned} E_{mec} &= \frac{1}{2}kA^2 \\ &= \frac{1}{2}\cdot 5\tfrac{N}{m}\cdot\left(0.1m\right)^2 \\ &= 0.025J \end{aligned}$$

3) Calculate the mechanical energy from the maximum speed

The maximum speed is

$$\begin{aligned} v_{\max} &= A\omega \\ &= 0.1m\cdot 5s^{-1} \\ &= 0.5\tfrac{m}{s} \end{aligned}$$

Thus, the energy is

$$\begin{aligned} E_{mec} &= \frac{1}{2}mv_{\max}^2 \\ &= \frac{1}{2}\cdot 0.2kg\cdot\left(0.5\tfrac{m}{s}\right)^2 \\ &= 0.025J \end{aligned}$$

Since the energy is constant, this value is the mechanical energy at any moment, not just at $t = 3$ s.

1.4 PENDULUM

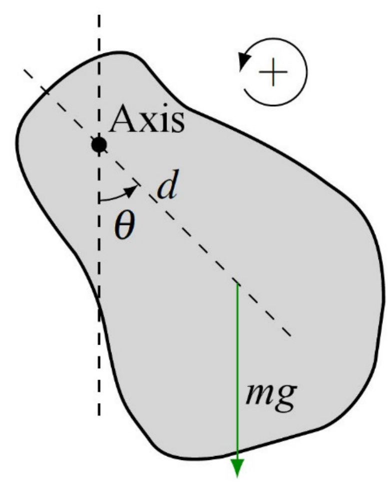
Proof that Pendulum Motion is a Harmonic Motion (for Small Amplitudes)

Is the pendulum complying with the condition to have a harmonic motion? A pendulum motion is a rotational instead of a linear motion. Without providing evidence, the harmonic motion condition for a rotational motion is

$$\alpha = -(\text{constant})\theta$$

where the constant is always ω^2.

Let's examine whether the pendulum respects this condition. A more complex case than a simple mass at the end of a rope is considered. We'll do the verification for an object oscillating around a fixed axis. Two forces are acting on the pendulum: the weight and the force exerted by the axis. This last force is exerted on the axis of rotation, and thus exerts no torque. Only the force of gravitation exerts a torque on the pendulum (the positive direction shown in the diagram will be used for the sign of this torque). Newton's second law for rotational motion gives



$$\sum \tau = I\alpha$$

$$-mgd\sin(180° - \theta) = I\alpha$$

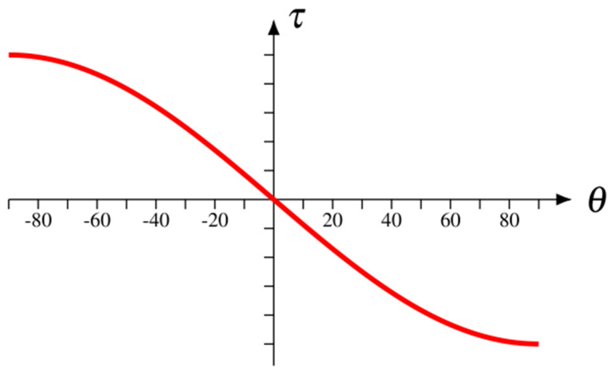
$$-mgd\sin\theta = I\alpha$$

where d is the distance from the axis of rotation to the centre of mass of the object. Solving this equation for the angular acceleration, we obtain

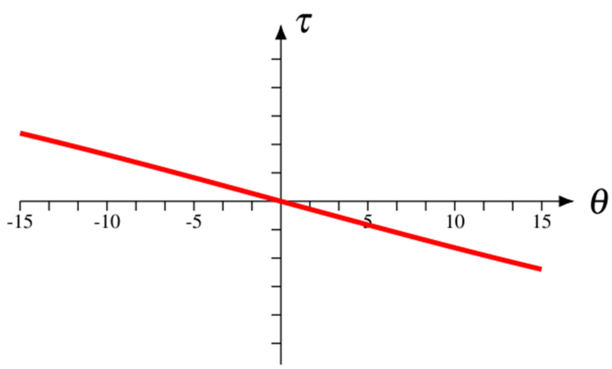
$$\alpha = -\left(\frac{mgd}{I}\right)\sin\theta$$

Unfortunately, this is not the desired result because of the sine function. The pendulum would meet the condition if, instead of $\sin\theta$, we would have had only θ.

This means that the pendulum motion is not a harmonic motion. It is an oscillation motion, but it is not a harmonic motion. The motion is described by another function more complex than a simple trigonometric function (the exact solution involves elliptic functions, which, normally, you do not know). The graph of the torque as a function of the angle (graph of $\tau = -\ mgd\ \sin\theta$) also confirms that we do not have the conditions to get a harmonic oscillation since the graph is not a straight line with a negative slope.



However, for angles less than 15°, the graph is.



Now, this is really close to a straight line with a negative slope. This means that, for small angles (smaller than about 15 °), the oscillation will effectively be harmonic.

This comes from the fact that at small angles, the following approximation can be made.

$$\sin\theta \approx \theta$$

With this approximation, the angular acceleration then becomes

$$\alpha = -\left(\frac{mgd}{I}\right)\theta$$

This is the condition for having a harmonic oscillation. This means that for small-amplitude oscillations, the motion of the pendulum looks a lot like a harmonic oscillation. The angular frequency can even be found since the value inside the parenthesis must be equal to ω^2. Therefore

$$\omega = \sqrt{\frac{mgd}{I}}$$

Angular Frequency of a Simple Pendulum

A simple pendulum is a point mass fixed at the end of a massless rope. The centre of mass of the pendulum is, therefore, at the centre of the point mass, and d is thus equal to the length of the rope.

$$d = L$$

As all the mass is concentrated in the point mass, the moment of inertia is

$$I = \sum mr^2$$
$$I = mL^2$$

Therefore, the angular frequency is

$$\omega = \sqrt{\frac{mgd}{I}}$$
$$= \sqrt{\frac{mgL}{mL^2}}$$

If this equation is simplified, it becomes

Angular Frequency of a Simple Pendulum (Amplitude < 15°)

$$\omega = \sqrt{\frac{g}{L}}$$

The period of the pendulum is, therefore,

$$T = \frac{2\pi}{\omega}$$
$$T = 2\pi\sqrt{\frac{L}{g}}$$

(As early as 1637, Galileo had indicated that $T \propto \sqrt{L}$. The complete formula was given by Huygens in 1673.)

It can be noted once again that the period does not depend on the amplitude. This is normal; this is what happens with any harmonic oscillations. Here, there is another curiosity: the period does not depend on the mass of the pendulum!

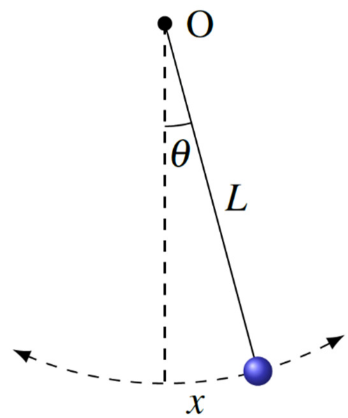
The period of a simple pendulum does not depend on its mass.

This is not so surprising if you think about it because the motion is caused by the force of gravity and this force gives the same acceleration to all objects, whatever their mass.

The change of period with the length of the rope can be seen in this clip. The effects produced are very nice.
http://www.youtube.com/watch?v=yVkdfJ9PkRQ

Description of the Motion of a Simple Pendulum

Knowing that the motion is a harmonic oscillation, the position can be described with a sine function. There are, however, two possibilities here to give this position. It can be given by the position x (measured along the arc of a circle) or by the angle θ (the angle between the rope and a vertical line).



The position can then be given by

Position of a Pendulum

$$x = A\sin\left(\omega t + \phi\right)$$

or

$$\theta = \theta_{\max}\sin\left(\omega t + \phi\right)$$

The angle θ can be in degrees or in radians as needed. If it in degrees, there is no need to put your calculator in degree mode to calculate the position, since this angle is not inside a trigonometric function. In fact, you must leave your calculator in radian mode because ω is always in radians per second. Since the angle inside the trigonometric function is in radians, the calculator must be in radian mode.

It is possible to switch easily from x to θ since the angle (in radians) is the length of the arc of a circle divided by the radius of the circle. This means that

Link between x and θ for a Pendulum

$$\theta_{(rad)} = \frac{x}{L}$$

With the maximum values of x (which is A) and θ (which is θ_{max}), this equation becomes

Link between A and θ_{max} for a Pendulum

$$\theta_{\max(rad)} = \frac{A}{L}$$

Common Mistake: Using the formulas of the previous sections and confusing θ and x

Note that you must use the position x in all the formulas from earlier sections and not θ. Sometimes, a person uses θ for the position instead of x for some calculations. This is a mistake.

Example 1.4.1

A pendulum consists of a mass attached at the end of a 1 m long rope. Initially (t = 0 s), the pendulum is at the position θ = 10 ° and is moving away from the equilibrium position with a speed of 50 cm/sec.

a) What is the period?

The angular frequency is

$$\begin{aligned}\omega &= \sqrt{\frac{g}{L}} \\ &= \sqrt{\frac{9.8\frac{N}{kg}}{1m}} \\ &= 3.13\tfrac{rad}{s}\end{aligned}$$

The period is then

$$\begin{aligned}T &= \frac{2\pi}{\omega} \\ &= \frac{2\pi}{3.13s^{-1}} \\ &= 2.007s\end{aligned}$$

(It is not a coincidence to have an answer so close to 2 seconds since, at one point, they tried to define the metre with a pendulum: the metre would have been the length of the pendulum having a period of 2 s. This idea was not followed, as g varies slightly from one place to another on Earth.)

b) What is the equation of θ (in degrees) as a function of time?

The equation of θ as a function of time is

$$\theta = \theta_{max}\sin(\omega t + \phi)$$

The value of ω is already known. The angular amplitude and the phase constant remains to be found, however. Those are found with

$$A^2 = x^2 + \left(\frac{v}{\omega}\right)^2$$

$$\tan(\omega t + \phi) = \frac{\omega x}{v}$$

To use these formulas, x and v at t = 0 must be known. The velocity is known (0.5 m/s) but not the position x. However, with the initial angular position (θ), the position can be calculated.

$$\begin{aligned} x &= \theta_{(rad)} L \\ &= 10° \cdot \frac{2\pi rad}{360°} \cdot 1m \\ &= 0.1745m \end{aligned}$$

Therefore, the amplitude is

$$A^2 = x^2 + \left(\frac{v}{\omega}\right)^2$$

$$A^2 = (0.1745m)^2 + \left(\frac{0.5\frac{m}{s}}{3.13s^{-1}}\right)^2$$

$$A = 0.2366m$$

Then, the angular amplitude (maximum angle) can be calculated.

$$\begin{aligned} \theta_{\max(rad)} &= \frac{A}{L} \\ &= \frac{0.2366m}{1m} \\ &= 0.2366rad \end{aligned}$$

In degrees, the maximum angle is

$$\begin{aligned} \theta_{\max(°)} &= 0.2366rad \cdot \frac{360°}{2\pi rad} \\ &= 13.55° \end{aligned}$$

The phase constant is

$$\tan(\omega t + \phi) = \frac{\omega x}{v}$$

$$\tan(\phi) = \frac{3.13s^{-1} \cdot 0.1745m}{0.5\frac{m}{s}}$$

$$\phi = 0.8295rad$$

The equation for θ is thus

$$\theta = 13.55° \cdot \sin\left(3.13\tfrac{rad}{s} \cdot t + 0.8295rad\right)$$

c) What is the speed of the pendulum when the angle is 5°?

When the position is known, the velocity can be calculated with

$$x^2 + \left(\frac{v}{\omega}\right)^2 = A^2$$

But first, the position x when the angle is 5° must be found. To begin, we calculate the angle in radians

$$\begin{aligned}\theta_{(rad)} &= 5° \cdot \frac{2\pi rad}{360°} \\ &= \frac{\pi}{36} rad\end{aligned}$$

And then the position can be calculated

$$\begin{aligned}x &= \theta_{(rad)} L \\ &= \frac{\pi}{36} rad \cdot 1m \\ &= 0.0873m\end{aligned}$$

The speed is then

$$x^2 + \left(\frac{v}{\omega}\right)^2 = A^2$$

$$(0.0873m)^2 + \left(\frac{v}{3.13s^{-1}}\right)^2 = (0.2366m)^2$$

$$v = 0.6883\tfrac{m}{s}$$

d) What is the maximum speed of the pendulum?

The maximum speed is

$$\begin{aligned}v_{\max} &= A\omega \\ &= 0.2366m \cdot 3.13s^{-1} \\ &= 0.741\tfrac{m}{s}\end{aligned}$$

Potential Energy of the Simple Pendulum

As for any harmonic oscillation, the potential energy (which is gravitational energy for a pendulum) is

$$U_g = \frac{1}{2} m\omega^2 x^2$$

Using the value of ω for a pendulum, the potential energy becomes

Potential Energy (good only for a pendulum)

$$U_g = \frac{1}{2} m \frac{g}{L} x^2$$

Using $x = \theta_{(\text{rad})} L$, this energy can also be written as

Potential Energy (good only for a pendulum)

$$U_g = \frac{1}{2} mgL\theta^2$$

The angle is in radians in this formula.

These formulas for the gravitational energy may seem a bit odd since we had seen in mechanics that the gravitational energy is *mgy*. However, it can easily be shown that from $U = mgy$, the aforementioned formulas of gravitational energy can be obtained.

First, the relationship between y and θ in a pendulum

$$y = L(1 - \cos\theta)$$

must be used. Then, the Taylor development for the cosine function (which can be used for oscillations with a small amplitude),

$$\cos\theta = 1 - \frac{\theta^2}{2} + \ldots$$

is used to obtain

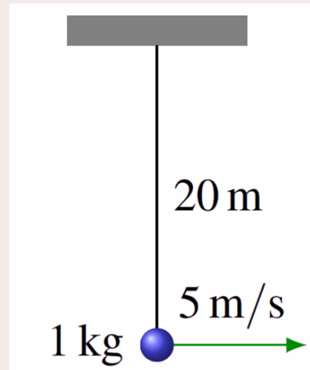
$$\begin{aligned} y &= L(1 - \cos\theta) \\ &= L\left(1 - \left(1 - \frac{\theta^2}{2}\right)\right) \\ &= \frac{L\theta^2}{2} \end{aligned}$$

Using this in *mgy*, the gravitational energy becomes

$$U_g = mgy = mg\frac{L\theta^2}{2}$$

which is the formula obtained previously.

Example 1.4.2



The pendulum in the diagram has a speed of 5 m/s when it is at the equilibrium position. What is the speed of the pendulum when the angle is 10°?

At instant 1 (when the pendulum is at the equilibrium position), the mechanical energy is

$$\begin{aligned} E &= E_k + U \\ &= \frac{1}{2}mv^2 + \frac{1}{2}mgL\theta^2 \\ &= \frac{1}{2}\cdot 1kg \cdot \left(5\tfrac{m}{s}\right)^2 + 0 \\ &= 12.5J \end{aligned}$$

At instant 2, (when the angle is 10° = π/18 rad), the mechanical energy is

$$\begin{aligned} E' &= \frac{1}{2}mv^2 + \frac{1}{2}mgL\theta^2 \\ &= \frac{1}{2}\cdot 1kg \cdot v^2 + \frac{1}{2}\cdot 1kg \cdot 9.8\tfrac{N}{kg}\cdot 20m \cdot \left(\frac{\pi}{18}rad\right)^2 \\ &= 0.5kg \cdot v^2 + 2.985J \end{aligned}$$

Mechanical energy conservation then gives

$$E = E'$$
$$12.5J = 0.5kg \cdot v^2 + 2.985J$$
$$v = 4.362\tfrac{m}{s}$$

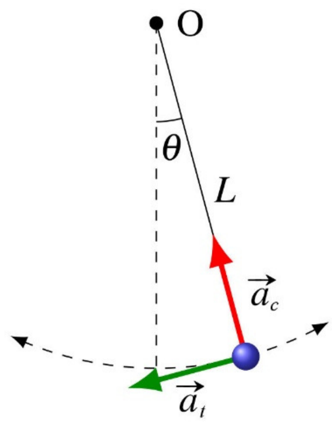
Acceleration of the Pendulum

Just a small note to tell you that the formula

$$a = -A\omega^2 \sin\left(\omega t + \phi\right)$$

will provide, for the pendulum, the tangential acceleration (a_t). Do not forget that the pendulum is also performing a circular motion and that there is also a centripetal acceleration.

$$a_c = \frac{v^2}{L}$$



The total acceleration of the pendulum is, therefore,

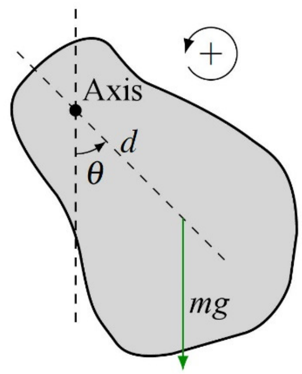
$$a = \sqrt{a_c^2 + a_t^2}$$

In the following clip, you can see the velocity and acceleration vectors of a pendulum, as well as the values of the energies.

http://www.youtube.com/watch?feature=fvwp&NR=1&v=jyHFXTZmWgI

Compound Pendulum

A compound pendulum is an object oscillating around an axis. The calculation of the angular frequency for this case has already been done.



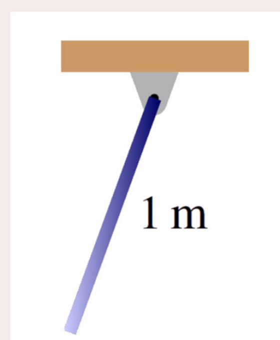
Angular Frequency of a Compound Pendulum

$$\omega = \sqrt{\frac{mgd}{I}}$$

It only remains to find *d* (the distance between the axis and the centre of mass) and *I* (the moment of inertia) according to the situation.

Example 1.4.3

A 1 m long rod oscillates. The axis of rotation is at one end of the rod, as shown in the diagram. What is the period of oscillation of the rod?



To find the period, which is $2\pi/\omega$, ω is needed. For a compound pendulum, ω is

$$\omega = \sqrt{\frac{mgd}{I}}$$

To calculate ω, *d* and *I* are needed.

Let's begin with *d*. As the centre of mass of the rod is in the middle of the rod and the axis is at the end of it, the distance between the centre of mass and the axis is

$$d = \frac{L}{2} = 0.5m$$

The moment of inertia I is

$$I = I_{c.m.} + md^2$$

As $I_{c.m.}$ of a rod is

$$I_{c.m.} = \frac{1}{12} mL^2$$

The moment of inertia is

$$I = \frac{1}{12} mL^2 + m\left(\frac{L}{2}\right)^2 = \frac{1}{3} mL^2$$

The angular frequency is therefore

$$\begin{aligned}
\omega &= \sqrt{\frac{mgd}{I}} \\
&= \sqrt{\frac{mgd}{\frac{1}{3} mL^2}} \\
&= \sqrt{\frac{3gd}{L^2}} \\
&= \sqrt{\frac{3 \cdot 9.8 \frac{N}{kg} \cdot 0.5m}{(1m)^2}} \\
&= 3.834 \tfrac{rad}{s}
\end{aligned}$$

and the period is

$$\begin{aligned}
T &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{3.834 s^{-1}} \\
&= 1.639s
\end{aligned}$$

The moment of inertia of objects can also be determined with the period if it is very difficult to calculate directly. Just make the object oscillate and calculate the value of I from the value of the period and d.

1.5 RESONANCE

So far, we have talked about free oscillating systems. This means that once the swinging has begun, there is no external force acting on the system. Therefore, mechanical energy is conserved, and the oscillation is perpetual.

Of course, frictional forces are always present. This is an external force that makes the mechanical energy of the system decrease. We would then deal with damped oscillations. With not too much friction, the amplitudes of the oscillations would just decrease with time (this corresponds to subcritical damping). We are not going to explore in detail what happens in this case.

An external force could also be acting on the system. The case of periodic forces exerted on the system is particularly interesting. Those are forces acting on the system cyclically. For example, a person who pushes a child on a swing is exerting a periodic force. Whenever the child returns, the person exerts a force on the child. Each boost then increases the amplitude of the motion.

Let's consider what happens if a periodic force is exerted on a mass-spring system that oscillates horizontally on a frictionless surface with a natural oscillation period of 1 second. Suppose initially that the force is exerted with a period that is not the same as that of the mass-spring system. For example, this can be a 5 N force acting towards the right during 0.1 second every 0.5 seconds. Initially, the force pushes the mass and puts it into motion. A half second later, the mass has done half of the oscillation and has returned to the point of departure, but is now moving towards the left. At this moment, the force towards the right acts again on the mass that is moving towards the left. This force will, therefore, stop the mass. Half a second later, the force to the right will put the mass in motion once again, but it will be stopped again half a second later. This is always what happens if the period of the force is different than the period of the oscillating system: sometimes, the force gives energy to the mass when the force is in the same direction as the velocity, and sometimes it removes energy from the mass when it acts in the opposite direction to the velocity. The overall effect is, therefore, not large.

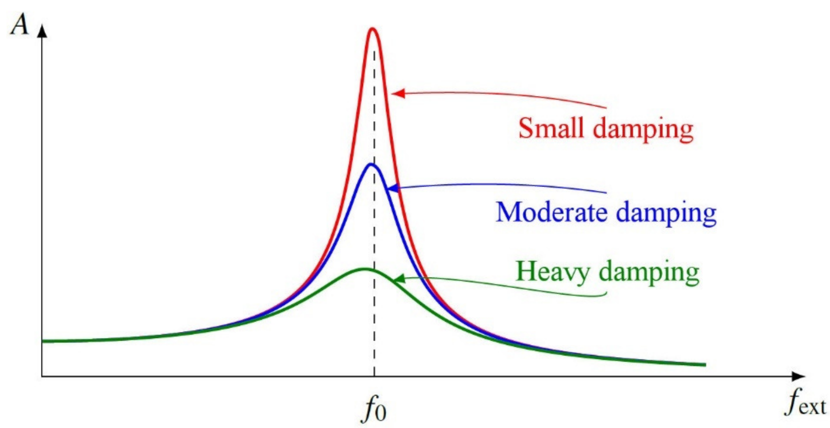
On the other hand, the effect is quite different if the period of the force is the same as the period of the oscillating system. If the force acts every second on our mass-spring system then, the amplitude will increase continuously. Initially, the force puts the mass in motion towards the right. When the force acts again a second later, the object has returned to the starting position and moving towards the right. A force acting towards the right on a mass travelling towards the right increases the speed of the mass, thereby increasing the amplitude. This increase in speed and amplitude will happen again every second, which means that the oscillation amplitude will go on increasing. The oscillation amplitude can become colossal when the force acts with a period equal to the period of oscillation of the system. If there's no friction, the amplitude will increase until the system is destroyed. With friction, the amplitude will increase until the energy provided by the force is equal to the energy dissipated by friction.

When a force acts on an oscillating system with a period equal to the natural period of oscillation of the system, the oscillation amplitude can become very large. This phenomenon is called resonance.

The oscillation amplitude of the system when there is a periodic force acting on the system can even be found. For example, you can have a look, in this document, at the calculation of the amplitude of a mass-spring system subjected to a force $F_0 \cos \omega t$ if there is a friction force proportional to the speed opposed ($F = -bv$) to the motion
https://physique.merici.ca/waves/proofamplitude.pdf
This calculation shows that the amplitude after a certain period of time is

$$A = \frac{F_0}{\sqrt{\left(-\omega^2 m + k\right)^2 + \left(\omega b\right)^2}}$$

Here is the graph of this amplitude as a function of the frequency of the force.



The dotted line is the natural frequency of oscillation of the system. It shows that when the force has a frequency equal to the natural frequency of oscillation of the system, the amplitude becomes enormous. There are actually three cases on this graph: little friction in red, more friction in blue and even more friction in green. Note that the amplitude of oscillation becomes very large for systems with little friction. The amplitude can become so great that the system can be destroyed. (In fact, resonance happens at a frequency a bit smaller than the natural frequency of oscillation, especially when friction is important. On the graph, it can be seen that the amplitude reaches a maximum value when the frequency is a bit smaller than the natural frequency for the curve with the largest friction.)

In this demonstration, a motor makes a mass-spring system oscillate. It can be seen that the amplitude is much larger when the motor exerts a force with a frequency identical to the natural frequency of the mass-spring system.
http://www.youtube.com/watch?v=aZNnwQ8HJHU

Sound waves arriving on an object make this object vibrate with the same frequency as the sound. If a sound arrives on a glass object with a frequency similar to the oscillation frequency of the glass, the amplitude may increase up until the glass breaks. (Yes, the edge

of a glass can oscillate. You can see this oscillation very well in the video. It seems to go slowly, but this is because a strobe was used.)
http://www.youtube.com/watch?v=17tqXgvCN0E
You can even try it at home.
http://www.youtube.com/watch?v=fbfjcEzFN2U
Remember that the intensity of the sound is not the only thing that matters, the frequency must be exactly right. It is, therefore, impossible for someone to break all the things made of glass around him by singing very loudly. This would require that all the glass objects have an oscillation frequency equal to the frequency of the sound, which is very unlikely.

In the following video, a helicopter is destroyed by resonance.
http://www.youtube.com/watch?v=ztBGCesBudE
Since the blades of the rear rotor are not balanced, the rotation of the propeller exerts a periodic force on the back of the helicopter. By rotating the propeller at the resonance frequency of the chopper, the oscillation amplitude steadily increases until the rear of the helicopter is destroyed. This would not have occurred if the propeller had spun with a greater or smaller period.

You can admire these young people trying to make a bridge collapse by applying a periodic force equal to one of the natural frequencies of the bridge (because there are several ways to swing a bridge).
http://www.youtube.com/watch?v=MNBun1JgDYo
http://www.youtube.com/watch?v=xlOS_31Ubdo

When the Millennium Bridge was opened in London, it was soon realized that there was a resonance problem. When the bridge started to oscillate, pedestrians, who tried to keep their balance, exerted a force on the bridge with the same frequency as the oscillation frequency of the bridge, and this amplified the oscillations.
http://www.youtube.com/watch?v=eAXVa__XWZ8
http://www.youtube.com/watch?v=gQK21572oSU (more detailed version)
The problem was solved by adding shock absorbers, thereby increasing the friction.

The same phenomenon was responsible for the collapse of the Basse-Chaine suspension bridge in Angers, France. On April 16, 1850, a violent storm was raging while the 11th regiment of light infantry was crossing the bridge. The wind of the storm then generated oscillations in the bridge. Just as on the Millennium Bridge, swinging motions were amplified by the soldiers trying to keep their balance on the bridge. The swings became so large that the cables holding the bridge gave way and 226 people were killed (some were killed by the bayonets of their fellow soldiers).

fr.wikipedia.org/wiki/Pont_de_la_Basse-Cha%C3%AEne#/media/File:Pont1839.jpg

Sometimes, it is said that the collapse of the bridge was caused by the soldiers walking in step with the same frequency as the resonant frequency of the bridge. However, the soldiers were not walking in step at this time. On the other hand, this is what caused the collapse of the Broughton Bridge which crosses the Irwel River in Manchester. On April 12, 1831, the soldiers felt the bridge swinging under their feet (even if they were not walking the steps at this time). Finding the feeling amusing, some began to whistle a marching tune to make everyone march at the same pace as the swinging of the bridge. Under the synchronized march, the amplitude of the oscillation increased until a part of the bridge table fell 6 m. As the river was very deep in this place, there were only injured people this time.

The collapse of the Tacoma Narrows Bridge (State of Washington), although spectacular, http://www.youtube.com/watch?v=3mclp9QmCGs
is not due to the resonance but to aeroelastic instabilities. In this case, the amplitude simply increases as the speed of the wind increases, there is no resonant frequency.

SUMMARY OF EQUATIONS

Link between *T* and *f*

$$T = \frac{1}{f}$$

Definition of Angular Frequency

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Phase Constant

$$\phi = \frac{\Delta t}{T} 2\pi$$

Position as a Function of Time for a Harmonic Oscillation

$$x = A \sin\left(\omega t + \phi\right)$$

Velocity as a Function of Time for a Harmonic Oscillation

$$v = A\omega \cos\left(\omega t + \phi\right)$$

Maximum Speed

$$v_{\max} = A\omega$$

Acceleration as a Function of Time for a Harmonic Oscillation

$$a = -A\omega^2 \sin(\omega t + \phi)$$

Maximum Acceleration

$$a_{max} = A\omega^2$$

Link between *x* and *v* at Some Instant

$$A^2 = x^2 + \left(\frac{v}{\omega}\right)^2$$

Link between *x* and *a* at Some Instant

$$a = -\omega^2 x$$

Condition for a Harmonic Oscillation

$$a = -(\text{constant})\,x$$

or

The potential energy must be proportional to x^2

ϕ calculation

$$\tan(\omega t + \phi) = \frac{\omega x}{v}$$

Angular Frequency for a Mass-Spring System

$$\omega = \sqrt{\frac{k}{m}}$$

Stretching of the Spring at the Equilibrium Position

$$kx_0 = mg$$

Mechanical Energy

$$E_{mec} = E_k + U$$

Kinetic Energy

$$E_k = \frac{1}{2}mv^2$$

Potential Energy (always good)

$$U = \frac{1}{2} m\omega^2 x^2$$

Potential Energy (good only for the mass-spring system)

$$U_S = \frac{1}{2} kx^2$$

Mechanical Energy

$$E_{mec} = \frac{1}{2} m\omega^2 A^2$$

$$E_{mec} = \frac{1}{2} mv_{\max}^2$$

Mechanical Energy (good only for the mass-spring system)

$$E_{mec} = \frac{1}{2} kA^2$$

Angular Frequency of a Simple Pendulum (Amplitude < 15°)

$$\omega = \sqrt{\frac{g}{L}}$$

Position of a Pendulum

$$x = A\sin\left(\omega t + \phi\right)$$

or

$$\theta = \theta_{\max} \sin\left(\omega t + \phi\right)$$

Link between x and θ for a Pendulum

$$\theta_{(rad)} = \frac{x}{L}$$

Link between A and $\theta_{\max}$ for a Pendulum

$$\theta_{\max(rad)} = \frac{A}{L}$$

Potential Energy (good only for a pendulum)

$$U_g = \frac{1}{2} m\frac{g}{L} x^2 \quad \text{or} \quad U_g = \frac{1}{2} mgL\theta^2 \quad \text{(The angle is in radians)}$$

Angular Frequency of a Compound Pendulum

$$\omega = \sqrt{\frac{mgd}{I}}$$

EXERCISES

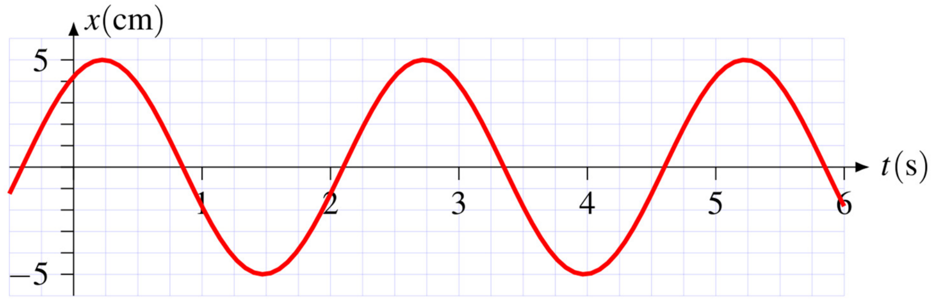
1.1 Harmonic Oscillations

1. The position as a function of time of an object in harmonic motion is given by the following formula.

$$x = 0.2m \cdot \sin\left(5\tfrac{rad}{s} \cdot t + \tfrac{\pi}{4}\right)$$

a) What is the amplitude?
b) What is the oscillation period?
c) What is the phase constant?
d) What is the maximal speed?
e) Write this equation in the form $x = A\cos(\omega t + \psi)$

2. A harmonic oscillation motion is described by this graph.



a) What is the amplitude of this motion (approximately)?
b) What is the period of this motion (approximately)?
c) What is the phase constant of this motion (approximately)?

3. The position as a function of time of an object performing a harmonic oscillation is given by the following formula.

$$x = 0.25m \cdot \sin\left(10\tfrac{rad}{s} \cdot t + \tfrac{3\pi}{4}\right)$$

a) What is the position at t = 1 s?
b) What is the velocity at t = 1 s?
c) What is the acceleration at t = 1 s?

4. The frequency of an object performing a harmonic oscillation is 5 Hz and the maximum acceleration of the object is 12 m/s². What is the maximum speed of the object?

5. The maximum speed of an object performing a harmonic oscillation is 32 m/s while the maximum acceleration of this object is 128 m/s².

   a) What is the period of the motion?
   b) What is the amplitude of motion?

6. At t = 0 s, an object in harmonic oscillation is at the position x = 10 cm and has a velocity of 24 cm/s. The period of the motion is 8 seconds. What is the equation of the motion $\left(x = A\sin\left(\omega t + \phi\right)\right)$?

7. At t = 0 s, an object in harmonic oscillation is at the position x = -20 cm and has a velocity of 0 m/s. The period of the motion is 8 seconds. What is the equation of motion $\left(x = A\sin\left(\omega t + \phi\right)\right)$?

8. At some instant during its oscillation motion, an object is at x = 6 cm, has a velocity of 1 m/s and an acceleration of -24 m/s².

   a) What is the period of the motion?
   b) What is the amplitude of motion?

9. The position as a function of time of an object performing a harmonic oscillation is given by the following formula.

$$x = 0.25m \cdot \sin\left(10\tfrac{rad}{s} \cdot t + \tfrac{3\pi}{4}\right)$$

   a) What is the velocity when the object is at x = 15 cm?
   b) What is the acceleration when the object is at x = 15 cm?

10. The position as a function of time of an object performing a harmonic oscillation is given by the following formula.

$$x = 0.2m \cdot \sin\left(5\tfrac{rad}{s} \cdot t + \tfrac{\pi}{4}\right)$$

   a) When is the mass at the position x = 12 cm for the first time?
   b) At what time is the velocity of the object equal to v = -60 cm/s for the first time?

11. The position as a function of time of an object performing a harmonic oscillation is given by the following formula.

$$x = 0.16m \cdot \sin\left(10\tfrac{rad}{s} \cdot t + \tfrac{\pi}{2}\right)$$

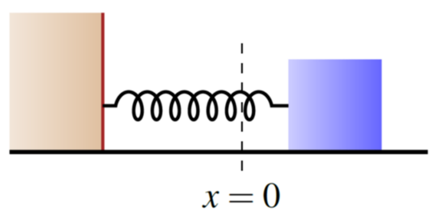
At what time is the mass at position x = 8 cm while having a positive velocity for the first time?

12. An object describes a harmonic oscillation with a period of 0.5 s. At t = 0 s, the magnitude of the acceleration of the object is maximum. At this instant, the acceleration reaches -32 m/s². What is the equation of motion $\left(x = A\sin\left(\omega t + \phi\right)\right)$?

13. In a harmonic motion having an amplitude of 20 cm and a period of 6 seconds, how much time does it take for the object to go from x = -10 cm to x = 10 cm?

1.2 Mass-Spring Systems

14. A 250 g mass is fixed at the end of a spring as shown in the diagram. The spring has a constant of k = 81 N/m. At t = 1 s, the mass is at x = 10 cm and has a velocity of - 2 m/s.



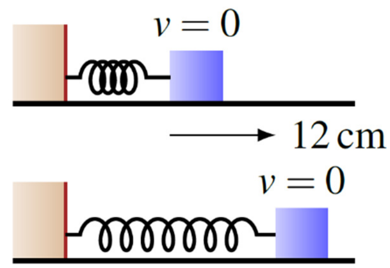
a) What is the amplitude of the motion?
b) What is the phase constant?
c) What is the equation of motion $\left(x = A\sin\left(\omega t + \phi\right)\right)$?

15. A mass-spring system has a frequency of 10 Hz. When the mass is changed for another mass which is 100 g greater, the frequency of oscillation decreases to 6 Hz.

a) What was the initial value of the mass?
b) What is the value of the spring constant?

16. When a 200 g mass is attached to a vertical spring, the spring extends 10 cm to reach the equilibrium position. What will the period of oscillation of this mass-spring system be?

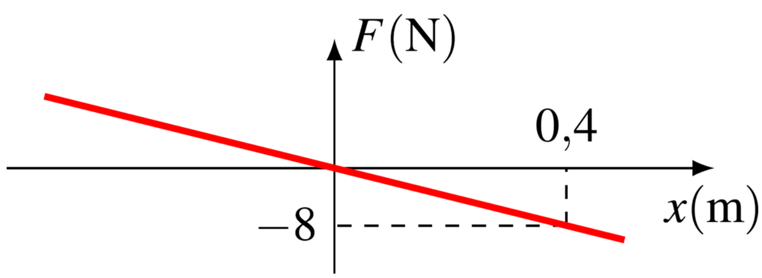
17. A mass takes 0.8 seconds to travel the 12 cm from one side to the other of its oscillation motion.



a) What is the amplitude of the motion?
b) What is the period of the motion?
c) What is the maximum speed of the mass?
d) What is the maximum acceleration of the mass?

18. An object is attached to the end of a spring with a constant of 250 N/m. The object is then set in motion, and the resulting motion has an amplitude of 20 cm while the maximum speed of the object is 4 m/s. What is the mass of the object?

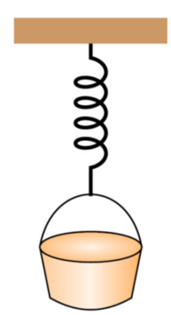
19. The force exerted by a spring on a mass of 200 g is given by this graph.



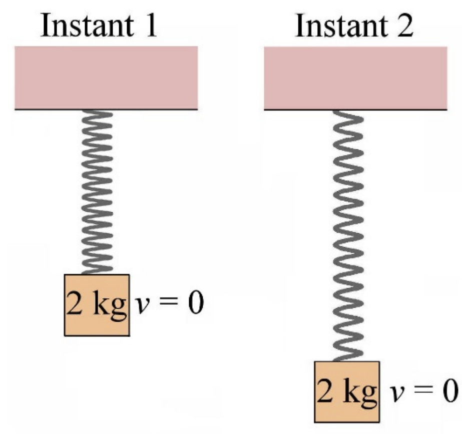
What is the period of the harmonic motion of this mass?

20. A mass-spring system oscillates with an amplitude of 12 cm and a 2 s period. At $t = 0$, the position of the mass is $x = -8$ cm, and the velocity is positive. What is the phase constant ϕ?

21. When water is added to the bucket shown in the diagram, the new equilibrium position is 12 cm lower. The empty bucket has a mass of 2 kg. Once the water is added, the system oscillates with a period of 2.4 s. What is the mass of the water added in the bucket?



22. A mass of 2 kg is suspended at the end of a spring. Initially, the spring is not stretched, and the mass is at rest. When we let the mass fall (without pushing it), it takes 0.6 seconds to reach its lowest point.



a) What is the spring constant?
b) What is the amplitude of this motion?
c) What is the maximum speed of the mass?
d) When will the mass have a speed of 1 m/s directed upward for the first time?

1.3 Mechanical Energy in Harmonic Motion

23. The position as a function of time of an object performing a harmonic oscillation is given by the following formula.

$$x = 0.2m \cdot \sin\left(5\tfrac{rad}{s} \cdot t + \tfrac{\pi}{4}\right)$$

The spring constant is 250 N/m.

a) What is the mechanical energy of this system?
b) What is the spring energy at t = 5 s?
c) What is the kinetic energy at t = 5 s?

24. At some instant, the kinetic energy of an oscillating mass-spring system is 10 J and the spring energy is 20 J. The period of the motion is 3 s and the amplitude is 80 cm.

a) What is the spring constant?
b) What is the mass?
c) What is the maximum speed of the mass?

25. The mechanical energy of a system performing a harmonic oscillation is of 5 J. The amplitude is 10 cm, and the period is 0.5 s.

a) What is the kinetic energy when x = 6 cm?
b) What is the spring energy when x = 4 cm?

26. The position as a function of time of an object performing a harmonic oscillation is given by the following formula.

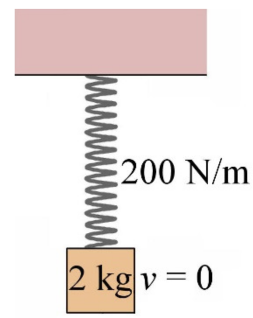
$$x = 0.25m \cdot \sin\left(10\tfrac{rad}{s} \cdot t + \pi\right)$$

When does $E_k = U$ for the first time if the mechanical energy is 50 J?

27. A mass-spring system oscillates with an amplitude of 12 cm.

a) What is the position of the object if the speed is equal to one-quarter of the maximum speed?
b) What is the position of the object if the kinetic energy is equal to half the spring energy?

28. The 2 kg mass in the diagram is suspended from the ceiling with a spring having a 200 N/m constant. Initially, the spring is neither stretched nor compressed. The mass is then released without pushing it. A harmonic motion is then obtained.



a) What will the period of the motion T be?
b) What will the amplitude of this motion be?

1.4 Pendulum

29. What is the length of the rope of a simple pendulum if the pendulum becomes vertical every 2 seconds?

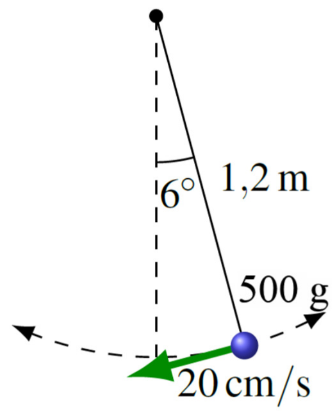
30. A pendulum has a period of 2 seconds on the surface of the Earth. What will the period of the pendulum be if it is brought to the surface of the Moon, where g is 1.6 N/kg?

31. The length of the rope of a simple pendulum is 2 m. The pendulum is placed in a position where the angle between the rope and the vertical is 10° and is then released (without pushing the pendulum).

a) What is the period of the motion?
b) What is the maximum speed of the pendulum?

32. A simple pendulum oscillates with an angular amplitude of 12° and a period of 1.6 seconds. What is the speed of the mass when the angle is 8°?

33. A simple pendulum consists of a 500 g mass and a 1.2 m long rope. Initially ($t = 0$ s), we have the following situation.



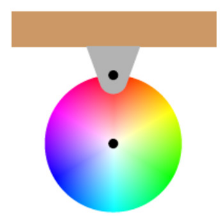
What is the equation of motion of the pendulum $\left(\theta = \theta_{max} \sin\left(\omega t + \phi\right)\right)$? (We want θ in degrees.)

34. The angular amplitude of a simple pendulum is 15°, and its maximum speed is 60 cm/s.

a) What is the length of the rope?
b) What is the period of the motion?

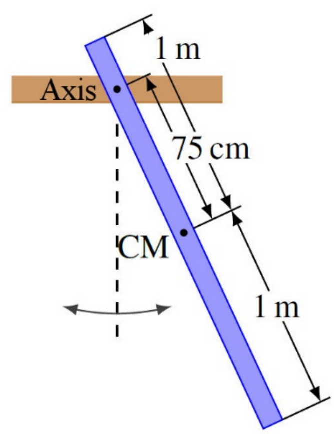
35. A disk is fixed to an axis of rotation as shown in the diagram. Knowing that the mass of the disk is 5 kg, that its radius is 20 cm, and that the moment of inertia of a disk rotating around an axis located at the edge of the disk is

$$I = \frac{3}{2} mR^2$$



determine the period of the motion if the disk is oscillating.

36. What is the period of oscillation of this 2 kg rod?



Reminders:

1) The moment of inertia of a rod is

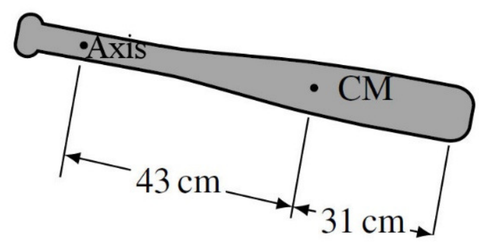
$$I_{cm} = \frac{1}{12} mL^2$$

2) The moment of inertia when the rotation axis is not at the centre of mass is

$$I = I_{cm} + mh^2$$

where h is the distance between the centre of mass and the axis of rotation.

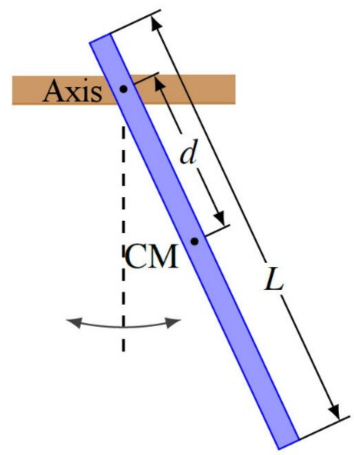
37. This 900 g baseball bat oscillates with a period of 1.335 seconds when the axis of rotation is at the position shown in the diagram. What is its moment of inertia when the axis is at this location?



Challenges

(Questions more difficult than the exam questions.)

38. What must be the value of d to have the smallest possible period of oscillation for this rod?



Reminders:

1) The moment of inertia of a rod is

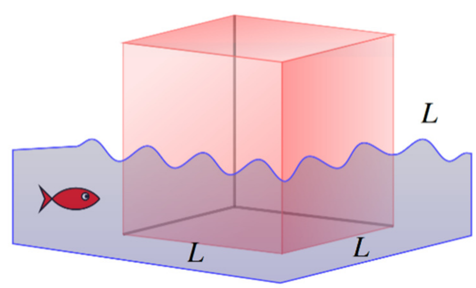
$$I_{cm} = \frac{1}{12} mL^2$$

2) The moment of inertia when the rotation axis is not at the centre of mass is

$$I = I_{cm} + mh^2$$

where h is the distance between the centre of mass and the axis of rotation.

39. A large cubic iceberg (L = 75 m) floats on the ocean. What is the period of oscillation of the iceberg if it slightly wobbles in a vertical direction at the surface of the ocean? (The density of water is 1000 kg/m³ and the density of ice is 920 kg/m³).



40. Suppose the force on an object is given by $F = -cx^3$ (instead of $F = -kx$). Show that with such a force, the period is proportional to $1/A$ (where A is the amplitude).

ANSWERS

1.1 Harmonic Oscillations

1. a) 20 cm b) 1.257 s c) π/4 d) 1 m/s e) $x = 0.2m \cdot \cos\left(5\frac{rad}{s} \cdot t - \frac{\pi}{4}\right)$
2. a) 5 cm b) 2.5 s c) 1 rad
3. a) -5.216 cm b) 2.445 m/s c) 5.216 m/s²
4. 0.382 m/s
5. a) 1.571 s b) 8 m
6. $x = 0.3215m \cdot \sin\left(\frac{\pi}{4}\frac{rad}{s} \cdot t + 0.3163\right)$
7. $x = 0.2m \cdot \sin\left(\frac{\pi}{4}\frac{rad}{s} \cdot t - \frac{\pi}{2}\right)$ or $x = 0.2m \cdot \sin\left(\frac{\pi}{4}\frac{rad}{s} \cdot t + \frac{3\pi}{2}\right)$
8. a) 0.3142 s b) 7.81 cm
9. a) ± 2 m/s b) -15 m/s²
10. a) 0.3425 s b) 0.2858 s
11. 0.5236 s
12. $x = 0.2026m \cdot \sin\left(4\pi\frac{rad}{s} \cdot t + \frac{\pi}{2}\right)$
13. 1 s

1.2 Mass-Spring Systems

14. a) 14.95 cm b) -15.591 rad c) $x = 0.1495m \cdot \sin\left(18\frac{rad}{s} \cdot t - 15.591rad\right)$
15. a) 56.25 g b) 222.1 N/m
16. 0.6347 s
17. a) 6 cm b) 1.6 s c) 0.2356 m/s d) 0.9253 m/s²
18. 625 g
19. 0.6283 s
20. -0.7297 rad
21. 183.2 g
22. a) 54.83 N/m b) 35.75 cm c) 1.872 m/s d) 0.7077 s

1.3 Mechanical Energy in Harmonic Motion

23. a) 5 J b) 1.844 J c) 3.156 J
24. a) 93.75 N/m b) 21.37 kg c) 1.676 m/s
25. a) 3.2 J b) 0.8 J
26. 0.0785 s
27. a) ±11.62 cm b) ±9.80 cm
28. a) 0.6283 s b) 9.8 cm
29. a) 0.7854 s b) 6 m/s c) 75 cm

1.4 Pendulum

30. 3.972 m
31. 4.95 s
32. a) 2.838 s b) 0.7727 m/s
33. 0.3896 m/s
34. $\theta = 6.868° \cdot \sin\left(2.858\frac{rad}{s} \cdot t + 2.079 rad\right)$
35. a) 0.536 m b) 1.47 s
36. 1.099 s
37. 2.194 s
38. 0.171 kgm²

Challenges

39. $L/\sqrt{12}$
40. 16.67 s