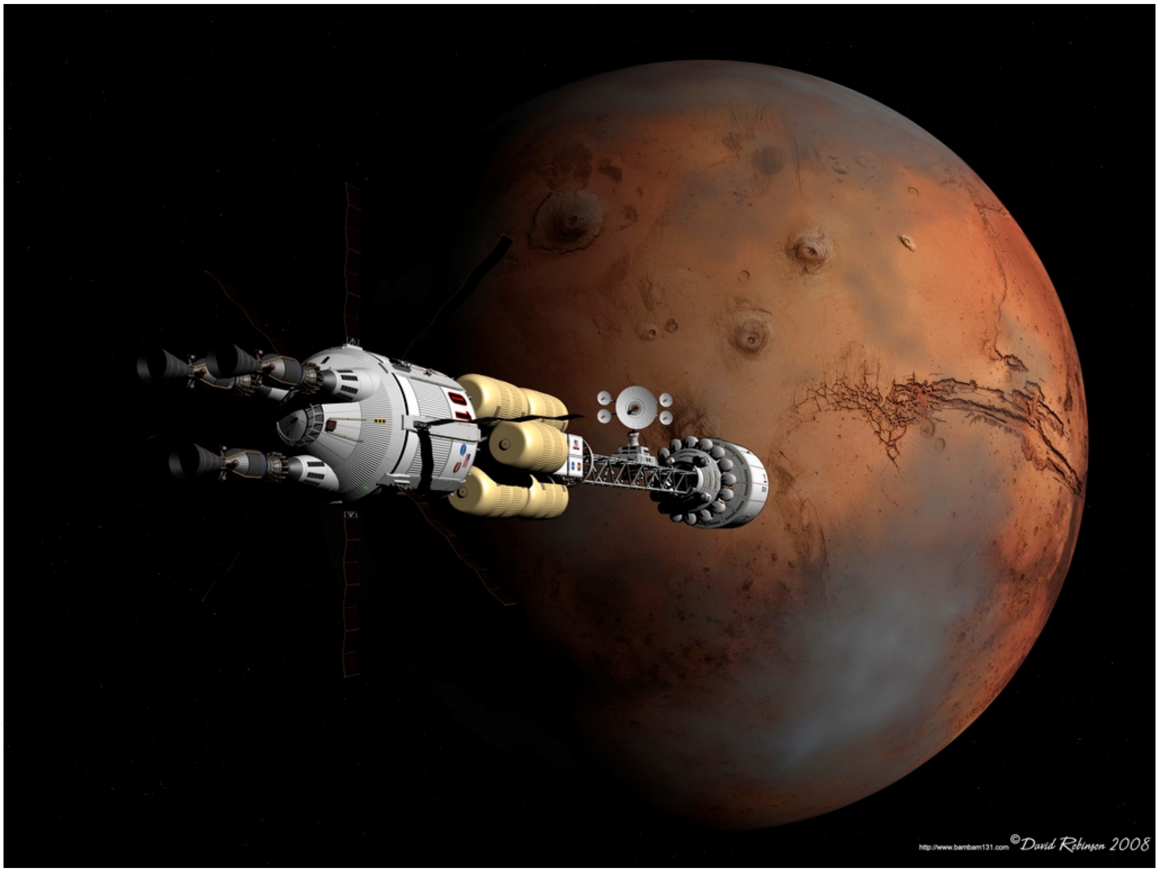
E1 GRAVITATION

To go to Mars, there is no need to use reactors during the whole journey (it would be very expensive to do so). The spacecraft simply need to be placed on a Hohmann transfer orbit with its perihelion at the Earth and its aphelion at Mars. How long will last this journey from the Earth to Mars?



spaceart1.ning.com/photo/destination-complete?context=user

Discover the answer to this question in this chapter.

The formula of the gravitational force was seen previously, in Chapter 4. It was a force of attraction between the masses whose magnitude is

$$F = G\frac{m_1 m_2}{r^2}$$

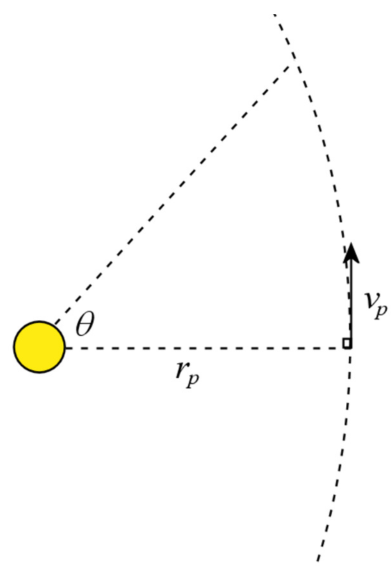
where G = 6.674 x 10^{-11} Nm²/kg². Note that r is the distance between the centres of the celestial objects.

From this law, it was found that an object can move in a circular orbit around a star or a planet. Actually, the circular orbit is only one possibility among many possible trajectories near a planet or a star.

E1.1 TRAJECTORIES NEAR A MASSIVE OBJECT

Shape of trajectory

Let's consider an object following a trajectory near a celestial object.



On its trajectory, the object will, at some point, pass at its closest position to the central mass (the star or the planet). The distance between the object and the central mass at this instant will be called r_p, and the speed at this point will be called v_p. Also, note that the velocity at this location is perpendicular to the distance. This position will be the reference point for the angle θ used to denote the position.

From the force, the shape of the path is sought. This means that r as a function of θ is sought. Details of the calculation of the shape of the path are beyond the scope of these notes, and the result is simply given here (however, the details of this calculation can be found in chapter 1 of the Astrophysics notes, in French). The shape of the path is given by

Distance as a Function of θ of an Object Near a Central Mass

$$r = r_p \frac{1+e}{1+e\cos\theta}$$

where e is a factor called *eccentricity* whose value is

Eccentricity

$$e = \frac{v_p^2 r_p}{GM_c} - 1$$

The exact shape of the path depends on the value of e. The different possibilities will be considered in the following sections.

Mechanical Energy

The mechanical energy of the object following the trajectory can be calculated. This energy is

$$E_{mec} = \frac{1}{2}mv_p^2 + \frac{-GM_c m}{r_p}$$

According to the eccentricity formula, the velocity is

$$e = \frac{v_p^2 r_p}{GM_c} - 1$$

$$v_p^2 = \frac{GM_c(1+e)}{r_p}$$

Thus, the energy becomes

$$\begin{aligned} E_{mec} &= \frac{1}{2}mv_p^2 + \frac{-GM_c m}{r_p} \\ &= \frac{1}{2}m\frac{GM_c(1+e)}{r_p} + \frac{-GM_c m}{r_p} \end{aligned}$$

$$\begin{aligned} E_{mec} &= \left(\frac{1+e}{2}-1\right)\frac{GM_c m}{r_p} \\ &= \left(\frac{1+e-2}{2}\right)\frac{GM_c m}{r_p} \\ &= \left(\frac{-1+e}{2}\right)\frac{GM_c m}{r_p} \end{aligned}$$

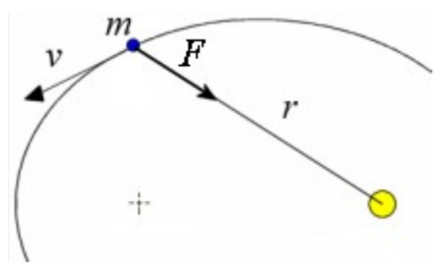
The end result is

Mechanical Energy

$$E_{mec} = -\frac{GM_c m(1-e)}{2r_p}$$

Angular Momentum

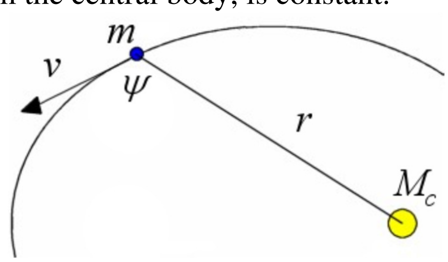
Remember that there is conservation of angular momentum if the sum of the external torque vanishes. There is only one force, gravity, acting on the object during its orbital motion. The torque, calculated from the central mass, is



$$\tau = Fr\sin\phi$$

where ϕ is the angle between the force and a line going from the central mass to the object in orbit (line r in the diagram). This torque vanishes because the force is pointing directly towards the central body, so in the same direction as the line r. The angle ϕ is, therefore, zero, and the sum of torques is zero.

This means that the angular momentum, calculated from the central body, is constant.



$$L = mrv\sin\psi = \text{constant}$$

$$rv\sin\psi = \frac{\text{constant}}{m}$$

Since the mass is constant, the term to the right is a constant.

$$rv\sin\psi = \text{constant}$$

The constant can easily be found by calculating its value at the point of closest approach to the central mass. At this point, the value is

$$r v \sin \psi = r_p v_p \sin 90°$$

This gives us a first angular momentum conservation equation.

Angular Momentum Conservation

$$r v \sin \psi = r_p v_p$$

From the formula of the eccentricity,

$$e = \frac{v_p^2 r_p}{G M_c} - 1$$

the speed can be obtained

$$v_p^2 = \frac{G M_c (1+e)}{r_p}$$

Using this value in the conservation equation, it becomes

$$\begin{aligned} r v \sin \psi &= r_p \sqrt{\frac{G M_c (1+e)}{r_p}} \\ &= \sqrt{G M_c r_p (1+e)} \end{aligned}$$

Angular Momentum Conservation

$$r v \sin \psi = \sqrt{G M_c r_p (1+e)}$$

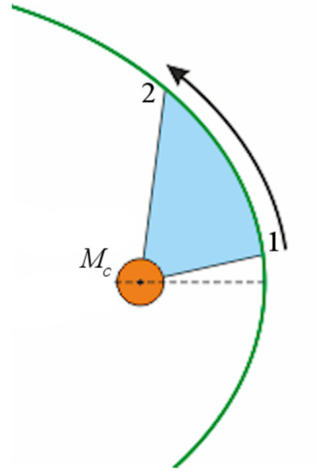
These equations of conservation of angular momentum are two equivalent formulations of what is known as *Kepler's second law*.

Swept Areas

Kepler gave a very different formulation of this law when he discovered it in 1608. Kepler's formulation was

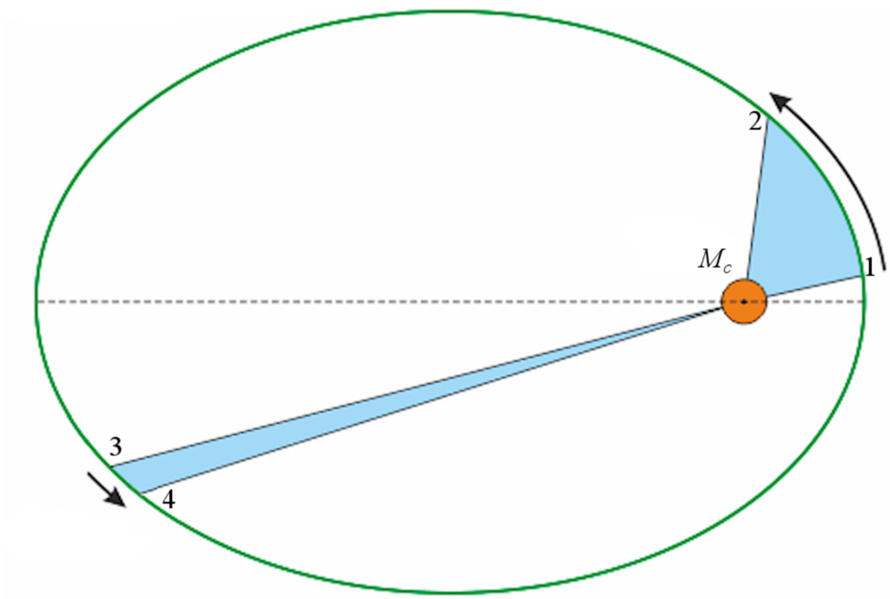
A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time

What does "swept area" mean? If the object moves from position 1 to position 2, the swept area is the area of the region bounded by the path, a line that goes from the central mass to position 1 and another line that goes from the central mass to position 2 (see diagram).



Kepler's second law means that this area is always the same if the time between positions 1 and 2 is the same.

Let's take an example to illustrate. One of the possible shapes for the orbit is an ellipse, such as shown in the following diagram.

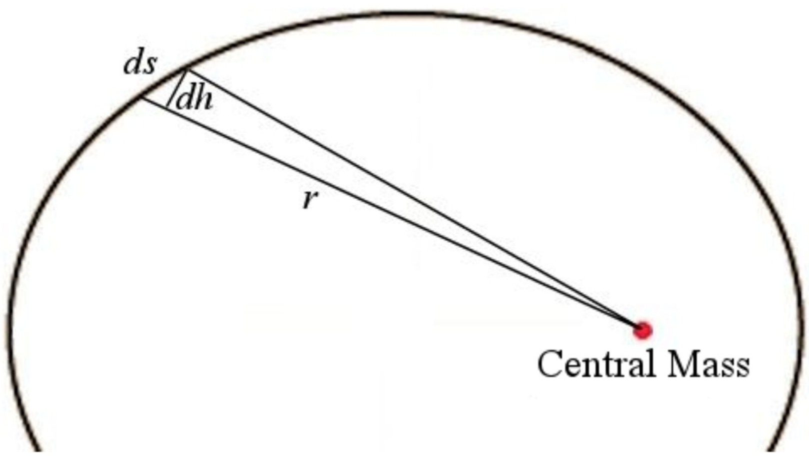


serge.bertorello.free.fr/astrophy/kepnew/kepnew.html

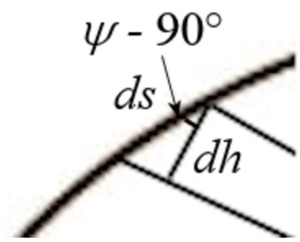
Kepler's second law then specifies that the area of the two regions shown in the diagram is the same, provided that the time it takes for the object to travel from position 1 to position 2 is the same as the time taken to travel from position 3 to position 4.

To prove this law, and show the link with the angular momentum conservation, the swept area will be calculated between two positions very close to each other. In fact, we'll take two points on the trajectory that are at a distance ds from each other.

Then, the swept area is a triangle whose height is dh.



As ψ is the angle between the velocity (the trajectory) and the distance r, the height of this triangle is



$$\begin{aligned} dh &= ds \cdot \cos(\psi - 90°) \\ &= ds \cdot \sin(\psi) \end{aligned}$$

Thus, the area of the triangle is

$$\begin{aligned} dA &= \frac{base \cdot height}{2} \\ &= \frac{rdh}{2} \\ &= \frac{rds \sin\psi}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{dA}{dt} &= \frac{rds \sin\psi}{2dt} \\ &= \frac{rv \sin\psi}{2} \end{aligned}$$

since $ds/dt = v$.

However, according to the conservation of angular momentum, the following relation holds

$$rv \sin\psi = \sqrt{GM_c r_p (1+e)}$$

This allows writing

$$\frac{dA}{dt} = \frac{\sqrt{GM_c r_p (1+e)}}{2}$$

The term on the right is a constant. When the rate of change is constant, *dA/dt* can be replaced by $\Delta A/\Delta t$ to finally obtain

Kepler's Second Law

$$\Delta A = \frac{\sqrt{GM_c r_p (1+e)}}{2} \Delta t$$

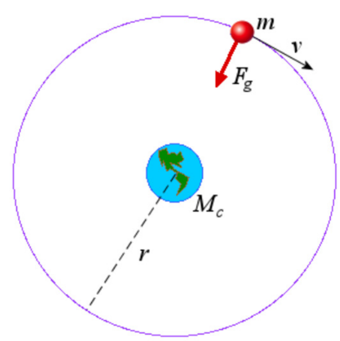
This result clearly shows that the swept area is always the same if the time is the same.

E1.2 CIRCULAR ORBITS ($e = 0$)

If the eccentricity is zero, then the formula of the path becomes

$$r = r_p$$

This indicates that r does not vary with the angle and that r is a constant. With a constant radius, the trajectory is a circular orbit.



www.ux1.eiu.edu/~addavis/3050/Ch09Gravity/Sat.html

To have a vanishing eccentricity, the speed must be

$$e = \frac{v_p^2 r_p}{GM_c} - 1$$

$$0 = \frac{v_p^2 r_p}{GM_c} - 1$$

$$v_p = \sqrt{\frac{GM_c}{r_p}}$$

This formula is actually the same equation for the speed on a circular orbit obtained in chapter 6. There was no index p then, but this makes no difference because the closest distance to the central mass is always the same with a circular orbit, and is equal to the radius of the orbit. Thus, the formula can be written as

Speed of an Object on a Circular Orbit

$$v = \sqrt{\frac{GM_c}{r}}$$

With a vanishing eccentricity, the mechanical energy of the object in orbit is

$$\begin{aligned} E_{mec} &= -\frac{GM_c m(1-e)}{2r_p} \\ &= -\frac{GM_c m}{2r_p} \end{aligned}$$

Since $r = r_p$ on a circular orbit, the energy is

Mechanical Energy of an Object on a Circular Orbit

$$E_{mec} = -\frac{GM_c m}{2r}$$

(Which is identical to the formula obtained in chapter 9.)

Finally, the period of revolution is found by dividing the circumference of the orbit by the speed of the object in orbit. This calculation had been made in Chapter 6, and the following result was obtained.

Period of an Object on a Circular Orbit (Kepler's 3rd Law)

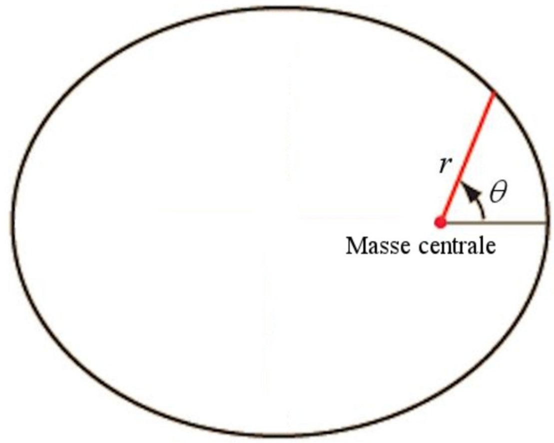
$$T = 2\pi\sqrt{\frac{r^3}{GM_c}}$$

E1.3 ELLIPTICAL ORBITS ($0 < e < 1$)

When the eccentricity is between 0 and 1, the value of r changes with the angle according to the following equation.

$$r = r_p \frac{1+e}{1+e\cos\theta}$$

This equation is the equation of an ellipse.



hyperphysics.phy-astr.gsu.edu/hbase/math/ellipse.html

To obtain such an ellipse, the speed v_p must be between two values. The minimum value is found with the minimum eccentricity.

$$e > 0$$

$$\frac{v_p^2 r_p}{GM_c} - 1 > 0$$

$$v_p > \sqrt{\frac{GM_c}{r_p}}$$

The maximum value is found with the maximum eccentricity.

$$e < 1$$

$$\frac{v_p^2 r_p}{GM_c} - 1 < 1$$

$$v_p < \sqrt{\frac{2GM_c}{r_p}}$$

Kepler's First Law

The elliptical shape of orbits was discovered in 1608 by Johannes Kepler. In 1600, he started a very thorough study of the orbit of Mars using the observation data of Tycho Brahe, the best observation data made before 1600.

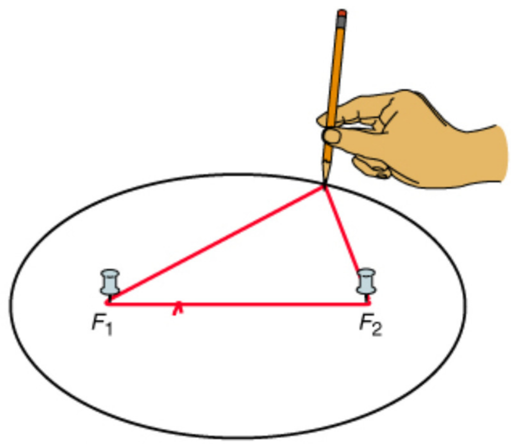
After 8 years of studies, Kepler arrived at a groundbreaking conclusion. While it was believed for nearly 20 centuries that the orbits of planets had to be perfect circles, on the pretext that the heavens must be perfect in order to reflect the perfection of the gods (or God), Kepler showed that the orbits are elliptical. It was a revolution in astronomy but it took almost a century before a majority of scientists were convinced of the truth of this law.

Kepler's First Law

The orbits are ellipses. The central mass occupies one of the foci.

Ellipse (or Ellipsis)

An ellipse looks like an oval, but it is a peculiar oval. To draw an ellipse, just use a loop of rope with two pushpins pressed into a board. Then take a pencil and then plot the diagram bounded by the rope as shown in the illustration.

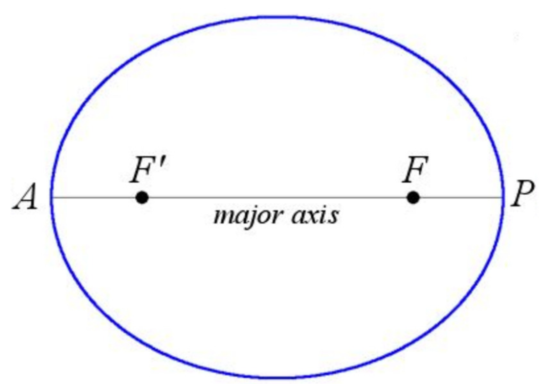


members.shaw.ca/len92/astronomy.htm

The two foci F_1 and F_2 corresponds to the two locations where the pushpins were placed.

This means that an ellipse is the set of all points on a plane whose distances from the two foci add up to a constant. The ellipse is more or less elongated depending on the distance between the pushpins and the length of the rope.

The points F and F' are the foci. Here, the central mass is located at the focus F. The line going from one side of the ellipse to the other passing through the foci is the major axis of the ellipse. The two points where the ellipse and the major axis intersect (points A and P) are the *apsides* (singular: *apsis*). (The major axis is also called the line of apsides.) Point A is the point on the orbit which is farthest from the central mass. The general name for this point is *apoapsis* (the terms apocenter, apofocus, and upper apse are also used). Point P is the point on the orbit nearest to the central mass. The general name for this point is *periapsis* (the terms pericenter, perifocus and lower apse are also used).



www.ck12.org/book/CK-12-PreCalculus-Concepts/r514/section/9.4/

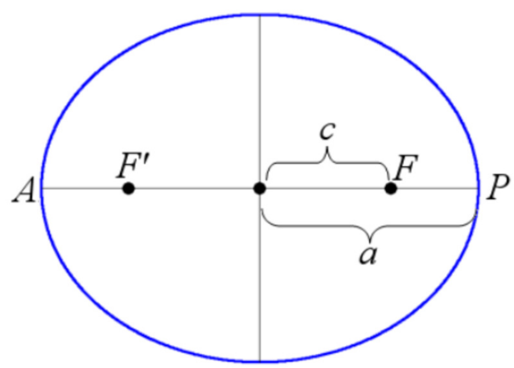
Actually, a large terminology is used to describe these apsides depending on the nature of the central mass. If the orbit is around the Sun, the nearest point is the perihelion and the farthest point is the aphelion. If the orbit is around the Earth, the nearest point is the perigee and the farthest point is the apogee. The following table shows you (as a curiosity) the names of these points according to the nature of the central mass. (But you should remember the underlined ones.)

Central Mass	Periapsis	Apoapsis
Galaxy	Perigalacticon	Apogalacticon
Black Hole	Perimelasma	Apomelasma
Star	Periastron	Apoastron
Sun	**Perihelion**	**Aphelion**
Mercury	Perihermion	Aphermion
Venus	Pericytherion	Apocytherion
Earth	**Perigee**	**Apogee**
Moon	Periselene	Aposelene
Mars	Periarerion	Apoareion
Jupiter	Perizene	Apozene
Saturn	Perikrone	Apokrone
Uranus	Periuranion	Apouranion
Neptune	Periposeidon	Apoposeidon
Pluto	Perihadion	Apohadion

Eccentricity

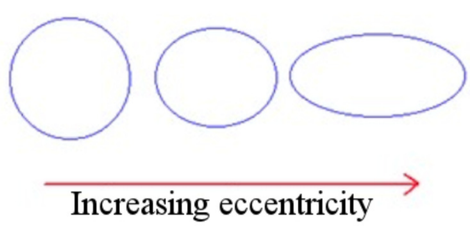
The eccentricity of an ellipse is defined by the ratio

$$e = \frac{FF'}{AP} = \frac{c}{a}$$



where c is the distance from the centre of the ellipse to one of the foci and a is the distance from the centre of the ellipse to one of the apsides (a is called the semi-major axis).

If both foci are at the centre, then the eccentricity is zero and the ellipse is a circle. The higher the eccentricity, the farther apart the foci are, and the more elongated the ellipse is.



www.astro-tom.com/technical_data/elliptical_orbits.htm

The value of the eccentricity of an ellipse is always less than 1 since the foci would not be inside the ellipse if the eccentricity were to be higher than 1.

As can be seen in the following table, the eccentricity of the planetary orbits is generally quite low.

Planet	Eccentricity of the Orbit
Mercury	0.206
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.056
Uranus	0.047
Neptune	0.009

With the exception of Mercury, planetary orbits deviate only a little from a circular shape. The eccentricity of the orbit of Mars is relatively high, and this allowed Kepler, who studied its orbit, to realize that the orbits are elliptical. However, these values of eccentricity are nothing compared to the eccentricity of the orbit of some objects that have very elongated orbits such as comets. For example, the orbit of the most famous comet, Halley's Comet, has an eccentricity of 0.970.

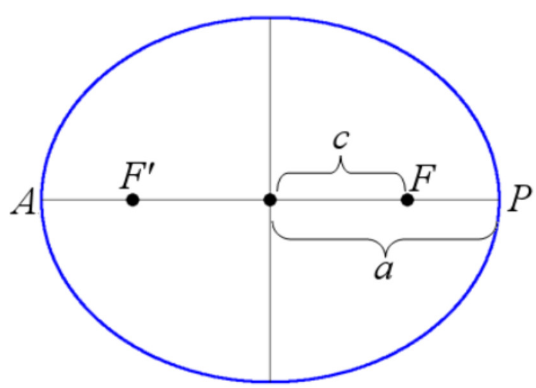
Distances between the Object in Orbit and the Central Mass

Distances at the Apoapsis and at the Periapsis

The relationships between the distances at the apoapsis and at the periapsis, on the one hand, and the eccentricity and the semi-major axis of the ellipse, on the other hand, can be found. The distances at the apoapsis and periapsis as functions of the semi-major axis *a* and the eccentricity *e* are

$$\begin{aligned} r_a &= a + c \\ &= a + ea \\ &= a(1+e) \end{aligned}$$

$$\begin{aligned} r_p &= a - c \\ &= a - ea \\ &= a(1-e) \end{aligned}$$



r_a and r_p as functions of a and e

$$r_a = a(1+e)$$
$$r_p = a(1-e)$$

These previous relationships can be inverted to get relations between the semi-major axis and the eccentricity as a function of the distances at the apoapsis and at the periapsis.

a and e as functions of r_a and r_p

$$a = \frac{r_p + r_a}{2} \qquad\qquad e = \frac{r_a - r_p}{r_a + r_p}$$

<u>Distance at any Point on the Orbit</u>

The formula giving the position of the object on the orbit as a function of the angle is already known.

$$r = r_p \frac{1+e}{1+e\cos\theta}$$

Since $r_p = a(1-e)$, the distance can also be written as

$$\begin{aligned} r &= a(1-e)\frac{1+e}{1+e\cos\theta} \\ &= \frac{a(1-e^2)}{1+e\cos\theta} \end{aligned}$$

Thus, two formulas can be used to find r as a function of θ.

Relationship Between r and θ for an Elliptical Orbit

$$r = r_p \frac{1+e}{1+e\cos\theta}$$

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

Orbital Speed

The Speed Must Change

It is obvious that the speed of an object cannot be constant on an elliptical orbit because the mechanical energy must be conserved. Indeed, for an object of mass m orbiting around another object of mass M_c , the mechanical energy is

$$E_{mec} = \frac{1}{2}mv^2 - \frac{GM_c m}{r} = \text{constant}$$

Thus, when the object in orbit approaches the central mass, its gravitational energy decreases and its kinetic energy must, therefore, increase. So the planet must go the fastest at the periapsis and go the slowest at the apoapsis.

The speed can be found with the mechanical energy. The general formula of the energy is

$$E_{mec} = -\frac{GM_c m(1-e)}{2r_p}$$

For an elliptical orbit, this equation becomes

$$\begin{aligned} E_{mec} &= -\frac{GM_c m(1-e)}{2r_p} \\ &= -\frac{GM_c m(1-e)}{2ra(1-e)} \end{aligned}$$

which leads to

Mechanical Energy of an Object on an Elliptical Orbit

$$E_{mec} = -\frac{GM_c m}{2a}$$

From this equation, the speed as a function of r can be found with the law of conservation of mechanical energy.

$$-\frac{GM_c m}{2a} = \frac{1}{2}mv^2 - \frac{GM_c m}{r}$$

Solving this equation for v, a formula for the speed as a function of r is obtained.

Speed of an Object on an Elliptical Orbit

$$v^2 = GM_c\left(\frac{2}{r}-\frac{1}{a}\right)$$

Speed at the Apoapsis and at the Periapsis

At the periapsis, $r = a(1-e)$ and the speed is

$$\begin{aligned}
v_p^2 &= GM_c\left(\frac{2}{a(1-e)}-\frac{1}{a}\right)\\
&= \frac{GM_c}{a}\left(\frac{2}{1-e}-1\right)\\
&= \frac{GM_c}{a}\left(\frac{2-(1-e)}{1-e}\right)\\
&= \frac{GM_c}{a}\frac{1+e}{1-e}
\end{aligned}$$

At the apoapsis, $r = a(1+e)$ and the speed is

$$\begin{aligned}
v_a^2 &= GM_c\left(\frac{2}{a(1+e)}-\frac{1}{a}\right)\\
&= \frac{GM_c}{a}\left(\frac{2}{1+e}-1\right)\\
&= \frac{GM_c}{a}\left(\frac{2-(1+e)}{1+e}\right)\\
&= \frac{GM_c}{a}\frac{1-e}{1+e}
\end{aligned}$$

Speed of an Object at the Periapsis or at the Apoapsis

$$v_p^2 = \frac{GM_c}{a}\frac{1+e}{1-e}$$

$$v_a^2 = \frac{GM_c}{a}\frac{1-e}{1+e}$$

Period

The period can be found with Kepler's second law.

$$\Delta A = \frac{\sqrt{GM_c r_p (1+e)}}{2} \Delta t$$

In one period, the full area of the ellipse is swept. As the area of an ellipse is

$$A = \pi a^2 \sqrt{1-e^2}$$

Kepler's second law gives

$$\pi a^2 \sqrt{1-e^2} = \frac{\sqrt{GM_c r_p (1+e)}}{2} T$$

Using the fact that $r_p = a(1-e)$, the equation becomes

$$\pi a^2 \sqrt{1-e^2} = \frac{\sqrt{GM_c a(1-e)(1+e)}}{2} T$$

$$\pi a^2 \sqrt{1-e^2} = \frac{\sqrt{GM_c a(1-e^2)}}{2} T$$

$$\pi a^2 = \frac{\sqrt{GM_c a}}{2} T$$

Solving for *T*, the formula for the period is obtained.

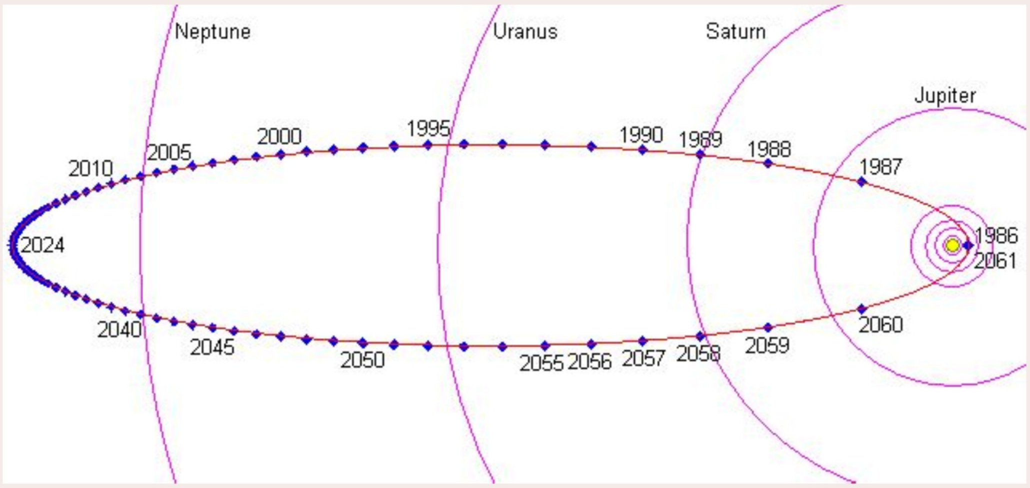
Period of an Object on an Elliptical Orbit (Kepler's 3rd Law)

$$T = 2\pi \sqrt{\frac{a^3}{GM_c}}$$

This formula is quite similar to the formula for a circular motion where *r* is simply substituted by *a*.

Example E1.3.1

Halley's Comet moves on an orbit whose distances at aphelion and perihelion are r_a = 35.295 AU and r_p = 0.587 AU (1 AU is a unit of distance equal to the average distance between the Earth and the Sun and which is about 1.5 x 10^{11} m). The mass of the Sun is 2 x 10^{30} kg.



www.uwgb.edu/dutchs/PLANETS/Comets.HTM

a) What is the eccentricity of this orbit?

The eccentricity is

$$\begin{aligned} e &= \frac{r_a - r_p}{r_a + r_p} \\ &= \frac{35.295AU - 0.587AU}{35.295AU + 0.587AU} \\ &= 0.9673 \end{aligned}$$

b) What is the semi-major axis (a) of the orbit?

The semi-major axis is

$$\begin{aligned} a &= \frac{r_a + r_p}{2} \\ &= \frac{35.295AU + 0.587AU}{2} \\ &= 17.941AU \end{aligned}$$

c) What is the period of this comet?

The period is

$$T = 2\pi\sqrt{\frac{a^3}{GM_c}}$$
$$= 2\pi\sqrt{\frac{\left(17.941AU \cdot 1.5\times10^{11}\,\frac{m}{AU}\right)^3}{6.674\times10^{-11}\,\frac{Nm^2}{kg^2}\cdot 2\times10^{30}kg}}$$
$$= 2.401\times10^9 s$$
$$= 76.08 \text{ years}$$

d) What is the speed at the perihelion?

The speed at the perihelion is

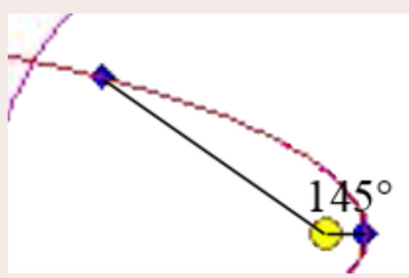
$$v_p = \sqrt{\frac{GM_c}{a}\frac{1+e}{1-e}}$$
$$= \sqrt{\frac{6.674\times10^{-11}\,\frac{Nm^2}{kg^2}\cdot 2\times10^{30}kg}{17.941UA\cdot 1.5\times10^{11}\,\frac{m}{UA}}\frac{1+0.9673}{1-0.9673}}$$
$$= 54.61\,\frac{km}{s}$$

e) What is the speed at the aphelion?

The speed at the aphelion is

$$v_a = \sqrt{\frac{GM_c}{a}\frac{1-e}{1+e}}$$
$$= \sqrt{\frac{6.674\times10^{-11}\,\frac{Nm^2}{kg^2}\cdot 2\times10^{30}kg}{17.941UA\cdot 1.5\times10^{11}\,\frac{m}{UA}}\frac{1-0.9673}{1+0.9673}}$$
$$= 0.908\,\frac{km}{s}$$

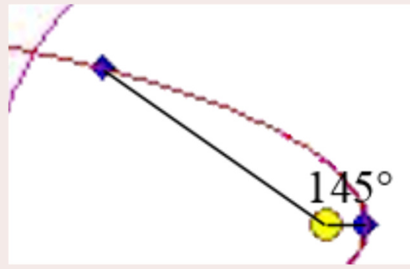
f) What is the distance between the Sun and the comet when it is at this position?



The distance at this position is

$$\begin{aligned} r &= \frac{a\left(1-e^2\right)}{1+e\cos\theta} \\ &= \frac{17.941AU\cdot\left(1-0.9673^2\right)}{1+0.9673\cdot\cos\left(145°\right)} \\ &= 5.559AU \end{aligned}$$

g) What is the speed of the comet when it is at this position?



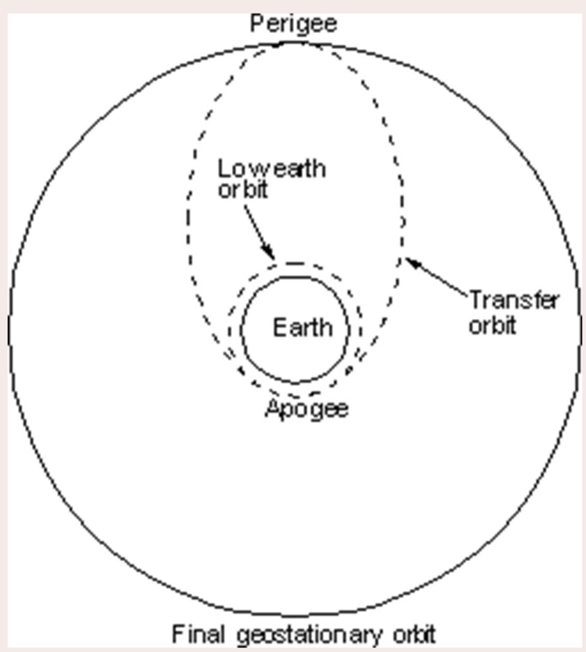
The speed at this position is

$$\begin{aligned} v &= \sqrt{GM_c\left(\frac{2}{r}-\frac{1}{a}\right)} \\ &= \sqrt{6.674\times10^{-11}\tfrac{Nm^2}{kg^2}\cdot2\times10^{30}kg\left(\frac{2}{5.559\cdot1.5\times10^{11}m}-\frac{1}{17.941\cdot1.5\times10^{11}m}\right)} \\ &= 16.45\tfrac{km}{s} \end{aligned}$$

Example E1.2.2

The transfer orbit

Whenever they want to place a satellite in a rather distant orbit around the Earth, they proceed by steps. First, the satellite is placed into a low circular orbit (low Earth orbit in the diagram). In the second step, reactors are fired once again to increase the speed of the rocket in such a way that its orbit becomes elliptical. This is the transfer orbit. Then, the distance at the apogee is adjusted to be exactly equal to the distance at which they wish to put the satellite into orbit. Finally, in the third step, the reactors are fired once again when the satellite is at the apogee in such a way that the orbit becomes circular again.



www.radio-electronics.com/info/satellite/satellite-orbits/satellite-launching.php

Find the speed and the mechanical energy of a 75 kg satellite on these orbits if the low Earth circular orbit is 300 km above the surface of the Earth and the large circular orbit is the orbit for the synchronous satellites, whose radius is 42 300 km. (For the elliptical orbit, just give the speeds at the perigee and the apogee.) The radius of the Earth is 6380 km, and the mass of the Earth is 5.98 x 10^{24} kg.

First step: The satellite is on an orbit 300 km above the surface of the Earth.

The speed on this orbit is

$$\begin{aligned} v &= \sqrt{\frac{GM_c}{r}} \\ &= \sqrt{\frac{6.674\times10^{-11}\,\frac{Nm^2}{kg^2}\cdot 5.98\times10^{24}\,kg}{6.68\times10^{6}\,m}} \\ &= 7.73\,\tfrac{km}{s} \end{aligned}$$

and the mechanical energy is

$$\begin{aligned} E &= -\frac{GM_c m}{2r} \\ &= -\frac{6.674\times10^{-11}\,\frac{Nm^2}{kg^2}\cdot 5.98\times10^{24}\,kg\cdot 75kg}{2\times6.68\times10^{6}\,m} \\ &= -2.24\times10^{9}\,J \end{aligned}$$

Second step: The satellite is on an elliptical orbit (transfer orbit)

To bring the satellite into a synchronous orbit, it must be on an elliptical orbit with the values r_a = 42 300 km and r_p = 6680 km.

The semi-major axis and the eccentricity of the orbit are

$$a = \frac{r_a + r_p}{2} = \frac{4.23\times10^{7}\,m + 6.68\times10^{6}\,m}{2} = 2.449\times10^{7}\,m$$

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{4.23\times10^{7}\,m - 6.68\times10^{6}\,m}{4.23\times10^{7}\,m + 6.68\times10^{6}\,m} = 0.7272$$

The speed at the perigee is then

$$v_p = \sqrt{\frac{GM_c}{a}\frac{1+e}{1-e}}$$
$$= \sqrt{\frac{6.674\times10^{-11}\frac{Nm^2}{kg^2}\cdot 5.98\times10^{24}kg}{2.449\times10^7 m}\frac{1+0.7272}{1-0.7272}}$$
$$= 10.16\tfrac{km}{s}$$

The speed of the satellite must then be increased from 7.73 km/s to 10.16 km/s in order to change the orbit from a circular shape to an elliptical shape that will bring the satellite 42 300 km from the Earth at the apogee.

The speed at the apogee is

$$v_a = \sqrt{\frac{GM_c}{a}\frac{1-e}{1+e}}$$
$$= \sqrt{\frac{6.674\times10^{-11}\frac{Nm^2}{kg^2}\cdot 5.98\times10^{24}kg}{2.449\times10^7 m}\frac{1-0.7272}{1+0.7272}}$$
$$= 1.61\tfrac{km}{s}$$

The energy on this new orbit is

$$E = -\frac{GM_c m}{2a}$$
$$= -\frac{6.674\times10^{-11}\frac{Nm^2}{kg^2}\cdot 5.98\times10^{24}kg\cdot 75kg}{2\cdot 2.449\times10^7 m}$$
$$= -6.11\times10^8 J$$

The energy must then be increased by $-6.11\times10^8 J - -2.24\times10^9 J = 1.63\times10^9 J$ to change from the small circular orbit to the transfer orbit.

Third step: The satellite is on a circular orbit with a 42 300 km radius.

The speed on this orbit is

$$v = \sqrt{\frac{GM_c}{r}}$$
$$= \sqrt{\frac{6.674\times10^{-11}\frac{Nm^2}{kg^2}\cdot 5.98\times10^{24}kg}{4.23\times10^7 m}}$$
$$= 3.07\tfrac{km}{s}$$

The speed of the satellite must then be increased from 1.61 km/s to 3.07 km/s in order to change the orbit from an elliptical shape to a circular shape at 42 300 km from the Earth.

The mechanical energy on this orbit is

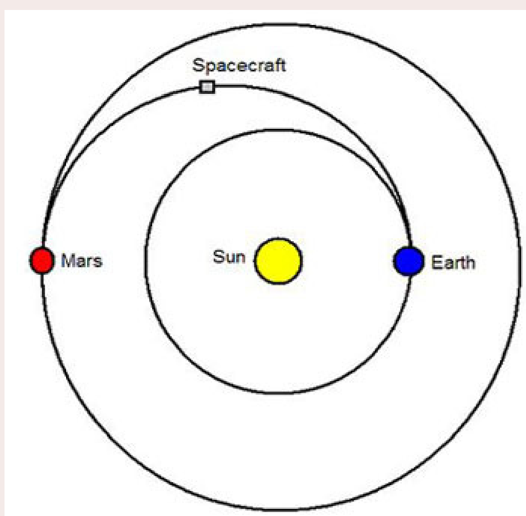
$$\begin{aligned} E &= -\frac{GM_c m}{2r} \\ &= -\frac{6.674\times10^{-11}\frac{Nm^2}{kg^2}\cdot 5.98\times10^{24}kg\cdot 75kg}{2\cdot 4.23\times10^7 m} \\ &= -3.54\times10^8 J \end{aligned}$$

The energy must then be increased by $-3.54\times10^8 J - -6.11\times10^8 J = 2.57\times10^8 J$ to change from the transfer orbit to the large circular orbit.

An elliptical orbit used to move from one circular orbit to another is called a *Hohmann transfer orbit*.

Example E1.2.3

To go to Mars, there is no need to use reactors during the whole journey (it would be very expensive to do so). The spacecraft simply need to be placed on a Hohmann transfer orbit with its perihelion at the Earth and its aphelion at Mars. How long will last this journey from the Earth to Mars? The mass of the Sun is 2 x 10^{30} kg. The distance between the Earth and the Sun 1.5 x 10^{11} m and the distance between Mars and the Sun is 2.3 x 10^{11} m.



www.teachengineering.org/view_activity.php?url=collection/cub_/activities/cub_mars/cub_mars_lesson04_activity1.xml

At the perihelion, the spaceship is at the same distance from the Sun than the Earth, so that r_p = 1.5 x 10^{11} m. At the aphelion, the spaceship is at the same distance from the Sun than Mars, so that $r_a = 2.3 \times 10^{11}$ m. The duration of the journey is equal to half the period since only half of the orbit is travelled.

The semi-major axis of the transfer orbit is

$$a = \frac{r_a + r_p}{2} = \frac{2.3\times10^{11}m + 1.5\times10^{11}m}{2} = 1.9\times10^{11}m$$

Therefore, the duration is

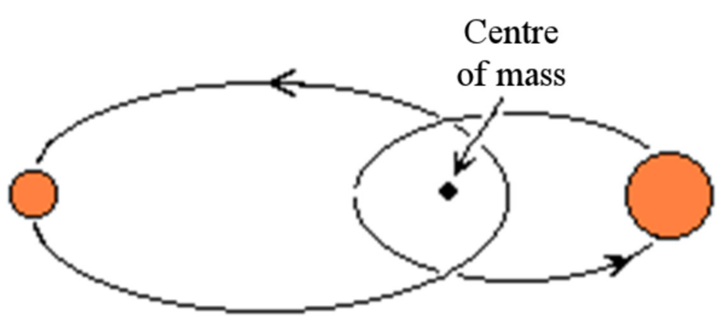
$$\begin{aligned} T &= \frac{1}{2}2\pi\sqrt{\frac{a^3}{GM_c}} \\ &= \pi\sqrt{\frac{\left(1.9\times10^{11}m\right)^3}{6.674\times10^{-11}\frac{Nm^2}{kg^2}\cdot 2\times10^{30}kg}} \\ &= 2.25\times10^7 s \\ &= 261\, days \end{aligned}$$

Of course, they must ensure that Mars is there when the ship arrives at the aphelion. The ship cannot be launched at any moment. The right period to launch is called the *launch window*.

Orbits of Two Objects of Similar Mass

Up to now, only orbits where the central mass is much greater than the mass of the orbiting object were examined. In this case, the central mass does not move much and the small mass describes an orbit around the largest mass.

However, what happens if the two objects have similar masses? The gravitational forces are then large enough to move the two masses, and the situation is not as simple as before. Yet, the solution to this problem is still relatively simple. Two objects describe elliptical orbits with the centre of mass located at one of the foci of each ellipse.



Although the formulas for studying such a system are not so much more complicated than that for a system where a mass is much greater than the other, this kind of system will not explore here.

E1.4 PARABOLIC TRAJECTORIES (e = 1)

If the eccentricity is exactly equal to 1, then the formula

$$r = r_p \frac{1+e}{1+e\cos\theta}$$

becomes

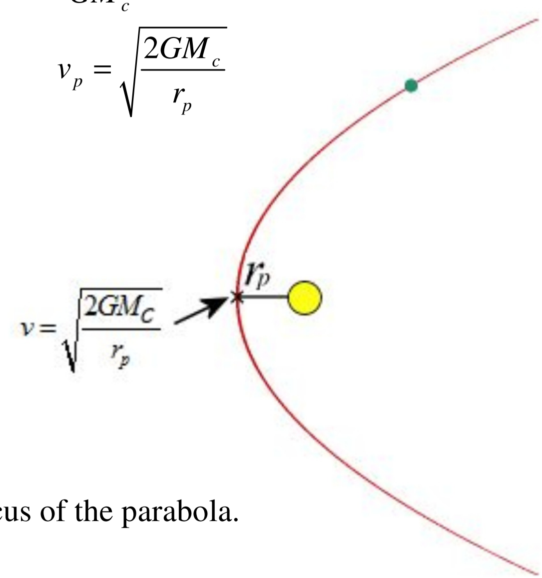
$$r = 2r_p \frac{1}{1+\cos\theta}$$

This equation is the equation of a parabola. To have this trajectory, the speed at the closest point from the central mass must be given by

$$e = 1$$

$$\frac{v_p^2 r_p}{GM_c} - 1 = 1$$

$$v_p = \sqrt{\frac{2GM_c}{r_p}}$$



The central mass is then at the focus of the parabola.

en.wikibooks.org/wiki/Astrodynamics/Orbit_Basics

Mechanical Energy

The mechanical energy of the object on a parabolic trajectory is

$$\begin{aligned} E_{mec} &= -\frac{GM_c m(1-e)}{2r_p} \\ &= 0 \end{aligned}$$

Mechanical Energy of an Object on a Parabolic Trajectory

$$E_{mec} = 0$$

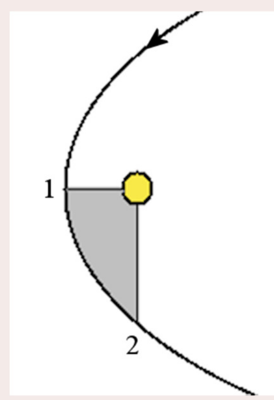
An object has this kind of orbit when its initial speed is close to zero when it is very far from the Sun. The object then approached the central mass to make a single pass. It then leaves the Sun to return to a very large distance where it will again have a very small speed. The object will not come back to the Sun because its energy is not negative, which means that the object is not bound to the central mass.

Speed

The speed of the object is easily found from the position with the mechanical energy.

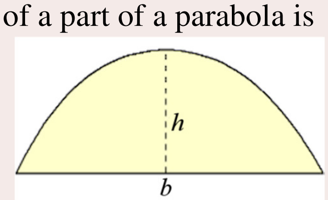
$$0 = \frac{1}{2}mv^2 - \frac{GM_c m}{r}$$

Example E1.4.1



A comet follows a parabolic trajectory around the Sun ($M = 2 \times 10^{30}$ kg). How long does it take for the comet to pass from position 1 to position 2, knowing that the speed at the point closest to the Sun is 100 km/s?

Hint: the area of a part of a parabola is



$$A = \frac{2bh}{3}$$

The time can be found with the Kepler's second law.

$$\Delta A = \frac{\sqrt{GM_c r_p(1+e)}}{2}\Delta t$$

However, in order to apply this formula, the distance of the comet at its closest position to the Sun must be known. This distance can be found with the mechanical energy.

$$0 = \frac{1}{2}mv_p^2 - \frac{GM_c m}{r_p}$$

$$0 = \frac{1}{2}v_p^2 - \frac{GM_c}{r_p}$$

$$0 = \frac{1}{2}\left(100\,000\tfrac{m}{s}\right)^2 - \frac{6.674\times10^{-11}\tfrac{kgm^2}{s^2}\cdot 2\times10^{30}kg}{r_p}$$

$$r_p = 2.6696\times10^{10}m$$

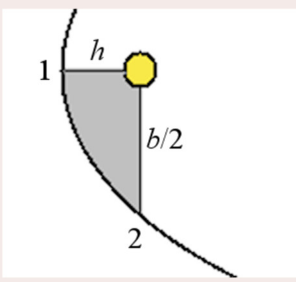
Therefore, Kepler's second law becomes

$$\Delta A = \frac{\sqrt{GM_c r_p\left(1+e\right)}}{2}\Delta t$$

$$\Delta A = \frac{\sqrt{6.674\times10^{-11}\tfrac{Nm^2}{kg^2}\cdot 2\times10^{30}kg\cdot 2.6696\times10^{10}m\left(1+1\right)}}{2}\Delta t$$

$$\Delta A = 1.3348\times10^{15}\tfrac{m^2}{s}\Delta t$$

For the area of the parabola, we have



$$h = r_p$$

$$\frac{b}{2} = r_p\frac{1+e}{1+r\cos 90°} = r_p\frac{2}{1+\cos 90°} = 2r_p$$

The area is half the area of the part of a parabola given by the formula.

$$\Delta A = \frac{1}{2}\frac{2bh}{3} = \frac{1}{2}\frac{2\left(4r_p\right)\cdot\left(r_p\right)}{3} = \frac{4r_p^2}{3} = \frac{4\left(2.6696\times10^{10}m\right)^2}{3} = 9.5024\times10^{20}m^2$$

Thus, the time is

$$\Delta A = 1.3348\times10^{15}\tfrac{m^2}{s}\Delta t$$

$$9.5024\times10^{20}m^2 = 1.3348\times10^{15}\tfrac{m^2}{s}\Delta t$$

$$\Delta t = 711{,}893s$$

$$\Delta t = 197.7h$$

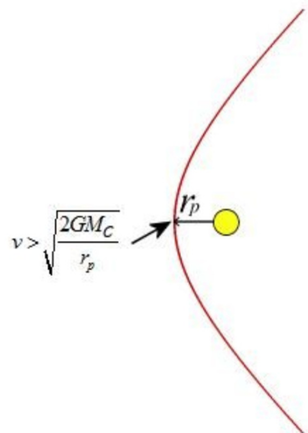
E1.5 HYPERBOLIC TRAJECTORIES (e > 1)

If the eccentricity is greater than 1, then the formula

$$r = r_p \frac{1+e}{1+e\cos\theta}$$

describes a hyperbola. To have this trajectory, the speed at the closest point from the central mass must be given by

$$e > 1$$
$$\frac{v_p^2 r_p}{GM_c} - 1 > 1$$
$$v_p > \sqrt{\frac{2GM_c}{r_p}}$$



en.wikibooks.org/wiki/Astrodynamics/Orbit_Basics

Then, the energy of the object on the hyperbolic trajectory is

$$E_{mec} = -\frac{GM_c m(1-e)}{2r_p} > 0$$

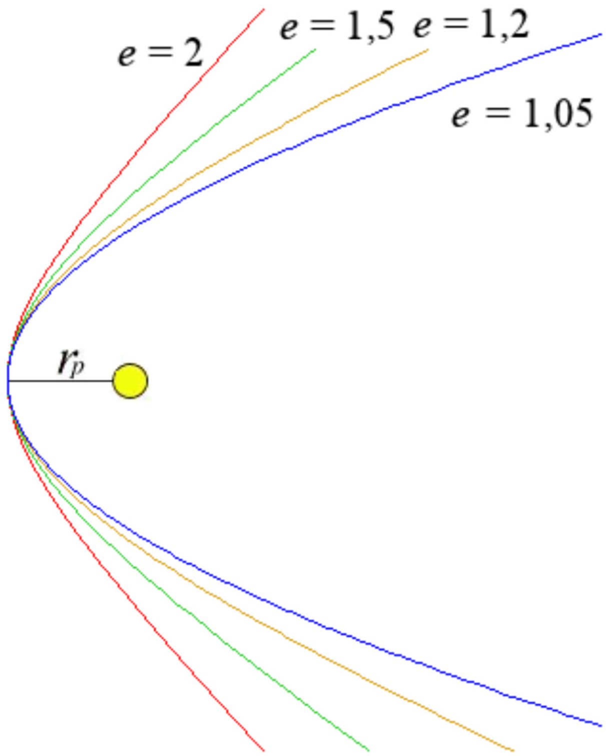
Mechanical Energy of an Object on a Hyperbolic Trajectory

$$E_{mec} > 0$$

The object does not come back because its energy is not negative, which means that the object is not bound to the central mass.

In this case, the object comes from a great distance with a certain speed and approaches the central mass to make a single pass. It then leaves to return to a very large distance where he will move with a certain speed.

Here's how the shape of the hyperbolic trajectory changes as a function of the eccentricity



Very few objects were observed to follow hyperbolic trajectories with a high eccentricity. For all the known comets, the greatest value of the eccentricity observed was 1.057.

E1.6 SUMMARY OF POSSIBLE TRAJECTORIES

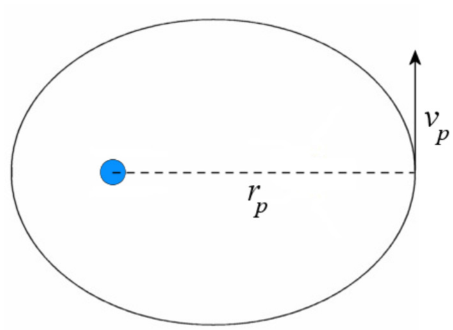
Here is a small summary of all possible trajectories according to the speed at the point of closest proximity to the central mass or the eccentricity.

Speed	Trajectory	Eccentricity	Mechanical Energy
$v_p = \sqrt{\frac{GM_c}{r_p}}$	Circular	$e = 0$	Negative
$\sqrt{\frac{GM_c}{r_p}} < v_p < \sqrt{\frac{2GM_c}{r_p}}$	Elliptical	$0 < e < 1$	Negative
$v_p = \sqrt{\frac{2GM_c}{r_p}}$	Parabolic	$e = 1$	Zero
$v_p > \sqrt{\frac{2GM_c}{r_p}}$	Hyperbolic	$e > 1$	Positive

One may then wonder what happens if

$$v_p < \sqrt{\frac{GM_c}{r_p}}$$

In fact, this situation is impossible. If you have a lower speed than the speed needed for a circular orbit, then there is too much centripetal force and the object moves closer to the central mass. The following elliptical orbit is then obtained.



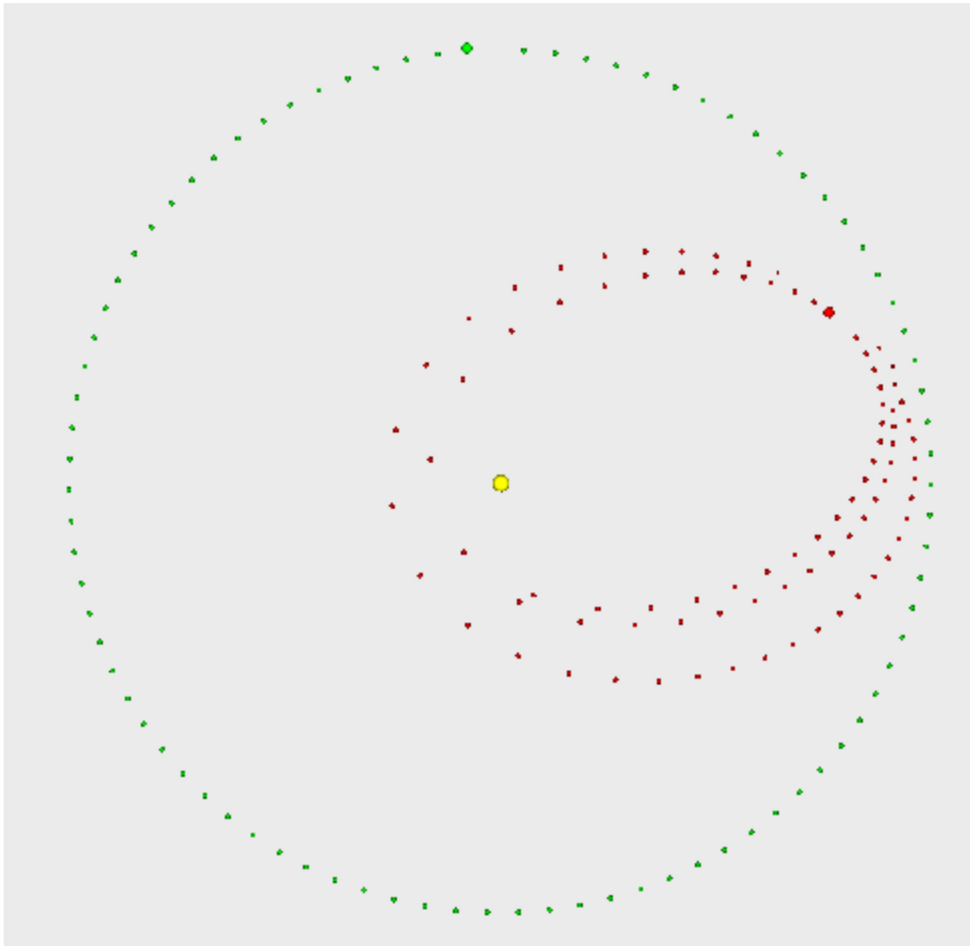
However, this situation is impossible because r_p was defined as being the distance of the point closest to the central mass, which is not the case in this diagram. It is impossible to have a speed lower than the speed needed for a circular orbit at the point closest to the central mass.

E1.7 PERTURBATIONS

In reality, the shape of the orbits is way more complex than previously seen and the orbits are never perfect ellipses, parabolas or hyperbolas. If there were only two rigid bodies in the universe, then the orbits would be perfect. As soon as there are other bodies nearby, the disturbances made by these other bodies slowly change the orbit of an object.

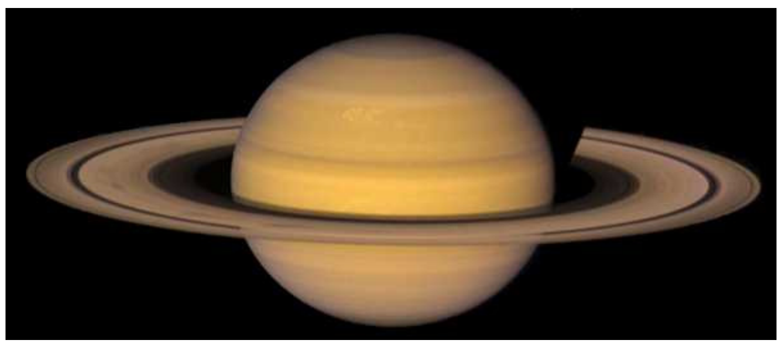
In the Solar System, the motion of each planet is perturbed by the gravitational force exerted by every other planets. Note that the motion of less massive objects is easier to change. The trajectories of comets, which are relatively light bodies, are easily altered, especially if they pass near a massive planet like Jupiter. Asteroids are also heavily influenced by the gravitational force of Jupiter. Even if they are not that close to Jupiter, the gravitational force exerted by the latter is large enough to have long-term effects.

This diagram shows how the orbit of an asteroid (in red) changes when it passes too close to Jupiter (in green) over a 50-year period.



Thanks to disturbance, the planet Neptune was discovered in 1846. William Herschel had discovered the planet Uranus quite by chance in 1781, and its orbit was studied from this moment on. In 1820, it was becoming clear that there was a problem with the orbit of this planet as it appeared to deviate from the predicted orbit. Some astronomers thought then that another unknown planet (which was eventually called Neptune) located beyond the orbit of Uranus was disrupting its motion. (Indeed, Uranus and Neptune were closest to each other in 1821, and the strength of disturbance was then at its peak.) John Couch Adams and Urbain Le Verrier both started calculating the position of the disturbing planet and both basically obtained the same position for Neptune. The result was a little approximate because there were some assumptions to make, such as the distance of this new planet, for example. When Adams sent its results to the Greenwich Observatory in September 1845, the astronomers there were unable to confirm the existence of the new planet. Actually, they put little efforts into the task, because the director of the Observatory was not so impressed by the method used. "Why should they put much effort for an uncertain outcome?" he said. Then, in June 1846, Le Verrier publishes the results of his calculations. The similarity of his results with those of Adams then convinced the observers in Greenwich that Adams's calculations were perhaps not as bad as they thought and they started a more sustained search of the planet. Meanwhile, le Verrier sent his results to the Berlin Observatory in September 1846, after having tried in vain to interest French astronomers. Fortunately, the Berlin Observatory had an excellent map of the area of the sky where Neptune should be according to Le Verrier, which allowed them to easily spot any change in this area of the sky. In less than an hour, the astronomers in Berlin discovered Neptune about a degree from the position predicted by Le Verrier. After the announcement of this discovery, astronomers in Greenwich found they had observed Neptune twice without realizing it… All that to say that the law of gravitation has allowed us to discover Neptune by calculation. This is one of the greatest successes of Newton's gravitational theory.

Most of the time, the effects of disturbances tend to cancel each other over time and the orbit, although disrupted, is relatively stable. However, if the largest disturbances always occur at the same place on the orbit at regular intervals, then the orbit is not stable. Thus, if an asteroid has a period of exactly half that of Jupiter, the maximum disturbance, which occurs when the asteroid and Jupiter are closest to each other, always happens at the same place on the orbit of the asteroid, every two revolutions of the asteroid in its orbit. Then, instead of cancelling each other over time, the disruptions add up and slowly move the asteroid out of that orbit. This is called gravitational resonance. Looking at the distribution of asteroids in the solar system, it can actually be noted that there is no asteroid on an orbit with a semi-major axis that corresponds to orbits with periods which are simple ratios of the period of Jupiter. The same phenomenon occurs in the rings of Saturn. Disturbances made by the satellites around Saturn prevent small rocks in the ring to have certain orbits, thereby explaining the dark gaps in the ring.



www.lasam.ca/billavf/nineplanets/saturn.html

Perturbations also affect more massive objects like the Earth. Because of disturbances made by other planets, the eccentricity of Earth's orbit varies over time. As the planets are not perfect spheres, other planets also exert a torque that can change the direction of the axis of rotation of the planets. The axis can even be completely inverted by this torque so that the planet rotates in a direction opposite to the rotation of all the other planets, as is currently the case for Venus. For the Earth, such a drastic change cannot happen because the Moon tends to stabilize the tilting angle of the axis of rotation of the Earth so that it may only change by a few degrees. All these changes to the motion of the Earth have a huge impact on Earth's climate and are mostly responsible for the onset of ice ages.

It is impossible to give the equations describing the motion of planets taking into account the perturbations made by other planets. It has been shown that it is impossible to give an exact analytical solution if there are 3 bodies or more interacting. The orbits are now found with computer simulations.

E1.8 GRAVITATIONAL FIELD

Definition of Gravitational Field

If an object is placed somewhere and a gravitational force acts on it, then there is a gravitational field at this location. As a force acts on any mass placed near the Earth, it follows that there is a gravitational field around the Earth.

This field by noted by the letter g. By definition, the field has the following characteristics:

1) If the field is stronger, the force is larger.
2) If a more massive object is placed in a field, the force on the object is larger.

The second characteristic is easily noticeable on the surface of the Earth. If a small rock is placed somewhere, a certain force acts on the rock. If a rock with twice the mass is placed at the same location, the force exerted on the rock is twice as large.

The strength of the field can vary from one location to another. It is for this reason that is called a field because, in mathematics, a field is a quantity whose value can change from one location to another.

According to the two characteristics aforementioned, the gravitational field can be defined by the following formula.

Force on an Object of Mass *m* in a Gravitational Field

$$\vec{F} = m\vec{g}$$

Since the force is a vector, g must also be a vector. This vector points in the direction of the force acting on an object placed at this location. Therefore, the gravitational field is a vector field.

The SI unit for the field is N/kg or m/s² (which are two equivalent units).

What Generates the Field?

The answer is not so complicated. If there is a gravitational field somewhere, then a gravitational force is exerted on a mass placed at this location. However, if there is a gravitational force, it is because other bodies attract the mass. So if there's a field somewhere, it is because there are masses in the vicinity this place. This means that

Masses create a gravitational field around them.

For example, there is a gravitational field in your room because there is a body with an enormous mass near your room which creates a field around it: the Earth.

Gravitational Field of a Point Mass of Mass *M*

If masses create a field around them, it must be possible to determine the magnitude of this field. Let's start with a simple case: the gravitational field made by a point mass of mass *M*.

With two point masses (*M* and *m*), the force between them is

$$F = G\frac{Mm}{r^2}$$

Another way to look at this force is to say that a force acts on the mass *m* because the mass *m* is in the gravitational field created by the mass *M*. Then, the force is

$$F = mg$$

As these two ways to see the gravitational force must give the same result, the following equation must be true.

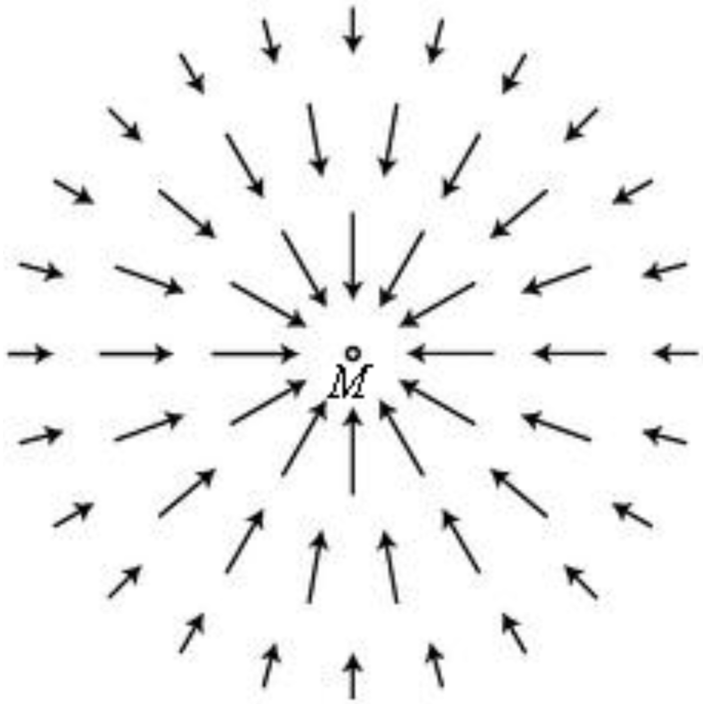
$$mg = G\frac{Mm}{r^2}$$

Thus, the field created by the mass *M* is

Magnitude of the Gravitational Field of a Point Mass of Mass *M*

$$g = \frac{GM}{r^2}$$

This formula indicates that the gravitational field decreases rapidly with the distance from the mass. Also, it indicates that gravitational field is stronger around more massive objects. The direction of the field at different locations around the mass *M* is shown in the diagram.

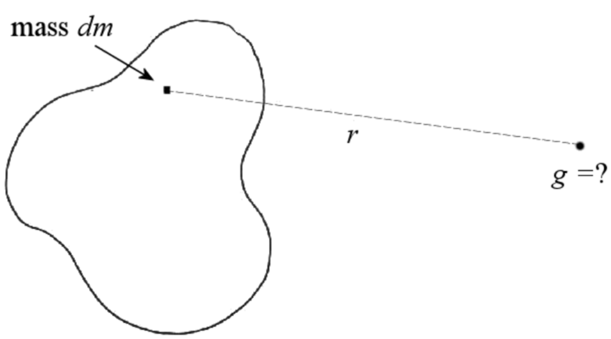


www.vias.org/physics/bk4_06_03.html

The field always points towards the mass *M*, because this is the direction of the force exerted by the mass *M* on the masses around it since the gravitational force is always attractive.

Gravitational Field Generated by an Extended Object

To calculate the field made by an extended object, the object is divided into infinitesimally small pieces. The fields made by each of these small pieces of mass *dm* are identical to the field made by a point mass. The magnitude of this field is then



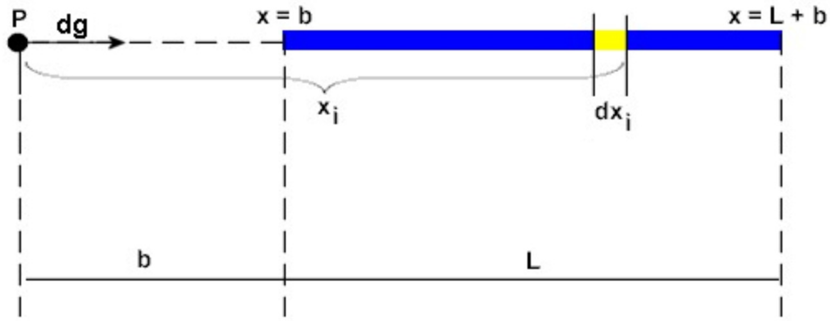
$$dg = \frac{Gdm}{r^2}$$

Finally, a vector sum of these fields created by the small masses is made to obtain for the total field. This sum of infinitesimal field is an integral.

As said previously, you are not quite ready to do integrals for objects in two or three dimensions. You can, however, do integrals for one-dimensional objects like a rod.

Gravitational Field of a Rod of Constant Density

Here, the gravitational field at point P at a distance b from the end of a straight uniform rod of length L is sought. To calculate this field, the rod is divided into small slices of length *dx.*



online.cctt.org/physicslab/content/phyapc/lessonnotes/Efields/EchargedRods.asp

The magnitude of the gravitational field made by one of these small pieces, which will be noted *dg*, is

$$dg = G\frac{dm}{x^2}$$

where *dm* is the mass of the small piece and *x* is the distance between the small piece and the point P. (The point P is at the origin)

If the rod has uniform density, then the mass is $dm = \lambda\, dx$, where λ is the linear density of the rod. Therefore, the gravitational field made by the piece is

$$dg = G\frac{\lambda dx}{x^2}$$

The magnitude of the total field is simply the sum of the fields made by each of the pieces.

$$\begin{aligned} g &= \int_b^{b+L} G\frac{\lambda dx}{x^2} \\ &= G\lambda\left(\frac{-1}{x}\right)\Bigg|_b^{b+L} \\ &= G\lambda\left[\frac{-1}{b+L} - \frac{-1}{b}\right] \\ &= G\lambda\left[\frac{-b}{b(b+L)} + \frac{b+L}{b(b+L)}\right] \\ &= \frac{G\lambda L}{b(b+L)} \end{aligned}$$

Since λL is the mass of the rod, the field is

Gravitational Field of a Uniform Rod

$$g = \frac{GM}{b(b+L)}$$

In the following direction

This calculation is by far the simplest that can be done to find the field of an extended object. In most cases, the vector dg must be resolved into components and three integrals must be calculated (one for each component). It is so beautiful.

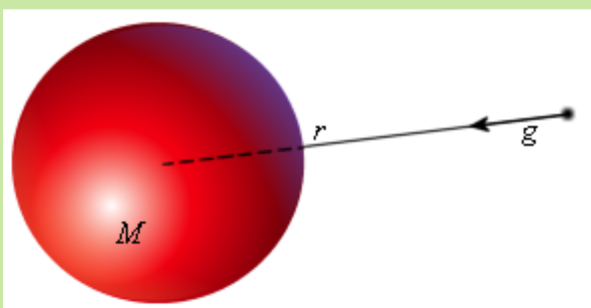
Gravitational Field of a Sphere

The gravitational field of a sphere is of paramount importance since this is the shape of planets and stars, the only objects to have sufficient mass to generate non-negligible gravitational forces. It could even be said that the calculation of the gravitational field for all shapes other than the sphere is a simple intellectual exercise whose sole purpose is entertainment.

The field of a sphere is found in the same way as what was done for the rod: the sphere is divided into small pieces and the field made by each of the pieces is found. The fields made by each of the small pieces are then added with an integral to get the total field. The result of this rather complex calculation (we'll spare you the details) is surprisingly simple. On the outside of a sphere, the gravitational field is identical to the field made by a point mass of the same mass which would be located in the centre of the sphere. In other words, the gravitational field outside a sphere is given by

Gravitational Field Outside a Sphere

$$g = \frac{GM}{r^2}$$

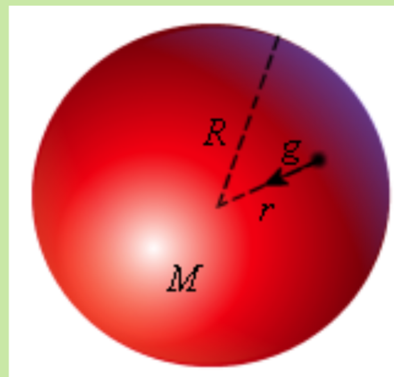


where r is the distance from the centre of the sphere. (This result, as well as those that will follow, is valid only if the sphere is symmetrical, which means that it must be identical in all directions from the centre. The density can change, but only as a function of the distance from the centre of the sphere.)

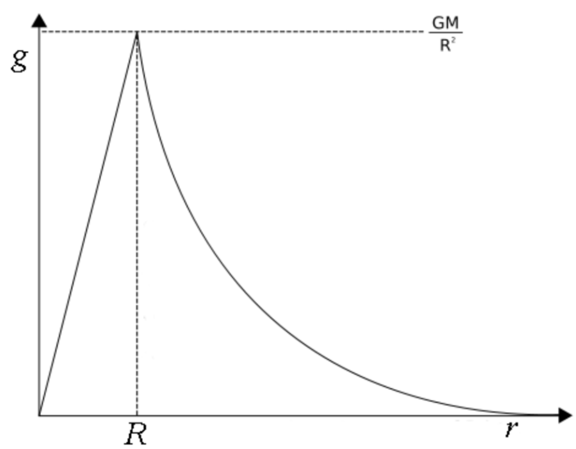
Inside the sphere, the gravitational field depends on how the mass is distributed. If the mass is evenly distributed, the field is

Gravitational Field Inside a Sphere of Constant Density

$$g = \frac{GMr}{R^3}$$



The graph of the strength of the field as a function of the distance from the centre of a uniform sphere is, therefore,

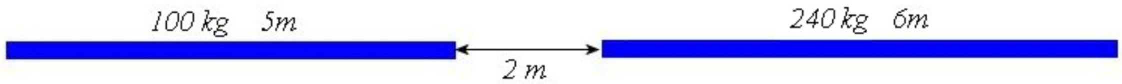


en.wikibooks.org/wiki/A-level_Physics_(Advancing_Physics)/Gravitational_Fields/Worked_Solutions

This means that the gravitational field of a uniform sphere reaches its maximum value at the surface of the sphere.

Why Use the Gravitational Field?

The field allows us to split the calculation of the force between two objects into two steps, which make the calculation of the force much easier. For example, it would be extremely difficult to directly calculate the gravitational force between these two rods with the law of gravitation.



The calculation of the force is instead done in two steps.

1) Calculate the gravitational field made by one of the objects

   The field is found with a formula for the field created by this object. If no formula is known, the field must be calculated with an integral.

   Here, the field made by the rod on the left is sought. We have a formula for the field created by a rod. The field is

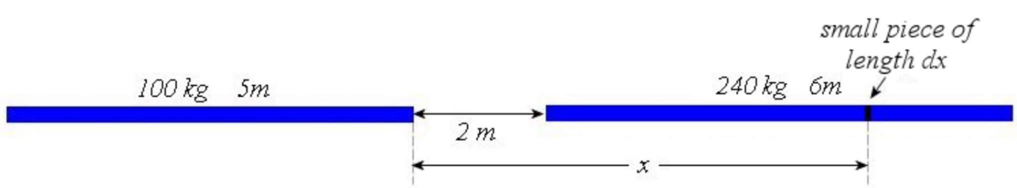
$$g = \frac{G \times 100kg}{x(x+5m)}$$

   where x is used for the distance since the atoms in the 240 kg rod are not always at the same distance from the rod on the left.

2) Calculate the force on the other object with $F = mg$

   This step can be easy, but it can also be complicated. Indeed, if the object is not in a uniform field, the object must be divided into smaller pieces and the force on each of the pieces must be calculated. The total force is then found by adding the forces on each piece with an integral.

   Here, the field made by the rod on the left is not uniform. The rod on the right must then be divided into infinitesimally small pieces of length dx.



   The force on one of these small pieces is

$$dF = -gdm$$

   It is negative because it is directed towards the left since the other rod attracts the piece. Using the mass of the piece ($dm = \lambda\, dx$) and the formula for the field, the force is

$$\begin{aligned} dF &= -g\,dm \\ &= -\frac{G\times 100kg}{x\left(x+5m\right)}\lambda_2 dx \end{aligned}$$

As the linear density of the rod on the right is λ_2 = 240 kg/6 m = 40 kg/m, the force is

$$dF = -\frac{G\times 100kg}{x\left(x+5m\right)} 40\tfrac{kg}{m}\,dx = -\frac{G\times 4000\frac{kg^2}{m}}{x\left(x+5m\right)}dx$$

The total force is then

$$F = \int_{2m}^{8m} \frac{-G\times 4000\frac{kg^2}{m}}{x\left(x+5m\right)}dx$$

The integral goes from 2 m to 8 m, because the rod on the right starts 2 m from the tip of the rod on the left and ends 8 m from the tip of the rod on the left. This integral gives

$$\begin{aligned} F &= \frac{-G\times 4000\frac{kg^2}{m}}{5m}\left[\ln\left(x\right)-\log\left(x+5m\right)\right]_{2m}^{8m} \\ &= \frac{-G\times 4000\frac{kg^2}{m}}{5m}\left[\ln\left(\frac{x}{x+5m}\right)\right]_{2m}^{8m} \\ &= \frac{-G\times 4000\frac{kg^2}{m}}{5m}\left[\ln\left(\frac{8}{13}\right)-\ln\left(\frac{2}{7}\right)\right] \\ &= -1.192\times 10^{-7}N \end{aligned}$$

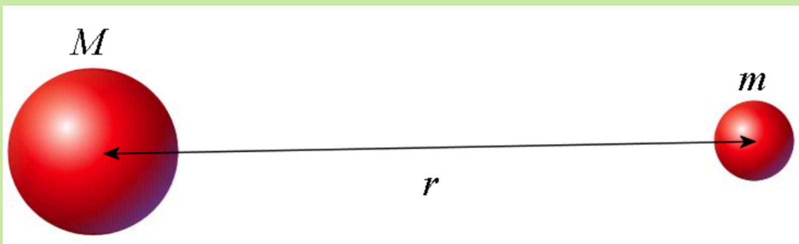
By using the gravitational field, the force was calculated by making two integrals one after the other. Imagine what this calculation would have looked like if we had attempted to calculate the force directly with the law of gravitation without using the field. Two nested integrals would then have to be calculated simultaneously… Please note that the example given was relatively simple since it is a problem in one dimension. Without the field in three dimensions, two simultaneous triple integrals have to be solved to calculate the force between two objects in three dimensions. The use of the field renders calculations a bit simpler.

The Force Between Two Spheres

The force between two spheres can now be calculated. The calculation is made in two steps, as the calculation of the force between two rods done previously. First, the gravitational field made by one of the spheres is calculated. The result is known in this case: a field identical to that of a point mass located at the centre of the sphere. Then, the force on the second sphere is obtained by calculating the force on each small pieces of the second sphere caused by the field of the first sphere and by summing all these forces by an integral. The result is remarkably simple once again. The force between the spheres is the same as it would be if the entire mass of each sphere were concentrated at the centre of the sphere! Two quite complex integrals (one for the calculation of the field of the first sphere and another for the force on the second sphere) had to be calculated, but the result is simple. The force between two spheres is

Gravitational Force between Two Spheres

$$F = \frac{GMm}{r^2}$$



where r is the distance between the centres of the spheres.

Application: Field at the Surface or Near the Surface of a Planet

Example E1.8.1

What is the magnitude of the gravitational field…

a) on the surface of the Earth if it has a mass of 5.972 x 10^{24} kg and a radius of 6378 km?

The surface is 6378 km from the centre of the Earth. Using the formula of the field made by a sphere, the field is

$$g = \frac{GM}{R^2} = \frac{6.674\times10^{-11}\,\frac{Nm^2}{kg^2}\cdot 5.972\times10^{24}\,kg}{\left(6.378\times10^{6}\,m\right)^2} = 9.80\,\frac{N}{kg}$$

b) 1000 km above the surface of the Earth if it has a mass of 5,972 x 10^{24} kg and a radius of 6378 km?

At 1000 km above the surface, the distance is 7378 km from the centre of the Earth. Using the formula of the field made by a sphere, the field is

$$g = \frac{GM}{R^2} = \frac{6.674\times10^{-11}\,\frac{Nm^2}{kg^2}\cdot 5.972\times10^{24}\,kg}{\left(7.378\times10^6\,m\right)^2} = 7.32\,\frac{N}{kg}$$

You can see that the gravitational field decreases as one moves farther away from the Earth.

c) on the surface of the Moon if it has a mass of 7.35 x 10^{22} kg and a radius of 1738 km?

$$g = \frac{GM}{R^2} = \frac{6.674\times10^{-11}\,\frac{Nm^2}{kg^2}\cdot 7.35\times10^{22}\,kg}{\left(1.738\times10^6\,m\right)^2} = 1.62\,\frac{N}{kg}$$

This field is about a sixth of the field on the surface of the Earth.

Example E1.8.2

A 100 kg satellite is located between the Earth and the Moon, at the place indicated in the diagram. The mass of the Moon is 7.35 x 10^{22} kg, and the mass of the Earth is 5.97 x 10^{24} kg.

a) What is the field at this place?

The total field is the sum of the fields made by each planet. As the field is always directed towards the planet causing the field, the field made by the Earth is towards the left and the field made by the Moon is towards the right.

The field is then

$$\begin{aligned} g &= -\frac{GM_{Earth}}{r_{Earth}^2} + \frac{GM_{Moon}}{r_{Moon}^2} \\ &= -\frac{6.674\times10^{-11}\,\frac{Nm^2}{kg^2}\cdot 5.97\times10^{24}\,kg}{\left(2\times10^8\,m\right)^2} + \frac{6.674\times10^{-11}\,\frac{Nm^2}{kg^2}\cdot 7.35\times10^{22}\,kg}{\left(1.85\times10^8\,m\right)^2} \\ &= -0.009961\,\frac{N}{kg} + 0.000144\,\frac{N}{kg} \\ &= -0.009817\,\frac{N}{kg} \end{aligned}$$

A negative answer means that the gravitational field is directed towards the Earth.

b) What is the force on the satellite?

The force is

$$\begin{aligned} F &= mg \\ &= 100kg \cdot \left(-0.009817 \tfrac{N}{kg}\right) \\ &= -0.9817N \end{aligned}$$

As the force is negative, it is directed towards the Earth.

Correction to the Gravitational Field at the Earth's Surface

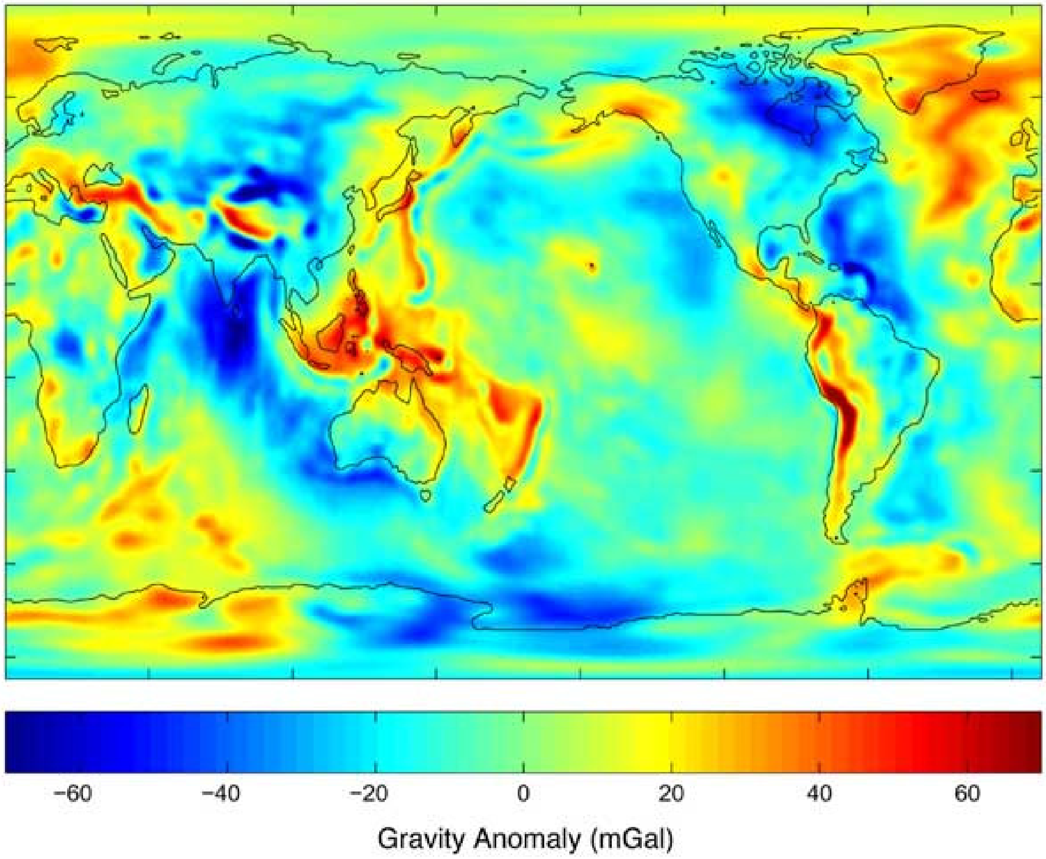
In a previous example, it was calculated that the gravitational field at the surface would be 9.80 N/kg everywhere if the Earth were a perfect sphere.

However, the Earth is not a perfect sphere. It rather has the shape of a slightly flattened sphere. Thus, the gravitational field strength varies with latitude. The calculation of the value of the field, in this case, is difficult enough but a good approximative result is

$$g = \left(9.780327 + 0.0516323\sin^2\varphi + 0.0002269\sin^4\varphi\right)\tfrac{N}{kg}$$

where ϕ is the latitude. As Quebec City lies at a latitude of about $\phi = 46^0$, the gravitational field should be 9.807105 N/kg in Quebec City.

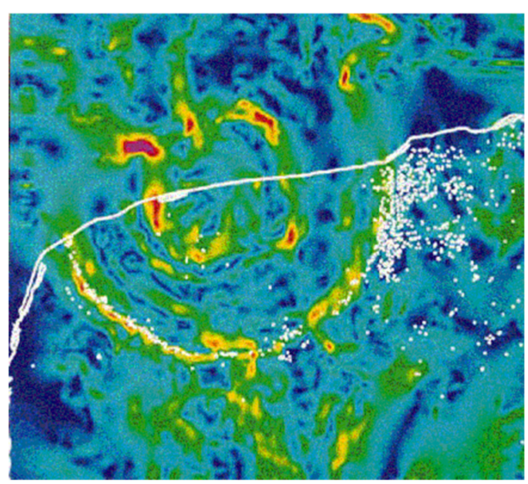
There can be other deviations of the order of 10^{-3} N/kg to the value given by this last formula since the Earth is not a perfect ellipsoid of revolution (there are changes in altitude) and does not have a perfectly uniform composition. The value of *g* can, therefore, vary depending on the local geological structure. This is what is called the gravity anomaly. The following map shows the anomaly at the surface of the Earth (the Gal is a unit worth 0.01 N/kg).



www.zonu.com/detail-en/2009-11-19-11208/Gravity-anomalies-in-the-world.html

It can be estimated that the anomaly in Quebec City is approximately -20 mGal and that the intensity of the field is about 9.80711 N/kg – 0.00020 N/kg = 9.80691 N/kg.

The gravity anomaly is never very large, of the order of 10^{-3} N/kg at most but it is sufficiently large so that certain geological structures of interest in the soil, such as oil reservoirs, can be detected. The gravity anomaly also lead to the discovery of the crater made by the meteorite responsible for the extinction of the dinosaurs 65 million years ago. It is on the Yucatan Peninsula in Mexico (image to the right).



planets.agu.org/Interview-with-Dr-Wasson.php

To discover these elements, very precise instruments are used. Currently, there are instruments capable of detecting variations as low as 10^{-8} N/kg in Earth's gravitational field. These instruments are so sensitive that they detect a change in the gravitational field if the device is only lifted 5 mm!

The Gravitational Field inside a Planet

Example E1.8.3

What is the gravitational field 500 km below the surface of the Earth (assuming that the density of the Earth is the same everywhere)? The mass of the Earth is 5.97 x 10^{24} kg, and its radius is 6380 km.

The field is

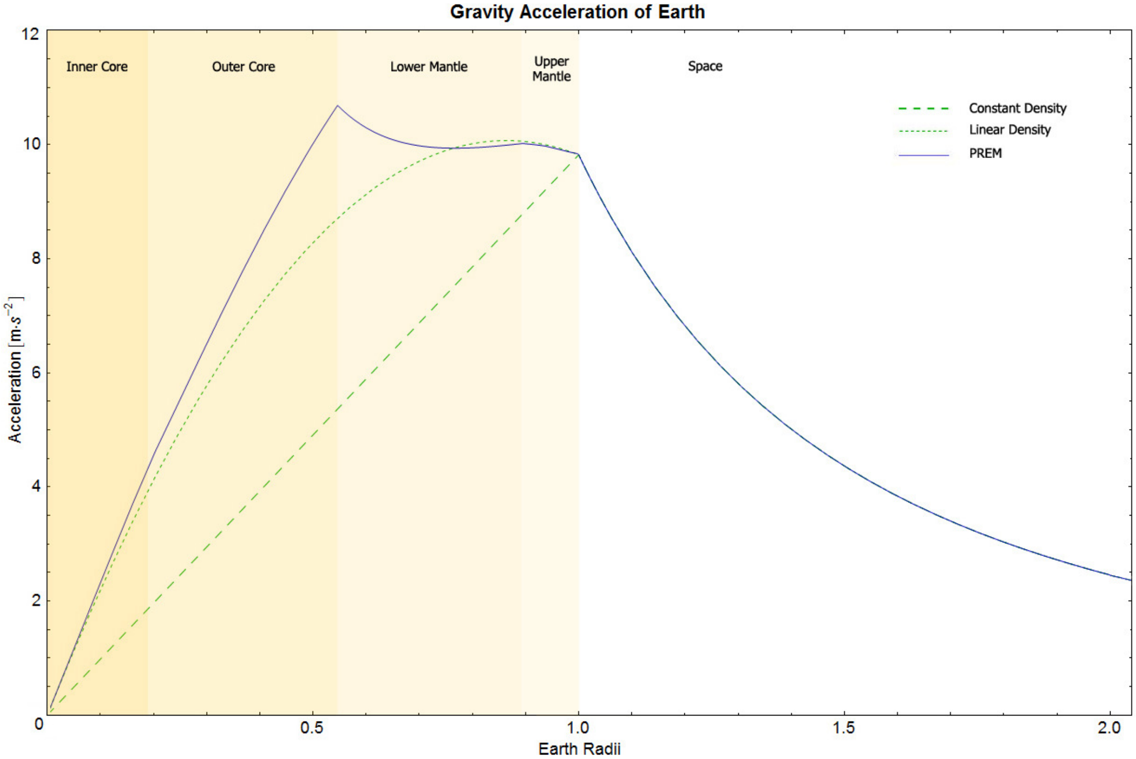
$$\begin{aligned} g &= \frac{GMr}{R^3} \\ &= \frac{6.674\times10^{-11}\,\frac{Nm^2}{kg^2}\cdot5.97\times10^{24}\,kg\cdot5.88\times10^6\,m}{\left(6.38\times10^6\,m\right)^3} \\ &= 9.02\,\frac{N}{kg} \end{aligned}$$

From this last formula, it is possible to think that the gravitational field always decreases as you get closer to the centre of the Earth. However, this is not the case since the Earth is not uniform. As its density is higher in the centre than near the surface, the gravitational field actually increases a little as one moves towards the centre of the Earth from the surface and only begins to decrease more deeply inside the Earth.

Depth km	g m/s²	Depth km	g m/s²
0	9.82	1400	9.88
33	9.85	1600	9.86
100	9.89	1800	9.85
200	9.92	2000	9.86
300	9.95	2200	9.90
413	9.98	2400	9.98
600	10.01	2600	10.09
800	9.99	2800	10.26
1000	9.95	2900	10.37
1200	9.91	4000	8.00

The exact variation of g for depth larger than 4000 km is not known, but it is known that it must be equal to 0 in the centre of the Earth (6380 km).

The graph of g as a function of depth is

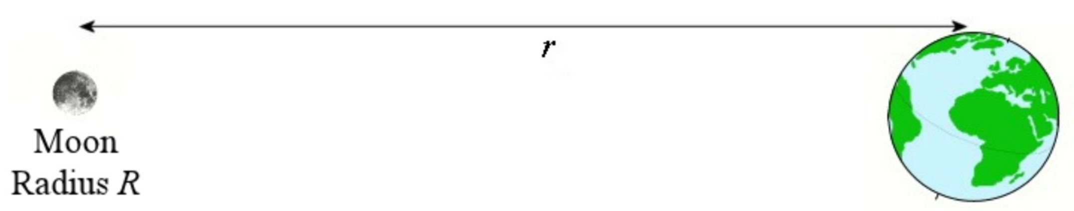


https://upload.wikimedia.org/wikipedia/commons/8/86/EarthGravityPREM.jpg

E1.9 TIDES

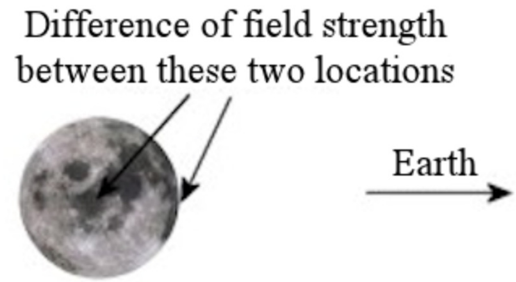
Magnitude of the Tidal Force

Tides are due to the fact that an extended object lies in a non-uniform gravitational field. To show the origin of tidal forces, the forces exerted on the Moon by the Earth will be examined.



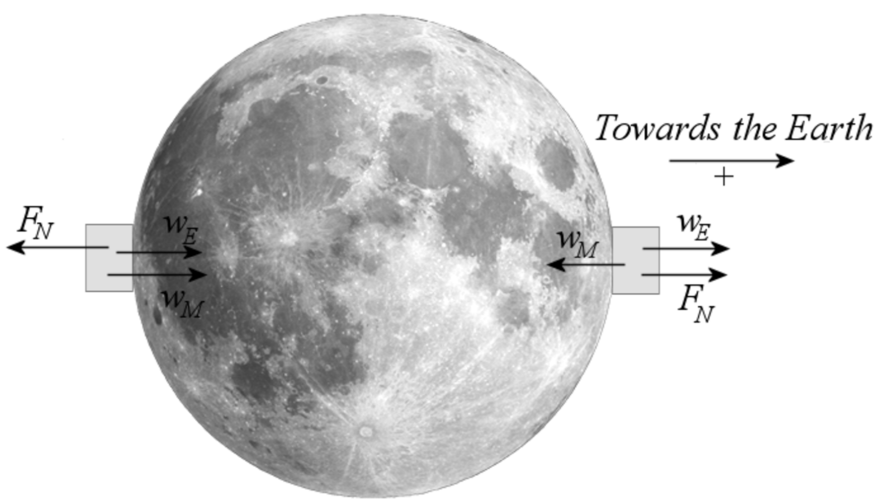
The Moon is in the gravitational field of the Earth, and this field decreases with distance. The difference in intensity of the gravitational field between the centre and a point on the surface of the Moon directly facing the Earth will be calculated. As the Moon is not very large, it will be assumed that the rate of change of the field is constant.

$$\begin{aligned}\Delta g &= (\text{rate of change of g}) \times (\text{distance}) \\ &= \left| \frac{dg}{dr} R \right| \\ &= \left| \frac{d\left(\frac{GM_{Earth}}{r^2}\right)}{dr} R \right| \\ &= \frac{2GM_{Earth}}{r^3} R\end{aligned}$$



As it is assumed that the rate is constant, this is also the difference between the field at the centre of the Moon and the field at the surface of the Moon directly opposite the Earth.

Now let's look at the effect of this difference in field intensity on two objects on the Moon surface. Each of these objects is attracted by the Moon and by the Earth. As the objects are on the surface of the Moon, a normal force is exerted on each of them.



In order to be able to write the force equation for these objects, the acceleration is needed. The Moon (and everything on it) accelerates towards the Earth since the Moon is making a circular motion around the Earth. This acceleration is caused by the force of gravitation exerted by the Earth and its magnitude is

$$\frac{GM_{Earth}M_{Moon}}{r^2} = M_{Moon}a$$

$$a = \frac{GM_{Earth}}{r^2}$$

For the object on the right side on the Moon (side facing the Earth), Newton's second law is

$$F_N + w_E - w_M = ma$$
$$F_N + mg_{Earth} - w_M = ma$$
$$F_N + m\left(g_{\text{Earth at the centre of the Moon}} + \Delta g\right) - w_M = ma$$
$$F_N + m\left(\frac{GM_{Earth}}{r^2} + \frac{2GM_{Earth}R}{r^3}\right) - w_M = m\frac{GM_{Earth}}{r^2}$$

The first term in the parenthesis is cancelled by the right side of the equation. The equation then becomes

$$F_N + \frac{2GM_{Earth}mR}{r^3} - w_M = 0$$
$$F_N = w_M - \frac{2GM_{Earth}mR}{r^3}$$

The normal force is, therefore, smaller than it would have been without the presence of the Earth (without the Earth, the normal force would have been simply equal to the weight). A person on the Moon (if there was one) would then have the impression that a force is acting upwards on the object. This force is the tidal force.

On this side of the Moon, the object makes a circular motion with a radius slightly smaller than the motion of the centre of the Moon. Then, the centripetal force required to make this motion is a little smaller. However, the gravitational force exerted by the Earth is slightly larger because the object is closer to the Earth. The force towards the centre is, therefore, greater than the centripetal force required to make the circular motion. This excessive force made by the Earth is offset by a decrease in the normal force, and this allows the object to follow the circular motion of the Moon.

For the object on the left side of the Moon (side opposite the Earth), Newton's second law is

$$-F_N + w_E + w_M = ma$$
$$-F_N + mg_{Earth} + w_M = ma$$
$$-F_N + m\left(g_{\text{Earth at the centre of the Moon}} - \Delta g\right) + w_M = ma$$
$$-F_N + m\left(\frac{GM_{Earth}}{r^2} - \frac{2GM_{Earth}R}{r^3}\right) + w_M = m\frac{GM_{Earth}}{r^2}$$

The first term in the parenthesis is again cancelled by the right side of the equation. The equation then becomes

$$-F_N - \frac{2GM_{Earth}mR}{r^3} + w_M = 0$$
$$F_N = w_M - \frac{2GM_{Earth}mR}{r^3}$$

Once again, the normal force is smaller than it would have been without the presence of the Earth. The tidal force is a force acting upwards on the object. We note that the magnitude of the tidal force is identical on both sides of the Moon.

On this side of the Moon, the object made a circular motion with a radius slightly larger than the motion of the centre of the Moon. Then, the centripetal force required to make this motion is a little larger. However, the gravitational force exerted by the Earth is slightly smaller because the object is farther away from the Earth. The force towards the centre is, therefore, smaller than the centripetal force required to make the circular motion. This lack of force made by the Earth is offset by a decrease in the normal force, and this allows the object to follow the circular motion of the Moon.

So far, you might think that this reasoning applies only to the Moon, but the situation is not that simple. The Moon does not simply revolve around an immobile Earth. In fact, the two planets revolve around the centre of mass of the system. The situation for the Earth is thus quite similar to that of the Moon, which means that tidal forces made by the Moon act on the Earth just like tidal forces made by the Earth act on the Moon.

In general, all the planets or stars near a body exert a tidal force on this body. The tidal force is

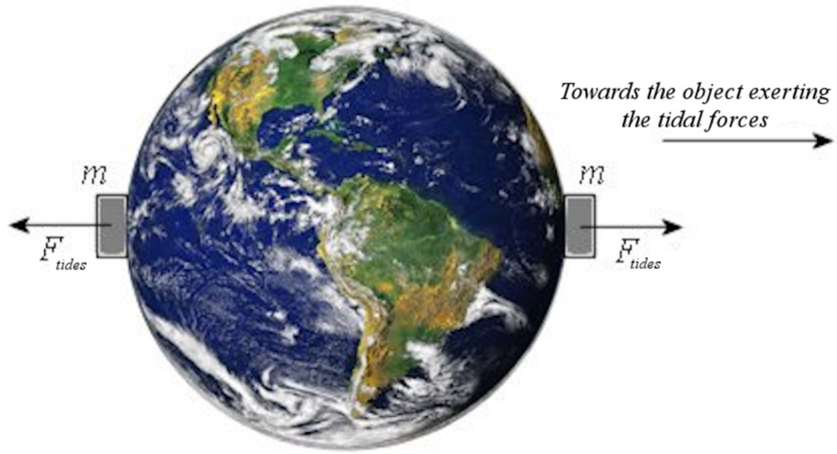
Tidal force

$$F_{tide} = \frac{2GMmR}{r^3}$$

where M is the mass of the planet (or star) that exerts the tidal forces, m is the mass of the object on which the tidal forces are exerted, R is the radius of the planet where is the object on which the tidal forces are exerted and r is the distance between the two bodies.

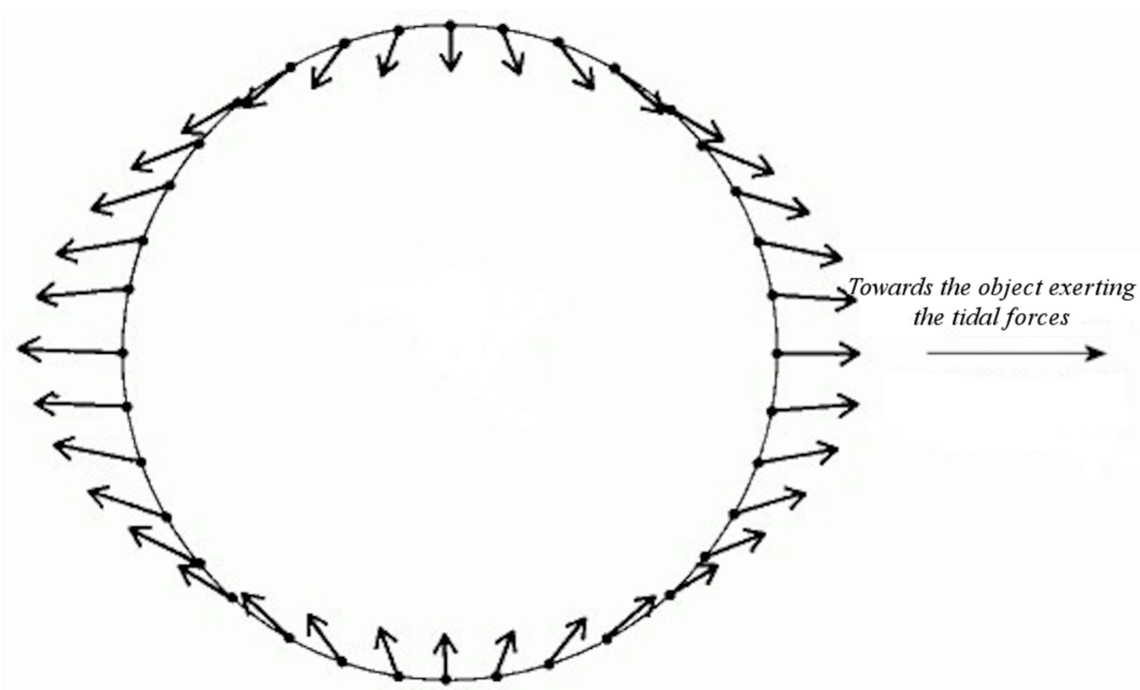
Therefore, the Sun also exerts tidal forces on the Earth, but they are about half as strong as those exerted by the Moon even though the mass of the Sun is much larger. These forces are weaker because Sun is farther away than the Moon and the tidal forces decrease quickly with distance.

According to our calculation, the directions of the tidal forces on each side of the planet is



www.oassf.com/en/earth.html

The calculation has been made for only two places on the surface of the planet, but this force exists everywhere inside the planet and on the planet's surface. When this calculation is made for any location on the surface, the following directions are obtained.



physics.stackexchange.com/questions/66400/how-you-feel-in-outer-space-vs-orbit

Example E1.9.1

What is the magnitude of the tidal force exerted by the Moon on a 100 kg object on the surface of the Earth (on the side towards the Moon or opposed to the Moon)? The mass of the Moon is 7.35 x 10^{22} kg, the Earth radius is 6378 km and the distance between the Moon and the Earth is 384 000 km.

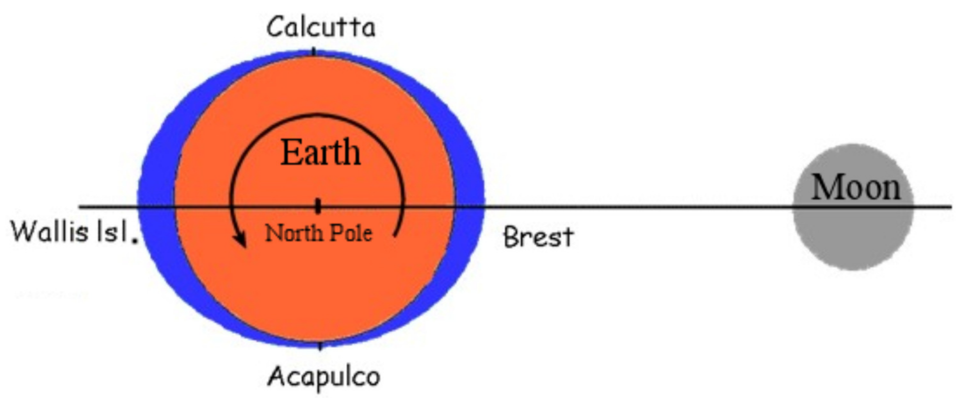
The force is

$$F_{tide} = \frac{2GMmR}{r^3}$$
$$= \frac{2 \times 6.674 \times 10^{-11} \frac{Nm^2}{kg^2} \times 7.35 \times 10^{22} kg \times 100kg \times 6.378 \times 10^6 m}{\left(3.84 \times 10^8 m\right)^3}$$
$$= 1.1 \times 10^{-4} N$$

It is not much (almost 10 million times smaller than the weight of the mass), but it is large enough so that some effects appears.

Changes in Sea Levels Caused by Tidal Forces

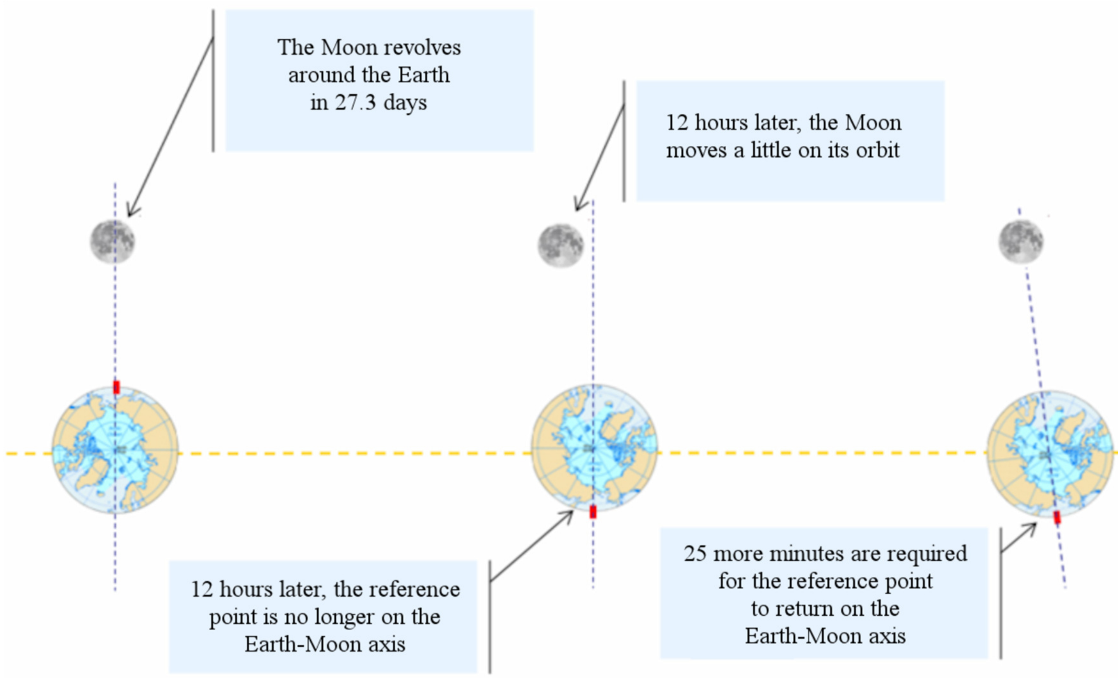
The previous image shows that the tidal forces seek to stretch the object on which the tidal forces act in a direction parallel to the direction of the body which exerts the tidal forces and to compress it in a perpendicular direction. This could result in a deformation of the planet on which the forces act. On Earth, water is displaced by these forces so that there are two areas where the water accumulates (the bulges) and there are areas where there is less water. We then have the following situation.



tpelesmarees.pagesperso-orange.fr/phenomene_maree.html

In this situation, there is plenty of water in Brest and it is high tide there. There is less water in Acapulco, and it is low tide there. As the Earth rotates, Acapulco will move to the area where there is plenty of water (facing the Moon) in 6 hours, then to the other area where there is less water 6 hours later, then to the other area where there is plenty of water (opposite to the Moon) 6 hours later to return finally to the area where there is less water another 6 hours later. In 24 hours, there were 2 high tides and 2 low tides.

In fact, the period is 24 hours and 50 min because the Moon revolves around the Earth and, therefore, changes its orientation relative to the Earth, as shown in this diagram.

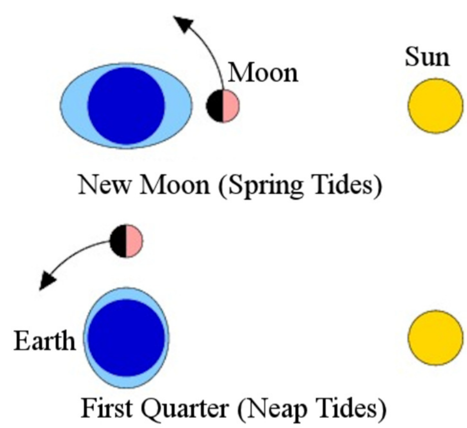


www.je-comprends-enfin.fr/index.php?/Eau-ondes-et-mouvement/eau-terre-lune-soleil-et-marees/id-menu-14.html

Actually, the Moon is not the only planet exerting tidal forces on the Earth. All other celestial bodies in the Solar System exert a force. In practice, the force made by all these bodies can be neglected, except for those exerted by the Moon and by the Sun, which exerts tidal forces half as strong as those exerted by the Moon. The extent of the accumulation of water will, therefore, be different depending on the configuration of the Moon, the Sun, and the Earth.

When the Moon, the Earth, and Sun are aligned (full moon or new moon), the tidal bulges made by the Moon and the Sun combine, resulting in a larger bulge. The tides then have a maximum amplitude. These are the spring tides (which do not necessarily occur in spring).

When the Moon, the Earth, and Sun form a 90° angle (it is then said that the Sun and the Moon are in quadrature), the tidal bulge made by the Sun forces cancel part of the bulge made by the Moon. The resulting tidal forces are weaker and the accumulation of water is less important. The amplitude of the tides is smaller. These are the neap tides.



www.ifremer.fr/lpo/cours/maree/forces.html

Moreover, the distance of the Moon is not always the same because its orbit is elliptical. The distance varies only by 7% compared to the average distance but this causes a variation in tidal forces of about 20% since the effects of tides vary with the cube of the distance.

Tides made by Moon should then be expected to be 20% larger when the Moon is closest to the Earth. If this occurs simultaneously with a full moon or a new moon, the amplitude of the tides can be exceptional.

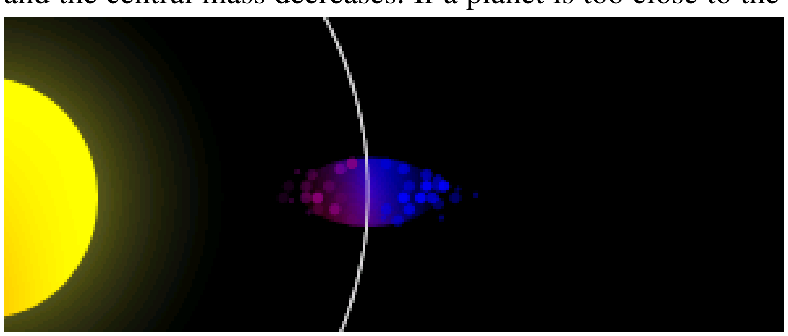
A calculation, taking into account the Moon and the Sun, shows that the variation in sea levels should be about 50 cm. Actually, the amplitude of the tides should be even lower because tidal forces not only rise the oceans, they also raise Earth's crust by about 20 cm. Thus, if the sea level rises 50 cm and the continent rises 20 cm, the difference is only 30 cm, which is basically the amplitude of the tides in the oceans. Sometimes the shape of the shoreline can amplify the tidal effect through various mechanisms, and the resulting amplitude may be much larger. The highest tides in the world are found in the Bay of Fundy, where there can be a variation of 17 metres in sea level between low tide and high tide.

bayoffundy.blogspot.ca/2010/09/biggest-tides-of-year-today.html

The water level changes in the Bay of Fundy can be seen in these clips.
http://www.youtube.com/watch?v=5W2sM1Ma7YA
http://www.youtube.com/watch?v=u3LtEF9WPt4

Roche Limit

It is worth noting that the tidal forces increase very rapidly if the distance between a planet and the central mass decreases. If a planet is too close to the central mass, the tidal forces on a rock on the surface of the planet may exceed the weight of this rock. Then, the rocks on the planet would be lifted from the surface and the planet would be slowly destroyed.



fr.wikipedia.org/wiki/Limite_de_Roche

Let's see how far from the central mass this will happen. The mass of the planet will be denoted M_p. At the limit, called the *Roche limit*, the tidal force, which seeks to lift the object, is equal to the gravitational force that attracts the object towards the ground. This means that

$$F_{tide} = F_g$$
$$\frac{2GM_c mR_p}{r^3} = mg$$
$$\frac{2GM_c mR_p}{r^3} = m\frac{GM_p}{R_s^2}$$
$$\frac{2M_c R_p}{r^3} = \frac{M_p}{R_p^2}$$
$$r^3 = \frac{2M_c R_p^3}{M_p}$$

If the densities of the central mass and the planet are constant, the masses are (assuming they are spherical)

$$M_c = \frac{4}{3}\pi R_c^3 \rho_c \qquad\qquad M_p = \frac{4}{3}\pi R_p^3 \rho_p$$

The equation then becomes

$$r^3 = \frac{2M_c R_s^3}{M_p}$$
$$r^3 = \frac{2\left(\frac{4}{3}\pi R_c^3 \rho_c\right) R_s^3}{\left(\frac{4}{3}\pi R_p^3 \rho_p\right)}$$
$$r^3 = \frac{2\rho_c R_c^3}{\rho_c}$$
$$r = \sqrt[3]{\frac{2\rho_c}{\rho_p}} R_c$$
$$r = 1.26\sqrt[3]{\frac{\rho_c}{\rho_p}} R_c$$

In fact, Édouard Roche made a better analysis taking into account the fact that the planets or stars would deform under the effect of tidal forces, losing their spherical shape. He then obtained a better solution, which is

Roche Limit

$$r = 2.42285\sqrt[3]{\frac{\rho_c}{\rho_p}}R_c$$

ρ_c is the density of the star or planet exerting the tidal forces
ρ_p is the density of the star of the planet subjected to the tidal forces
R_c is the radius of the star or planet exerting the tidal forces

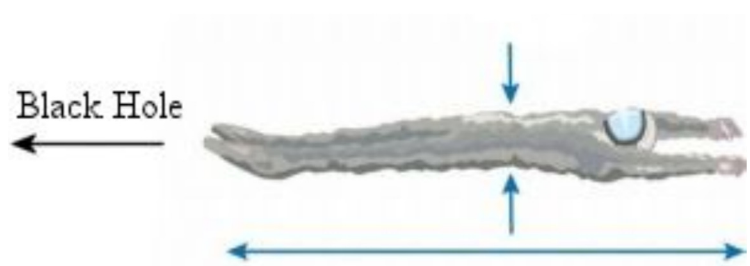
If the densities of the central mass and the planet are identical, the Roche limit is simply 2.42285 times the radius of the central mass. For the Earth (R_c = 6378 km), this limit is 15 450 km from the centre of the Earth.

Thus, if the distance between the Moon and the Earth were smaller than 15 450 km, the Moon would be slowly destroyed by tidal forces since the forces seeking to stretch the Moon would be greater than the gravitational force that seeks to keep the Moon together. As the Moon is 384 400 km away, it is far from the Roche limit.

Tidal forces also prevent rocks from clumping together with the force of gravity to form a bigger satellite if these rocks is inside the Roche limit.

Notice that you are currently inside the Roche limit of the Earth. Yet, you are not torn apart by tidal forces because the gravitational force is not the force that keeps the cells of your body together. Instead, they are held together by electrical forces that are much greater than the tidal forces acting on your body.

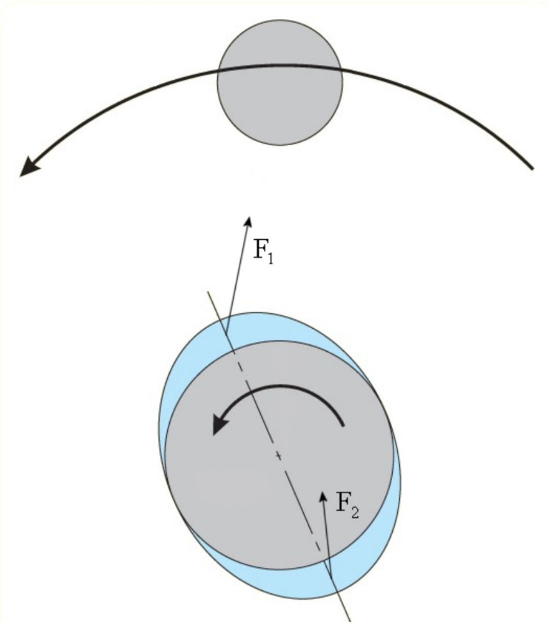
Tidal forces can become even stronger in areas where the gravitational field varies very quickly. Near a black hole, the tidal forces are so large that they would destroy any astronaut too close to the black hole. As forces seek to stretch the object in on direction and compress the object in the other direction, an astronaut would be stretched as a result of these forces, a process called "spaghettification".



community.emc.com/people/ble/blog/2011/11/06/holographic-principle-to-multiverse-reality

Long-Term Effects of Tidal Forces

Lengthening of the Day



www.aerospaceweb.org/question/astronomy/q0262.shtml

The rotation of the Earth drags the two tidal bulges in the direction of the rotation. Hence, the two tidal bulges are not directly in line with the Moon but are instead offset by about 3° from the direction of the Moon. In the diagram, the gravitational forces made by the Moon on these bulges are shown.

Each of these forces exerts a torque on the Earth, but the torque on the bulge facing the Moon is larger because the force is greater since this bulge is closer to the Moon. Thus, the two torques do not vanish a net torque is exerted on the Earth. This torque is opposed to the rotation of the Earth and, therefore, slows down Earth's rotation. The period of rotation changes very slowly so that the day currently lengthens by about 2 ms per century. However, this can represent a considerable variation on a geological scale, especially since the effect was somewhat more significant in the past. As the effect is cumulative, it can be calculated that the 20^{th} century lasted approximately 0.1 seconds longer than the 19^{th} century. When the Earth was formed 4 billion years ago, the day had a duration of about 15 hours. 380 million years ago, the day lasted 22 hours and it now lasts 24 hours. The day will continue to lengthen and would reach a length of 47 days in 50 billion years (but this will not happen because the Earth and the Moon will be destroyed by the Sun in 5 billion years).

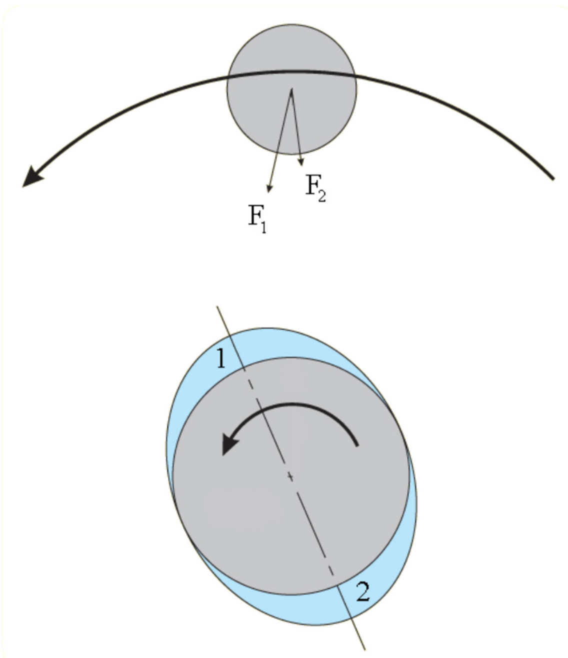
The same thing happened to the Moon, but with greater intensity. Even if there's no water on the Moon, there are still tidal bulges resulting from ground uplifts. The slowdown of the rotation of the Moon was much faster, however, since its moment of inertia is significantly smaller. The slowdown was so fast that the Moon finally arrived at what the tidal forces are attempting to do: to stop the rotation relative to the other body. That is why the Moon always has the same side facing the Earth.

This situation is quite common. The four largest moons of Jupiter also have the same side facing Jupiter. This is the equilibrium position to achieve and, given enough time, all two-body systems will eventually have the same side facing each other. Pluto and its satellite Charon is the only known example where both bodies have reached this equilibrium position. In theory, the next system in the Solar System that should reach this equilibrium is the Earth and the Moon (but they will not because the Sun will die before this happens).

With an elliptical orbit, slightly different situations may occur. Mercury, the planet on which the Sun exerts the largest tidal forces, does not have the same side always facing the Sun. Instead, the period of rotation of Mercury is exactly two thirds of the period of revolution around the Sun so that two opposite locations on Mercury alternate to face the Sun at perihelion.

The Moon Moves Farther From the Earth

If the Moon exerts a force on the tidal bulges, then the tidal bulges also exert a force on the Moon. Much of this force is directed towards the Earth and helps to make the centripetal force, but a small tangential component is heading in the same direction as the motion of the Moon around the Earth.



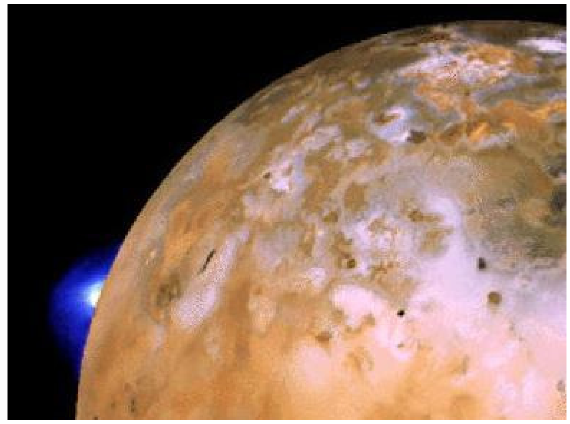
www.aerospaceweb.org/question/astronomy/q0262.shtml

This force accelerates the Moon and allows it to gain energy. The Moon then moves slowly away from the Earth at a rate of 3 to 4 cm per year. The radius of the orbit of the Moon increases continuously, thereby increasing the duration of the month. This process will continue until the period of revolution of the Moon around the Earth reaches 47 days in 50 billion years. The distance between the Moon and the Earth will then be of a little over 550 000 km. (It is 384 400 km now.)

If the period of revolution of a moon around a planet is smaller than the period of rotation of the planet, the two effects made by the tidal forces are in the opposite direction: the planet rotates faster and faster and the satellite gets closer and closer to the planet until it crashes on it. This is what is happening right now with the largest moon of Mars, Phobos.

Tidal forces generate a great quantity of heat. In the case of the Earth, the heat is generated by the constant raising and lowering of the oceans and the continents. This generates about 2% of the internal heat of the Earth (estimated at 2×10^{19} J per year compared to 10^{21} J per year for radioactive disintegrations). This contribution was more important in the past when the Moon was closer to the Earth. The heat generated by tidal forces is more important in moons although it decreases a lot once the satellite reaches its equilibrium position by always having the same side facing the planet. Therefore, the tidal forces do not generate much heat in the Moon now.

For the moons of Jupiter, it could be expected that the tidal forces do not generate much heat since they all have reached the equilibrium position. However, these four large moons perturb each other motion so that the tidal bulges oscillates around the equilibrium position. This oscillation generates much heat. So much heat is generated in Io, the moon closest to Jupiter (on which the largest tidal forces are exerted), that there are volcanoes on its surface.



planetarygeomorphology.wordpress.com

SUMMARY OF EQUATIONS

Distance as a Function of θ of an Object Near a Central Mass

$$r = r_p \frac{1+e}{1+e\cos\theta}$$

Eccentricity

$$e = \frac{v_p^2 r_p}{GM_c} - 1$$

Mechanical Energy

$$E_{mec} = -\frac{GM_c m(1-e)}{2r_p}$$

Angular Momentum Conservation

$$r v \sin\psi = r_p v_p$$

$$r v \sin\psi = \sqrt{GM_c r_p (1+e)}$$

Kepler's Second Law

$$\Delta A = \frac{\sqrt{GM_c r_p (1+e)}}{2} \Delta t$$

Speed of an Object on a Circular Orbit

$$v = \sqrt{\frac{GM_c}{r}}$$

Mechanical Energy of an Object on a Circular Orbit

$$E_{mec} = -\frac{GM_c m}{2r}$$

Period of an Object on a Circular Orbit (Kepler's 3rd Law)

$$T = 2\pi \sqrt{\frac{r^3}{GM_c}}$$

Kepler's First Law

The orbits are ellipses. The central mass occupies one of the foci.

r_a and r_p as Functions of a and e

$$r_a = a(1+e)$$
$$r_p = a(1-e)$$

a and e as Functions of r_a and r_p

$$a = \frac{r_p + r_a}{2} \qquad e = \frac{r_a - r_p}{r_a + r_p}$$

Relationship Between r and θ for an Elliptical Orbit

$$r = r_p \frac{1+e}{1+e\cos\theta}$$

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

Mechanical Energy of an Object on an Elliptical Orbit

$$E_{mec} = -\frac{GM_c m}{2a}$$

Speed of an Object on an Elliptical Orbit

$$v^2 = GM_c\left(\frac{2}{r} - \frac{1}{a}\right)$$

Speed of an Object at the Periapsis or at the Apoapsis

$$v_p^2 = \frac{GM_c}{a}\frac{1+e}{1-e}$$

$$v_a^2 = \frac{GM_c}{a}\frac{1-e}{1+e}$$

Period of an Object on an Elliptical Orbit (Kepler's 3rd Law)

$$T = 2\pi\sqrt{\frac{a^3}{GM_c}}$$

Parabolic Trajectories

$$e = 1$$
$$E_{mec} = 0$$

Hyperbolic Trajectories

$$e > 1$$
$$E_{mec} > 0$$

Force on an Object of Mass *m* in a Gravitational Field

$$\vec{F} = m\vec{g}$$

Magnitude of the Gravitational Field of a Point Mass of Mass *M*

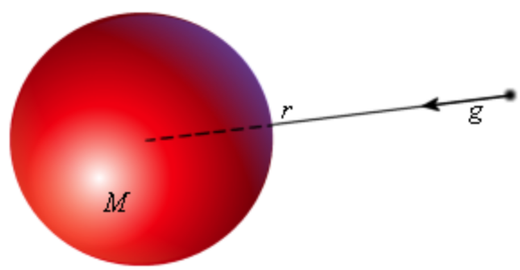
$$g = \frac{GM}{r^2}$$

Gravitational Field of a Uniform Rod

$$g = \frac{GM}{b\left(b+L\right)}$$

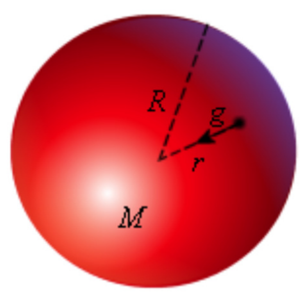
Gravitational Field outside a Sphere

$$g = \frac{GM}{r^2}$$



Gravitational Field inside a Sphere of Constant Density

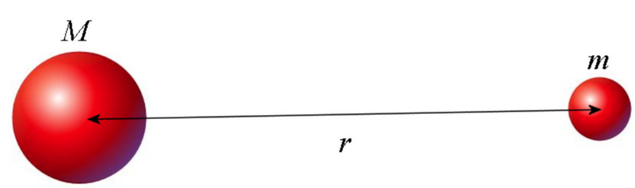
$$g = \frac{GMr}{R^3}$$



Gravitational Force between Two Spheres

$$F = \frac{GMm}{r^2}$$

where r is the distance between the centres of the spheres.



Tidal Force

$$F_{tide} = \frac{2GMmR}{r^3}$$

Roche Limit

$$r = 2.42285\sqrt[3]{\frac{\rho_c}{\rho_p}}R_c$$

ρ_c is the density of the star or planet exerting the tidal forces
ρ_p is the density of the star of the planet subjected to the tidal forces
R_c is the radius of the star or planet exerting the tidal forces

EXERCISES

Use the following data for these exercises.

Earth
Mass = 5.98 x 10^{24} kg
Radius = 6380 km
Semi-major axis of the orbit (a) = 1 UA = 149 600 000 km
Eccentricity of the orbit = 0.01671

Moon
Mass = 7.35 x 10^{22} kg
Radius = 1738 km
Distance between the Earth and the Moon = 384 400 km

Sun
Mass = 1.9885 x 10^{30} kg

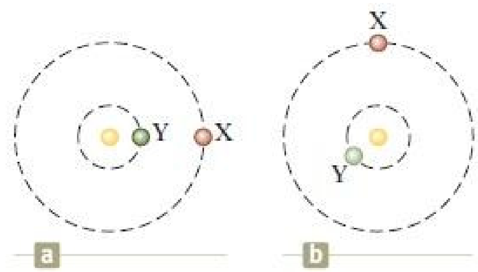
E1.1 Trajectories Near a Massive Object

1. At its closest point to the Sun, an object has a speed of 70 km/s. At this instant, the distance between the object and the Sun is 50 million km.

   a) What is the eccentricity of its orbit?
   b) Is the orbit circular, elliptic, parabolic or hyperbolic?
   c) How far from the Sun will the object be when the angle is 90 degrees from its position of closest proximity to the central mass?

E1.2 Circular Orbits

Two planets are in a circular orbit around a star. The radius of the orbit of planet X is 3 times larger than the radius of planet Y. In 5 years, planet X has travelled a quarter of its orbit around the star (change between the diagrams a and b). From what angle has the planet Y moved during that time?



www.chegg.com/homework-help/questions-and-answers/two-planets-x-y-travel-counterclockwise-circular-orbits-star-shown-figure--radii-orbits-ra-q900222

E1.3 Elliptical Orbits

2. What is the distance between the Earth and the Sun at perihelion?

3. What is the distance between the Earth and the Sun at aphelion?

4. What is the speed of the Earth at perihelion?

5. What is the speed of the Earth at aphelion?

6. What is the period of revolution of the Earth around the Sun?

7. What is the mechanical energy of the Earth on its orbit?

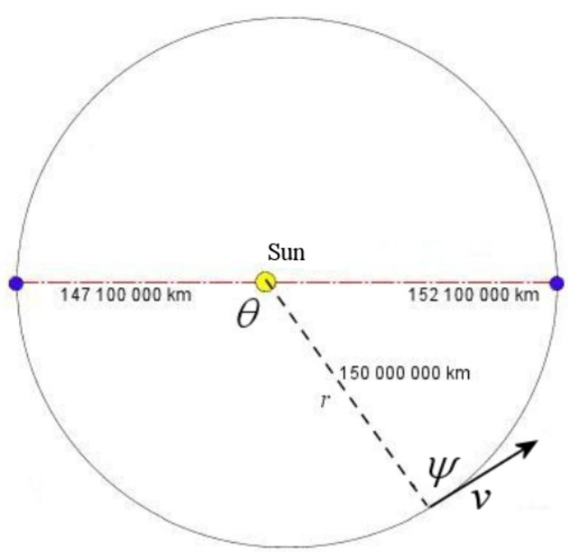
8. What is the angular momentum of the Earth on its orbit?

9. At some point on its orbit, the Earth is at a distance of 150 000 000 km from the Sun.

   a) What is the speed of the Earth?

   b) What is the angle θ in the diagram?

   c) What is the angle between the speed of the Earth and the line going from the Earth to the Sun (ψ in the diagram)?



la.climatologie.free.fr/atmosphere/atmosphere1.htm

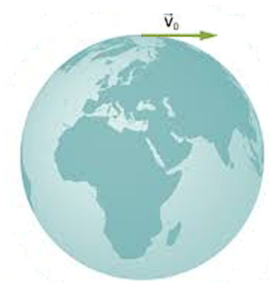
10. The newly created Quebec space agency's goal is to increase the international influence of Quebec by organizing the first mission with astronauts to Jupiter. Since the budget is somewhat limited, they decide to hire college students to calculate the best path to get to Jupiter. They're looking for the path that will take the least possible time.

    In fact, they hesitate between two possibilities:

    1) Give the ship an elliptical trajectory that will move the ship from the Earth to Jupiter. This orbit has its perihelion at a distance equal to the radius of the orbit of the Earth and its aphelion at a distance corresponding to the radius of the orbit of Jupiter.
    2) Give the ship an elliptical path that will take it first to Venus and then change its speed to give it a new elliptical orbit that will take it up to Jupiter. In the case of the first orbit to Venus, the aphelion is equal to the radius of the orbit of the Earth and the perihelion is equal to the radius of the orbit of Venus. For the second orbit, the perihelion is still equal to the radius of the orbit of Venus but the aphelion is equal to the radius of the orbit of Jupiter.

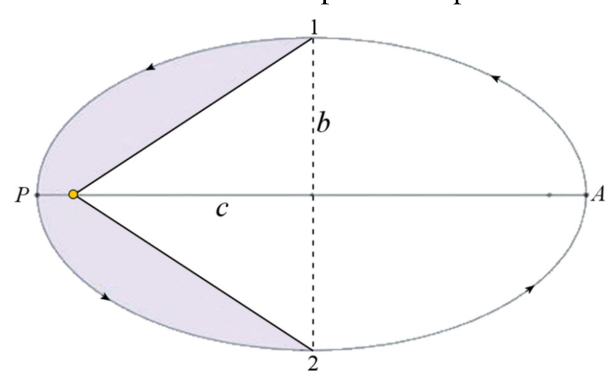
    Of course, the second path is longer but it is travelled at a higher speed. So, it is unclear which of the two trajectories takes the least time. Find the time required to reach Jupiter by following these two paths knowing that the radius of the orbit of Jupiter is 7.8×10^{11} m and the radius of the orbit of Venus is 1×10^{11} m.

An object is launched tangentially from the surface of the Earth with a speed equal to 85% of its escape velocity. What will be the maximum distance between the Earth and the object?



cnx.org/contents/59891349-7823-4303-8e80-672146b479cb%404/projectile-motion

11. A comet orbiting the Sun has a period of 50 years. The eccentricity of the orbit is 0.87. How long does take for the comet to travel from the point 1 to point 2 in its orbit? (Hint: this time is calculated with the grey area. In addition, the dimensions of the ellipse on the following diagram are $c = ae$ and $b = a\sqrt{1-e^2}$, and the area of an ellipse is $A = \pi a^2\sqrt{1-e^2}$.)



cseligman.com/text/history/kepler2.htm

12. A small moon revolves around a planet. The maximum speed of the moon is of 2.2 km/s, and the minimum speed is 1.7 km/s. The period of the moon around the planet period is 25 days.

a) What is the semi-major axis (a) of the orbit of the moon?
b) What is the mass of the planet?
c) What is the eccentricity of the orbit of the moon?

E1.4 Parabolic Trajectories

13. An asteroid following a parabolic trajectory is 50 million km from the Sun when it is at its point of closest proximity from the Sun.

a) What is the speed of the asteroid at its closest point to the Sun?
b) What will be the speed of the asteroid when it will be 200 million km from the Sun?

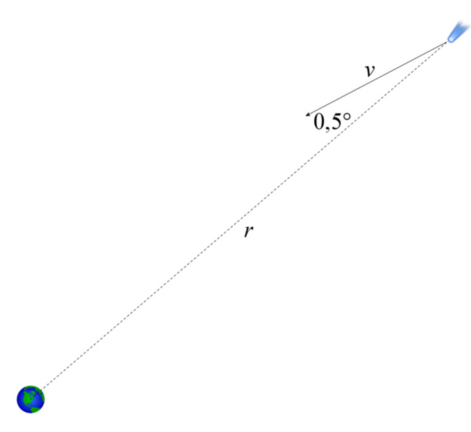
E1.5 Hyperbolic Trajectories

14. An asteroid following a hyperbolic trajectory is 50 million km from the Sun when it is at its point of closest proximity from the Sun. The eccentricity is 1.2.

a) What is the speed of the asteroid at its closest point to the Sun?
b) What will be the speed of the asteroid when it will be 200 million km from the Sun?

E1.6 Summary of Possible Trajectories

15. A comet is 100 000 000 km from the Earth. Then, its speed is 100 m/s and the angle between its speed and its distance is barely 0.5 °. (For this problem, we'll assume that there is nothing but the Earth and the comet in the universe.)



a) Is this comet on an elliptical, parabolic or hyperbolic orbit?
b) Will this comet strike the Earth? (To find out, calculate the value of r_p. If r_p is smaller than the radius of the Earth, the comet hits the Earth.)
c) What is the eccentricity of the orbit of the comet?

E1.8 Gravitational Field

16. What is the gravitational field 100 km above Earth's surface?

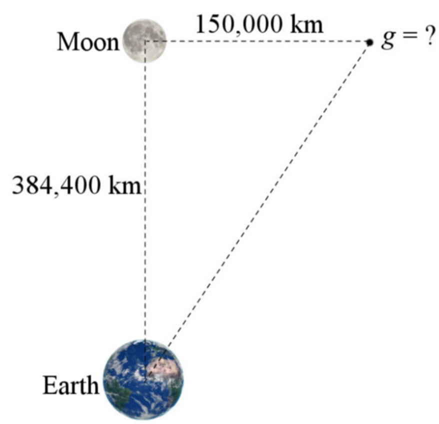
17. What is the gravitational field 1000 km beneath the Earth's surface if the density of the Earth is considered to be constant?

18. A 70 kg person is on the surface of the Moon.

    a) What is the gravitational field at the surface of the Moon?
    b) What would be the weight of the person on the surface of the Moon?
    c) This weight represents what percentage of the weight of the person on Earth?

19. How far from the Earth does the gravitational field vanishes between the Earth and the Moon?

20. What is the magnitude of the gravitational field at the position shown in the diagram?



21. a) What is the force exerted by the Earth on the Moon?
    b) What is the force exerted by the Moon on the Earth?

E1.9 Tides

22. Calculate the exact ratio between the magnitude of tidal forces exerted by the Moon on Earth and the magnitude of tidal forces exerted by the Sun on Earth?

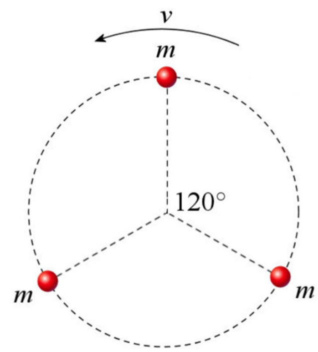
23. What should be the distance between the Earth and the Moon for the tidal force exerted by the Moon on your body to be equal to 1% of your weight?

24. How far from the Sun must Mercury approach to be destroyed by the Sun's tidal forces? (Density of the Sun = 1408 kg/m³, density of Mercury = 5427 kg/m³, radius of the Sun = 695 000 km.) Knowing that the smallest distance between Mercury and the Sun is 46 000 000 km, can we say that Mercury is in danger?

Challenges

(Questions more difficult than the exam questions.)

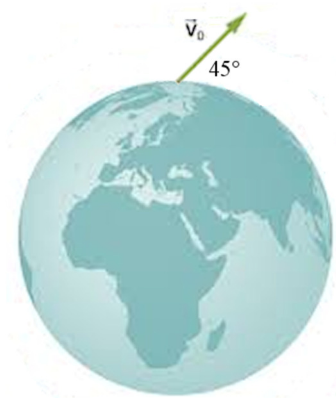
25. Three planets that are attracting each other with the gravitational force are following a circular path as shown. What is the period of revolution of these planets? The mass of each planet is 10^{24} kg and the radius of the circular path is 10 000 000 km.



26. An object is launched with a 45° angle from the surface of the Earth with a speed equal to 85% of its escape velocity.

    a) What will be the maximum distance between the Earth and the object?

    b) At what distance from the starting point will the object fall back on Earth (in other words, the range is sought)?



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ANSWERS

E1.1 Trajectories Near a Massive Object

1. a) 0.8355 b) elliptic c) 91.77 million km

E1.2 Circular Orbits

2. 467.7°

E1.3 Elliptical Orbits

3. 147 000 000 km
4. 152 100 000 km
5. 30.286 km/s
6. 29.291 km/s
7. 365.26 days
8. -2.652×10^{33} J

9. 2.664×10^{40} kg m²/s
10. a) 29.705 km/s b) 100.2° c) 89.0° or 91.0°
11. path 1:997.9 days path 2:1057.7 days
12. 10,231 km
13. 11.15 years
14. a) 1 196 691 km b) 6.706×10^{25} kg c) 0.1282

E1.4 Parabolic Trajectories

15. a) 73.03 km/s b) 36.53 km/s

E1.5 Hyperbolic Trajectories

16. a) 76.64 km/s b) 36.63 km/s

E1.6 Summary of Possible Trajectories

17. a) hyperbolic b) No (It passes 9540 km from the centre of the Earth)
    c) 1.0000482

E1.8 Gravitational Field

18. 9.50 N/kg
19. 8.27 N/kg
20. a) 1.624 N/kg b) 113.7 N c) 16.6%
21. 346 040 km from the centre of the Earth
22. $2{,}432 \times 10^{-3}$ N/kg
23. a) 1.985×10^{20} N b) 1.985×10^{20} N

E1.9 Tides

24. Tidal forces exerted by the Moon are 2.18 times greater than those exerted by the Sun.
25. 8612 km (distance between the centres of the planets, there would, therefore, be only 494 km between the surfaces of the planets!)
26. 1 074 743 km

Challenges

27. 32.07 years
28. a) 14 013 km b) 15 364 km