9 MECHANICAL ENERGY

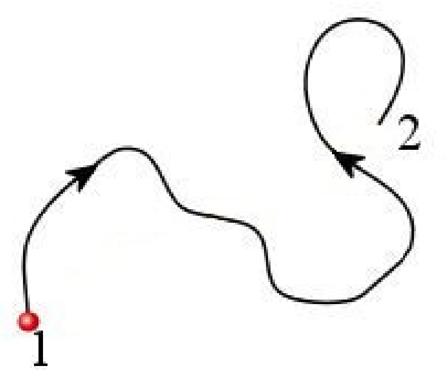
A roller coaster car initially at rest at the top of a slope descends the slope and then moves into a loop as illustrated in this image. The radius of the loop is 25 m. If the initial height of the car is 100 m, what is the speed of the car at the highest point of the loop?

moon.com/2011/05/summertime-thrills-part-2/

Discover the answer to this question in this chapter.

9.1 GRAVITATIONAL ENERGY

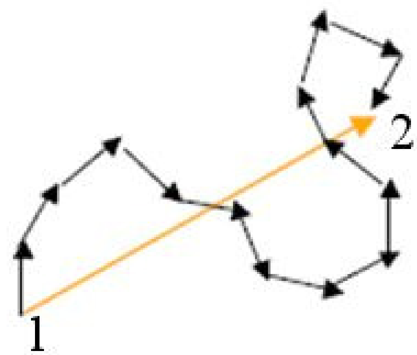
Work Done by the Gravitational Force



Suppose that an object moves from position 1 to position 2 as shown in the diagram.

The work done by the force of gravity on the object will be calculated. As the angle between the motion and the force change constantly, the work must be calculated with an integral.

$$W_g = \int_1^2 \vec{F} \cdot \overrightarrow{ds}$$



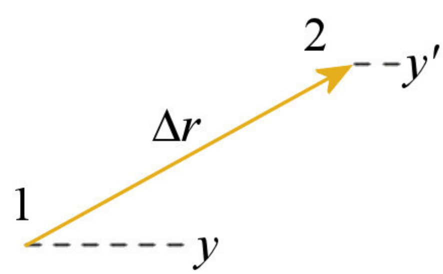
Since the force is constant, the force can be separated from the integrand to get

$$W_g = \vec{F} \cdot \int_1^2 \overrightarrow{ds}$$

However, this integral is the sum of all the infinitesimal displacement vectors along the path. This sum can be seen in the diagram (although there would be many more vectors in reality).

As you have learned, the sum of these vectors is simply a vector going directly from point 1 to point 2. The work thus becomes

$$W_g = \vec{F} \cdot \overrightarrow{\Delta r}$$



where the vector Δr is a vector going straight from point 1 to point 2. The work is therefore

$$\begin{aligned} W_g &= \vec{F}_g \cdot \overrightarrow{\Delta r} \\ &= F_x \Delta x + F_y \Delta y + F_z \Delta z \end{aligned}$$

With axes oriented conventionally (y directed upwards), the x and z components of the force are zero. The work then becomes

$$\begin{aligned} W_g &= F_y \Delta y \\ &= -mg\Delta y \\ &= -mg\left(y' - y\right) \\ &= -\left(mgy' - mgy\right) \end{aligned}$$

The work done by the force of gravity between points 1 and 2 is, therefore, linked to the variation of the quantity *mgy*. This quantity was given the name *gravitational energy* and is noted U_g.

Gravitational Energy

$$U_g = mgy$$

This quantity can also be called the *gravitational potential energy*. In these notes, the term potential is often omitted. (We'll see why later.)

The work done by the gravitational force is thus

$$\begin{aligned} W_g &= -\left(mgy' - mgy\right) \\ &= -\left(U_g' - U_g\right) \end{aligned}$$

Finally, the work can be written as

Work Done by the Gravitational Force

$$W_g = -\Delta U_g$$

A very fast way to calculate the work done by the force of gravitation is thus obtained.

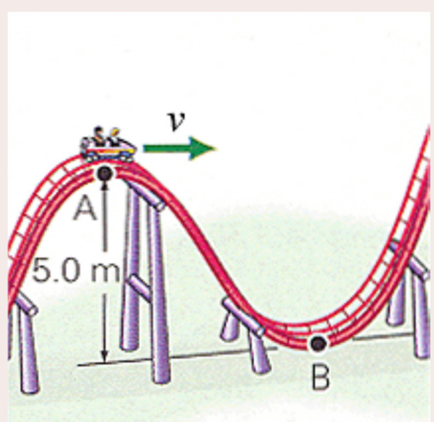
Note that the gravitational energy could also have been

$$U_g = mgy + \text{constant}$$

This would give exactly the same results because the only thing that matters when work is calculated is the variation of the energy. Then, the constants cancel each other, and the same result is obtained. However, the addition of this constant has the same effect as making the position of the origin $y = 0$ arbitrary. For each problem, the position of the origin $y = 0$ must, therefore, be chosen.

Example 9.1.1

A roller coaster car, whose mass is 1000 kg including the passengers, moves down a slope such as illustrated in the diagram. What is the work done by the force of gravity on the car when it goes from point A to point B?



morkphysics.wordpress.com/question-3

The work will be calculated with

$$W_g = -\Delta U_g$$

To calculate the U_g's, the position of the origin $y = 0$ must be chosen first. Here, this $y = 0$ is set at the ground (point B).

Thus, the gravitational energy at point A is

$$\begin{aligned} U_g &= mgy \\ &= 1000kg \cdot 9.8\tfrac{N}{kg} \cdot 5m \\ &= 49\,000J \end{aligned}$$

The gravitational energy at point B is

$$\begin{aligned} U_g' &= mgy' \\ &= 1000kg \cdot 9.8\tfrac{N}{kg} \cdot 0m \\ &= 0J \end{aligned}$$

The work done by the gravitational force is thus

$$\begin{aligned} W_g &= -\Delta U_g \\ &= -\left(U_g' - U_g\right) \\ &= -\left(0J - 49\,000J\right) \\ &= 49\,000J \end{aligned}$$

If the $y = 0$ had been set at point A, the gravitational energies would have been U_g = 0 J and U_g' = -49 000 J and the gravitational energy variation would have been the same.

This relatively easy calculation would have been quite complicated with $W = F\Delta s \cos\theta$ because the angle between the force and the displacement changes continually during the descent as it is not done in a straight line. The roller coaster car rather follows a curve of variable inclination, and this implies that the angle changes constantly. The calculation of the work with $W = F\Delta s \cos\theta$ would, therefore, have been done with an integral while a simple subtraction is sufficient when the gravitational energy is used.

9.2 CONSERVATIVE FORCES

Can an Energy U Be Found for Other Forces?

If it is possible to simplify the calculation of the work done by the gravitational force by defining a gravitational energy, can it be done for other forces? This would allow us to find the work done with a potential energy with the following formula.

Work Done from Potential Energy

$$W = -\Delta U$$

It will be explained later why U bears the name of "potential energy".

Unfortunately, the potential energy exists only for a few forces. For many forces, it is not possible to calculate work using potential energy. This is because there are two types of force.

1) Conservative forces: forces for which the work done can be calculated with a potential energy.

2) Non-conservative forces: forces for which the work done cannot be calculated with a potential energy.

In 1 Dimension

For the potential energy associated with a force to exist in one dimension, there is only on condition.

Condition for the Potential Energy of 1-Dimensional Force to Exist

For the potential energy of a 1-dimensional force to exist, the force must depend only on the position.

In this case, the formula for the potential energy can be found by assuming that the displacement of the object is very small (dx). Then

$$W = Fdx$$

But since the work is equal to -dU for a small displacement, the equation becomes

$$-dU = Fdx$$

This leads directly to

Relationship between Potential Energy and Force in One Dimension

$$F = -\frac{dU}{dx} \qquad \text{and} \qquad U = -\int Fdx$$

Example 9.2.1

The force exerted on an object is given by the following equation.

$$F_x = 8\tfrac{N}{m^3}x^3$$

a) What is the formula of the potential energy associated with this force?

The formula is

$$\begin{aligned} U &= -\int F dx \\ &= -\int 8\tfrac{N}{m^3}x^3 dx \\ &= -2\tfrac{N}{m^3}x^4 + cst \end{aligned}$$

The value of the constant is arbitrary. Any value can be used.

b) What is the difference of potential energy of this object if it moves from x = -1 m to x = 3 m?

The change in potential energy is

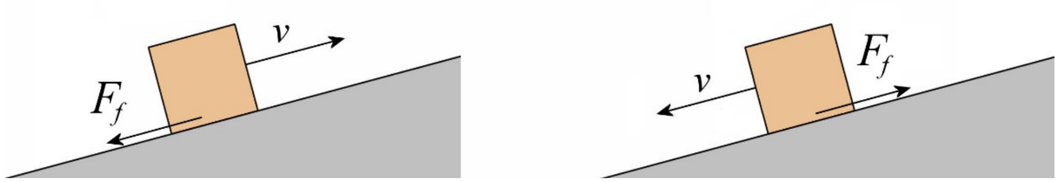
$$\begin{aligned} \Delta U &= \left(-2\tfrac{N}{m^3}\cdot(3m)^4 + cst\right) - \left(-2\tfrac{N}{m^3}\cdot(-1m)^4 + cst\right) \\ &= -162J - -2J \\ &= -160J \end{aligned}$$

Therefore, the potential energy has decreased by 160 J. (It can also be seen that the constant disappeared in the end result. This is why its value does not matter).

c) What is the work done on the object by this force?

Since $W = -\Delta U$, the work is 160 J.

If the force does not depend exclusively on the position, the force is not conservative, and there is no potential energy associated with this force. For example, imagine that a block moves up a slope and that there is some friction. The block goes up the slope, and then comes back down.



When the block returns to the same position on the slope on its way down, the force is not the same since the direction is different. This means that the force does not depend exclusively on the position as it is different when the object is at the same position. This means that frictional forces are not conservative, and there is no potential energy associated with them.

In 2 Dimensions

Link between U and F in Two Dimensions

For the potential energy to exist in two dimensions, the force must still depend exclusively on the position. But there is another condition in two dimensions.

If an object has an infinitesimal displacement, the work done on the object is

$$W = \vec{F} \cdot \overrightarrow{ds}$$

If the potential energy exists, the following equation must hold

$$W = -dU$$

This means that

$$dU = -\vec{F} \cdot \overrightarrow{ds}$$

Since, in two dimensions,

$$\begin{aligned}\vec{F} \cdot \overrightarrow{ds} &= \left(F_x \vec{i} + F_y \vec{j}\right) \cdot \left(dx\vec{i} + dy\vec{j}\right) \\ &= F_x dx + F_y dy\end{aligned}$$

The following equation must be true if the potential energy exists.

$$dU = -F_x dx - F_y dy$$

But dU is also

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

By comparing these last two equations, it can be inferred that the following equations must be true if the potential energy exists.

Components of *F* From *U* in Two Dimensions

$$F_x = -\frac{\partial U}{\partial x} \qquad\qquad F_y = -\frac{\partial U}{\partial y}$$

Example 9.2.2

Find the force (magnitude and direction) exerted on an object at the position x = 1 m and y = 2 m if the potential energy of the object is given by

$$U = -1\tfrac{N}{m^2}x^3 + 1\tfrac{N}{m^2}y^2 - 1\tfrac{N}{m}xy + 2J$$

The x-component of the force is

$$\begin{aligned} F_x &= -\frac{\partial U}{\partial x} \\ &= 3\tfrac{N}{m^2}x^2 + 1\tfrac{N}{m}y \end{aligned}$$

At x = 1 m and y = 2 m, this force is

$$\begin{aligned} F_x &= 3\tfrac{N}{m^2}\cdot(1m)^2 + 1\tfrac{N}{m}\cdot 2m \\ &= 5N \end{aligned}$$

The y-component of the force is

$$\begin{aligned} F_y &= -\frac{\partial U}{\partial y} \\ &= -2\tfrac{N}{m^2}y + 1\tfrac{N}{m}x \end{aligned}$$

At x = 1 m and y = 2 m, this force is

$$\begin{aligned} F_y &= -2\tfrac{N}{m^2}\cdot 2m + 1\tfrac{N}{m}\cdot 1m \\ &= -3N \end{aligned}$$

Therefore, the magnitude of the force is

$$\begin{aligned} F &= \sqrt{F_x^2 + F_y^2} \\ &= \sqrt{(5N)^2 + (3N)^2} \\ &= 5.831N \end{aligned}$$

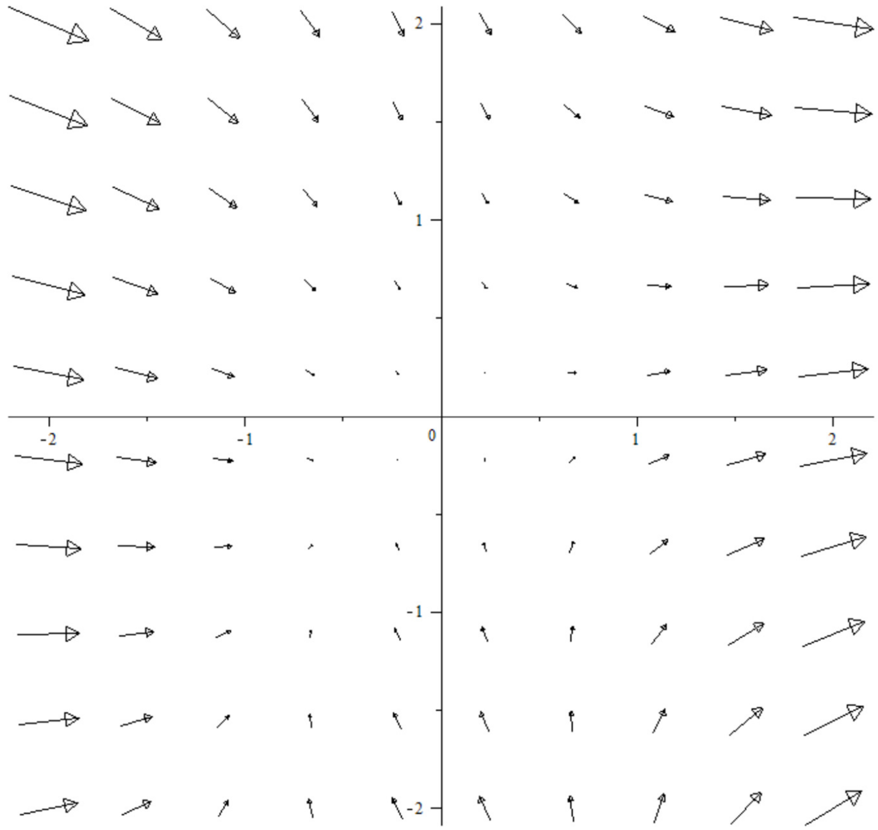
and the direction of the force is

$$\begin{aligned} \theta &= \arctan\tfrac{F_y}{F_x} \\ &= \arctan\tfrac{-3N}{5N} \\ &= -30.96° \end{aligned}$$

In fact, we can have fun and calculate the force at several places in the xy plane. Here's a graph showing the force associated with the potential energy

$$U = -1\tfrac{N}{m^2}x^3 + 1\tfrac{N}{m^2}y^2 - 1\tfrac{N}{m}xy + 2J$$

at different places.



It can easily be seen that the force is in the direction calculated at the position (1 m, 2 m).

The formulas

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y}$$

also suggest that U can be found from the components of the force by doing the opposite of the partial derivative, i.e. by calculating partial integrals.

U From the Components of F in Two Dimensions

$$U = -\int F_x dx \qquad U = -\int F_y dy$$

With the partial integral with respect to x, the terms depending only on x and those depending on both x and y are found. With the partial integral with respect to y, the terms depending only on y and the terms depending on both x and y are found. By combining the results of these two integrals, the formula of the potential energy is found.

Example 9.2.3

Find the potential energy associated with the following force.

$$\vec{F} = \left(6\tfrac{N}{m^2}x^2 + 2\tfrac{N}{m}y\right)\vec{i} + \left(2\tfrac{N}{m}y + 2\tfrac{N}{m}x\right)\vec{j}$$

The partial integrals are

$$\begin{aligned} U &= -\int\left(6\tfrac{N}{m^2}x^2 + 2\tfrac{N}{m}y\right)dx \\ &= -2\tfrac{N}{m^2}x^3 - 2\tfrac{N}{m}yx + C_1 \end{aligned}$$

$$\begin{aligned} U &= -\int\left(2\tfrac{N}{m^2}y + 2\tfrac{N}{m}x\right)dy \\ &= -1\tfrac{N}{m^2}y^2 - 2\tfrac{N}{m}yx + C_2 \end{aligned}$$

where the C's are constants. The first term of the first integral is the part of U that depends only on x. The first term of the second integral is the part of U that depends only on y. The second terms of both integrals are the part of U that depends on both x and y (which must obviously be identical). Therefore, U is equal to

$$U = -2\tfrac{N}{m^2}x^3 - 1\tfrac{N}{m^2}y^2 - 2\tfrac{N}{m}yx + cst$$

Condition for U to Exist

This way of calculating U clearly indicates that for U to exist, there is an additional condition in 2 dimensions. As the two integrals must give the same potential energy formula (more precisely, the part that depends on both x and y must be identical for the two integrals), the following condition must be respected.

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

In this document, you can see why this condition must be true.
http://physique.merici.ca/mechanics/proofU2D.pdf

Thus, here are the conditions for the existence of potential energy in 2 dimensions.

Condition for the Existence of the Potential Energy of a 2-Dimensional Force

The force must depend only on the position

and

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

If these conditions are met, then U exists, and the force is conservative. If one of these conditions is not met, then U does not exist, and the force is not conservative.

Example 9.2.4

Determined if the following force is conservative.

$$\vec{F} = \left(3\tfrac{N}{m^2}x^2 + 1\tfrac{N}{m}y\right)\vec{i} + \left(2\tfrac{N}{m}y + 1\tfrac{N}{m}x\right)\vec{j}$$

It is conservative since the two derivatives of the condition are equal.

$$\frac{\partial F_x}{\partial y} = 1\tfrac{N}{m} \qquad \frac{\partial F_y}{\partial x} = 1\tfrac{N}{m}$$

Therefore, U exists for this force.

Example 9.2.5

Determined if the following force is conservative.

$$\vec{F} = \left(3\tfrac{N}{m^2}x^2 + 1\tfrac{N^2}{m}y^2\right)\vec{i} + \left(2\tfrac{N}{m}y + 1\tfrac{N}{m^2}x^2\right)\vec{j}$$

It is not conservative since the two derivatives of the condition are not equal.

$$\frac{\partial F_x}{\partial y} = 2\tfrac{N}{m^2}y \qquad \frac{\partial F_y}{\partial x} = 2\tfrac{N}{m^2}x$$

Thus, U does not exist for this force.

Calculation of W when U Exists

When U exists, the calculation of the work becomes very easy with

$$W = -\Delta U$$

Example 9.2.6

What is the work done to move from the point (1 m, 1 m) to the point (2 m, 2 m) if the force is

$$\vec{F} = \left(6\tfrac{N}{m^2}x^2 + 2\tfrac{N}{m}y\right)\vec{i} + \left(2\tfrac{N}{m}y + 2\tfrac{N}{m}x\right)\vec{j}$$

Note that we already know (from a previous example) that this force is conservative and that the potential energy is

$$U = -2\tfrac{N}{m^2}x^3 - 1\tfrac{N}{m^2}y^2 - 2\tfrac{N}{m}yx + cst$$

The work is

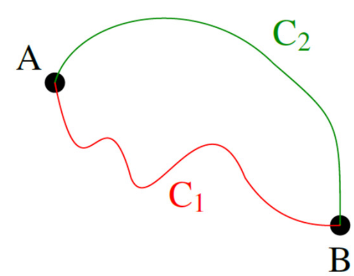
$$\begin{aligned} W &= -\Delta U \\ &= -\left(\left[-2\tfrac{N}{m^2}\cdot(2m)^3 - 1\tfrac{N}{m^2}\cdot(2m)^2 - 2\tfrac{N}{m}\cdot 2m\cdot 2m + cst\right]\right. \\ &\qquad\left. -\left[-2\tfrac{N}{m^2}\cdot(1m)^3 - 1\tfrac{N}{m^2}\cdot(1m)^2 - 2\tfrac{N}{m}\cdot 1m\cdot 1m + cst\right]\right) \\ &= -\left(\left[-28J + cst\right] - \left[-5J + cst\right]\right) \\ &= 23J \end{aligned}$$

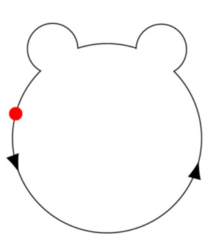
(Note that the constants cancel out and that its value has no influence on the value of the work.)

The Work Does Not Depend on the Path for a Conservative Force

When U exists, the work depends only on the difference of potential energy between the starting and the finishing position.

This means that, for a conservative force, the work done by the force on an object moving from one place to another is the same for all paths (as trajectories C_1 and C_2 in the diagram) if all these paths have the same starting point and the same ending point. If the force is not conservative, the work can depend on the path taken to get from one point to another.

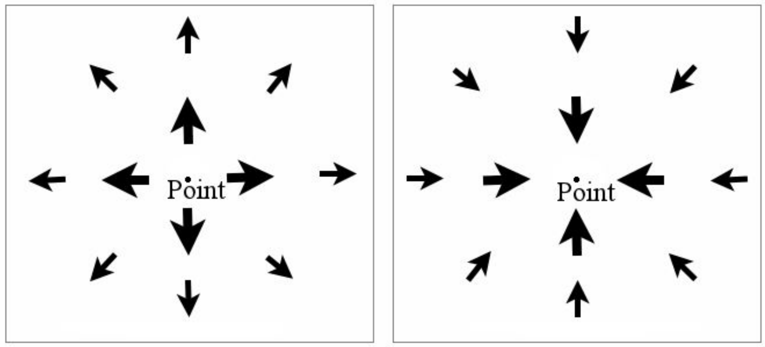




(This also means that the work done by a conservative force is zero if an object travels along a closed path (as in the diagram to the left).)

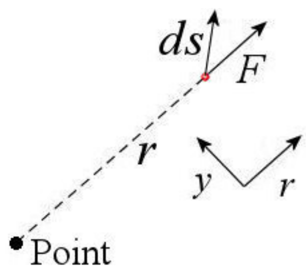
Central Forces

A central force is a force whose magnitude depends only on the distance *r* (meaning that the magnitude of the force must change in the same way for every direction) and which is always directed towards or away from a fixed point, as shown in the diagram.



www.maxwells-equations.com/fields/electric.php

It can be shown that a central force in two dimensions is conservative, regardless of the formula *F* giving the magnitude of the force as a function of the distance.
http://physique.merici.ca/mechanics/proofcentralF.pdf



For a central force, we have, for an infinitesimal displacement,

$$\begin{aligned} dU &= -\vec{F} \cdot \overrightarrow{ds} \\ &= -\left(F\vec{i}\right) \cdot \left(dr\vec{i} + dy\vec{j}\right) \\ &= -Fdr \end{aligned}$$

(In this last calculation, an *x*-axis in the radial direction (dotted line) was used and the position is called *r* instead of *x*).

This leads to

Link between the Potential Energy and the Force for a Central Force

$$F = -\frac{dU}{dr} \qquad \text{and} \qquad U = -\int F dr$$

In 3 Dimensions

By doing the same reasoning as in two dimensions, this link between *F* and *U* can be found.

Link between *U* and the Components of a 3-Dimensional Force

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z}$$

$$U = -\int F_x dx \qquad U = -\int F_y dy \qquad U = -\int F_z dz$$

This time, the 3 integrals must give the same formula for U. This adds a few conditions for the existence of U. These conditions are

The force must depend only on the position

and

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \qquad \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x} \qquad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}$$

Again, the work is the same for all paths between two points for a conservative force.

It can also be shown that central forces are still conservative in three dimensions. This result is very important since the gravitational force and the electric force are central forces in three dimensions. Moreover, the formulas

$$F = -\frac{dU}{dr} \qquad \text{and} \qquad U = -\int F dr$$

are still valid for central forces.

Which Forces Are Conservative?

It remains to determine which of the forces used in the previous chapters are conservative.

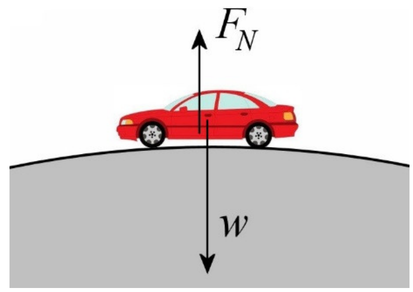
Friction Force

It has already been shown that the frictional force is not conservative since the force depends on the direction of the motion of the object. As the force can be different when the object is at the same position, it is not a conservative force.

Normal Force

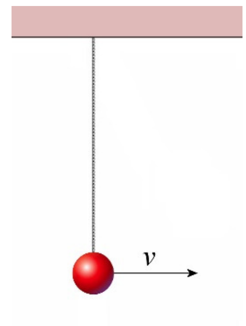
The example of a car passing over a bump will be used to determine whether this force is conservative. If the speed of the car is v, then the normal force is given by

$$F_N = mg - \frac{mv^2}{r}$$



The normal force depends on the speed of the car. If the car goes faster, the normal force gets smaller. As the normal force can be different even if the car is at the same position, it is not a conservative force.

Tension Force



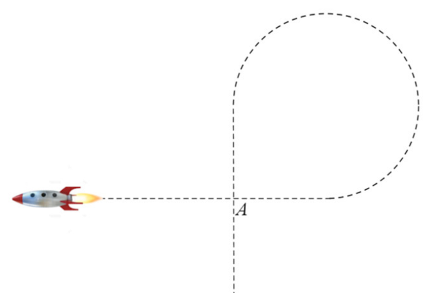
Let's take the example of a swinging pendulum to determine if this force is conservative. If the speed of the bob is v at the lowest point, the tension force is given by

$$T = mg + \frac{mv^2}{r}$$

The tension depends on the speed of the pendulum. If the pendulum goes faster, the tension gets greater. As the tension force can be different even if the bob is at the same position, it is not a conservative force.

Force Made by a Rocket

Imagine that the rocket follows the path shown in the diagram. When the rocket is at point A for the first time, the force made by the engine is upwards. When the rocket is at point A for the second time, the force is directed towards the left. Thus, the force was different even if the position was the same, which shows that the force does not depend exclusively on the position. Therefore, the force made by the engine of the rocket is not conservative.



Gravitational and Electrical Forces

These two forces are central forces whose magnitude decreases with the distance. So, these are conservative forces.

Springs

The magnitude of the force exerted by a spring is kx. An object attached to a spring is always subjected to the same force when it is at a specific distance from the equilibrium position of the spring. As the force depends only on position, it is a conservative force.

The Total Potential Energy Formula

Thus, only the following forces are conservative:

- Gravitational force
- Force made by a spring
- Electrical force

A potential energy U can then be found for each of these conservative forces. These energies are noted as follows:

U_g : Gravitational Energy

U_{sp} : Spring Energy

U_E : Electrical Energy

In mechanics, electrical energy will not be considered. Therefore, the formula of the total potential energy will be

Potential Energy of a System

$$U = U_g + U_{sp}$$

The gravitational energy formula is already known. Only the spring energy formula remains to be found.

Spring Energy

For a spring, the motion is one-dimensional, and the force is $F = -kx$. (The minus sign is there to indicate the direction of the force.)

As the force depends only on the position, the condition is met. Thus, the potential energy of the spring is

$$\begin{aligned} U &= -\int -kx\,dx \\ &= \frac{1}{2}kx^2 + Cst \end{aligned}$$

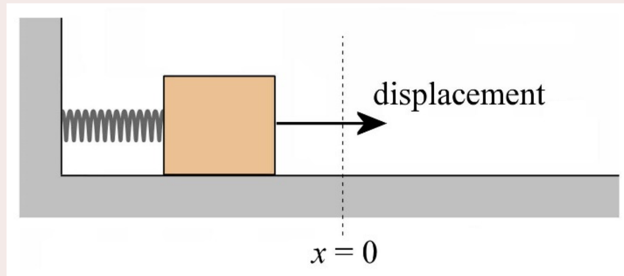
Since the constant is arbitrary and is useless, let's forget it. Therefore, the potential energy of the spring is

Spring Energy

$$U_{sp} = \frac{1}{2}kx^2$$

With the gravitational energy mgy, the constant of integration allows us to change the position of the origin $y = 0$. The constant allows us to do that only if the energy is a linear function of the position. Here, the constant does not enable us to change the position of the origin $x = 0$ because x is squared. The constant is actually useless and is set to zero. The origin $x = 0$ must then be at the position where the spring is neither stretched nor compressed.

Example 9.2.7



A spring with a constant of 120 N/m is attached to a 5 kg block. What is the work done by the spring on the block when the spring changes from a 3 cm compression to a 1 cm stretching?

In this problem, we have to calculate the work done between these two instants:

Instant 1 : spring having a 3 cm compression.
Instant 2 : spring having a 1 cm stretching.

The spring energy at instant 1 is

$$\begin{aligned} U_{sp} &= \frac{1}{2}kx^2 \\ &= \frac{1}{2}\cdot 120\tfrac{N}{m}\cdot(0.03m)^2 \\ &= 0.054J \end{aligned}$$

The spring energy at instant 2 is

$$\begin{aligned} U'_{sp} &= \frac{1}{2}kx'^2 \\ &= \frac{1}{2}\cdot 120\tfrac{N}{m}\cdot(0.01m)^2 \\ &= 0.006J \end{aligned}$$

Therefore, work done by the spring is

$$\begin{aligned} W_{sp} &= -\Delta U_{sp} \\ &= -(0.006J - 0.054J) \\ &= 0.048J \end{aligned}$$

9.3 MECHANICAL ENERGY CONSERVATION

Conservation Law

Consider what happens if there are only conservative forces doing work on an object. Then, the work is

$$W_{net} = -\Delta U$$

where U is the sum of all the potential energies.

This is the net work because the work done by each force acting on the object is included in the potential energy.

According to the previous chapter, the net work is also

$$W_{net} = \Delta E_k$$

By combining these two equations, it is found that

$$-\Delta U = \Delta E_k$$

This equation becomes

$$\Delta E_k + \Delta U = 0$$
$$E_k' - E_k + U' - U = 0$$
$$\left(E_k' + U'\right) - \left(E_k + U\right) = 0$$
$$\Delta\left(E_k + U\right) = 0$$

With the following definition

Mechanical Energy

$$E_{mec} = E_k + U$$

the following result is obtained

$$\Delta E_{mec} = 0$$

Equivalently, this equation is

$$E_{mec}' - E_{mec} = 0$$
$$E_{mec} = E_{mec}'$$

From this, the following rules are obtained.

Law of Conservation of Mechanical Energy

$$\Delta E_{mec} = 0$$

or

$$E_{mec} = E'_{mec}$$

or

$$E_{mec} = \text{constant}$$

Conservation means that something remains unchanged, as is the case here. According to the second equation, the value of the mechanical energy remains constant. The first equation tells us that the variation of mechanical energy is zero, which is another way of saying that it does not change.

Remember this restriction, however:

The law of conservation of mechanical energy is only valid if the work is done exclusively by conservative forces.

There may be non-conservative forces acting, but the work done by theses force must vanish.

Suddenly, the expression *non-conservative forces* can be understood. If there is a non-conservative force doing work, the mechanical energy is not conserved.

Galileo is the first to use the principle of mechanical energy conservation in a primitive form. The idea was then developed by Leibniz in 1686, but it was not until 1750 that a complete formulation was available. If you want to know more about the history of the discovery of the law of energy conservation, click on this link.
http://physique.merici.ca/mechanics/discoveryEmec.pdf

Problem-Solving Method With Mechanical Energy

The law of conservation of mechanical energy can now be used to solve problems. The method is relatively straightforward.

1) Write the formula for the mechanical energy (see next section).

2) Find the mechanical energy at some moment (instant 1). This energy is denoted *E*.

3) Find the mechanical energy at some other moment (instant 2). This energy is denoted *E'*.

4) Use these two energies in the law of conservation of mechanical energy.

$$E = E'$$

5) Solve this equation.

The Formula for the Mechanical Energy of a System

To write down the formula for the mechanical energy of a system, the rules are pretty simple because there are not many conservative forces considered in this course. The mechanical energy is simply the sum of the kinetic energies, the gravitational energies and the spring energies. Therefore, the rules to write down the formula of mechanical energy are

1) $\frac{1}{2}mv^2$ for each object in the system.

2) mgy for each object in the system.

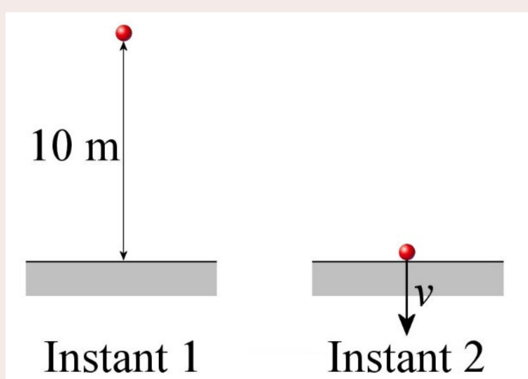
3) $\frac{1}{2}kx^2$ for each spring in the system.

As the number of objects and springs does not change from one moment to the next, the formula of the mechanical energy at the instant 1 is exactly the same as the formula of the mechanical energy at instant 2.

Example 9.3.1

A 4 kg object is dropped from a height of 10 m, starting with a zero initial velocity. How fast will it hit the ground?

The two following instants are considered.



Mechanical Energy Formula

As the system is composed of only one object, the mechanical energy is

$$E_{mec} = \tfrac{1}{2}mv^2 + mgy$$

Energy at Instant 1

At instant 1, the mechanical energy is

$$\begin{aligned} E &= \frac{1}{2}mv^2 + mgy \\ &= 0 + 4kg \cdot 9.8\tfrac{N}{kg} \cdot 10m \\ &= 392J \end{aligned}$$

We have chosen to set the origin $y = 0$ at the ground level. Its position cannot be changed thereafter.

Energy at Instant 2

At instant 2 (just before the object hits the ground), the mechanical energy is

$$\begin{aligned} E' &= \tfrac{1}{2}mv'^2 + mgy' \\ &= \tfrac{1}{2} \cdot 4kg \cdot v'^2 + 0 \\ &= 2kg \cdot v'^2 \end{aligned}$$

The instant 2 must be just before the ball hits the ground because the normal force will do some work on the ball as soon as the ball hits the ground. As the normal force is not a conservative force, the mechanical energy of the ball will cease to be conserved once the ball touches the ground.

Mechanical Energy Conservation

The law of conservation of mechanical energy is now used to find the speed.

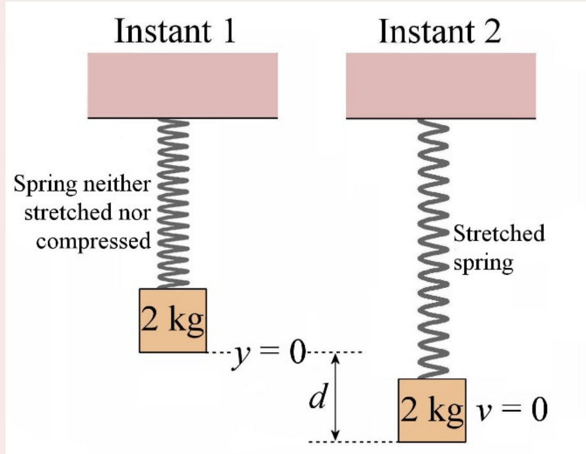
$$\begin{aligned} E &= E' \\ 392J &= 2kg \cdot v'^2 \\ v' &= 14\tfrac{m}{s} \end{aligned}$$

Therefore, the ball hits the ground at 14 m/s. Note that the speed (magnitude of the velocity) is obtained. Energy conservation does not give the direction of velocity.

Note that this is not the only way to solve this problem. It might also have been solved with the equations of kinematics, and the same result would have been obtained.

Example 9.3.2

A 2 kg block is suspended from the ceiling with a spring with a constant of 10 N/m. Initially, the spring is neither stretched nor compressed, and the block has no speed. The block is then released.



a) What will the maximum stretching of the spring be?

Mechanical Energy Formula

As the system is composed of a mass and a spring, the energy of the system is

$$E_{mec} = \tfrac{1}{2}mv^2 + mgy + \tfrac{1}{2}kx^2$$

Energy at Instant 1

At instant 1, the mechanical energy is

$$\begin{aligned} E &= \tfrac{1}{2}mv^2 + mgy + \tfrac{1}{2}kx^2 \\ &= 0 + 0 + 0 \\ &= 0 \end{aligned}$$

The origin $y = 0$ was set at the initial position of the block.

Energy at Instant 2

At instant 2, the mass has fallen a distance d. As the velocity of the mass is zero when the spring has its maximum extension, the mechanical energy at instant 2 is

$$\begin{aligned} E' &= \tfrac{1}{2}mv'^2 + mgy' + \tfrac{1}{2}kx'^2 \\ &= 0 + 2kg \cdot 9.8\tfrac{N}{kg} \cdot (-d) + \tfrac{1}{2} \cdot 10\tfrac{N}{m} \cdot d^2 \\ &= -19.6N \cdot d + 5\tfrac{N}{m} \cdot d^2 \end{aligned}$$

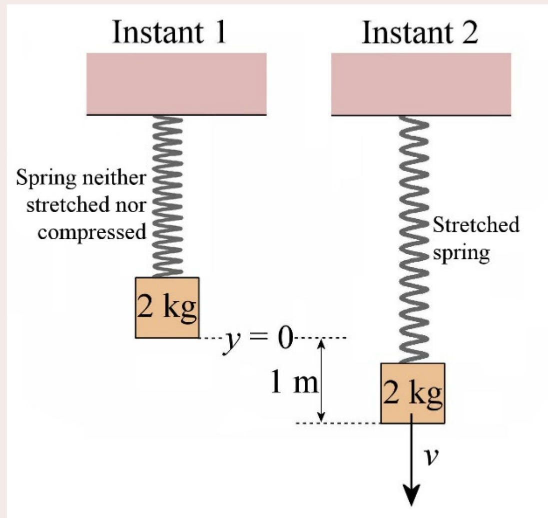
Mechanical Energy Conservation

Therefore, the law of energy conservation gives

$$E = E'$$
$$0 = -19.6N \cdot d + 5\tfrac{N}{m} \cdot d^2$$

The solutions to this equation are d = 0 and d = 3.92 m. In fact, the equation gives us a little more than what is needed. When the speed is set to zero in E', we are bound to find all the positions where speed is zero. That is why the initial position (d = 0) is also a solution (since the mass has no speed at this position). The other solution is the answer sought: the maximum stretching of the spring is 3.92 m.

b) What will the speed of the block be when the spring is stretched 1 m?



Energy at Instant 1

At instant 1, the energy is the same as in a), so E = 0 J.

Energy at Instant 2

When the spring is stretched 1 m, the energy of the system is

$$\begin{aligned} E' &= \tfrac{1}{2}mv'^2 + mgy' + \tfrac{1}{2}kx'^2 \\ &= \tfrac{1}{2}2kg \cdot v'^2 + 2kg \cdot 9.8\tfrac{N}{kg} \cdot (-1m) + \tfrac{1}{2} \cdot 10\tfrac{N}{m} \cdot (1m)^2 \\ &= 1kg \cdot v'^2 - 19.6J + 5J \\ &= 1kg \cdot v'^2 - 14.6J \end{aligned}$$

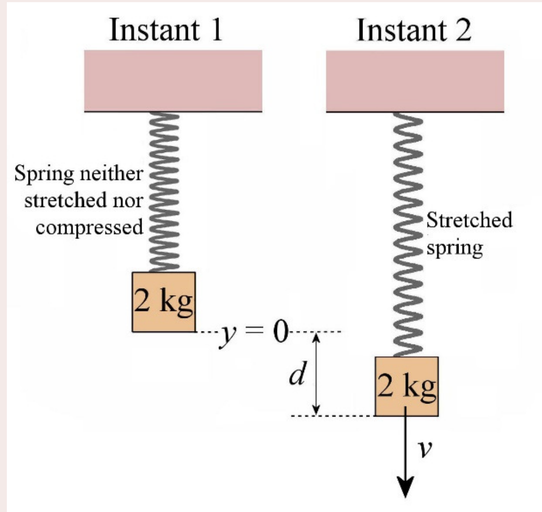
Mechanical Energy Conservation

The law of energy conservation gives

$$E = E'$$
$$0 = 1kg \cdot v'^2 - 14.6J$$
$$v' = 3.82\tfrac{m}{s}$$

Note that this speed may be directed upwards or downwards since the mass will oscillate after it has been released. Sometimes it will go up, sometimes it will go down when the spring is stretched by 1 m.

c) What will, be the stretching of the spring when the spring energy is equal to the kinetic energy of the block?



Energy at Instant 1

At instant 1, the energy is the same as in a), so $E = 0$ J.

Energy at Instant 2

At instant 2, the spring is stretched a distance d and the block is at the height $-d$, Therefore, the energy of the system is

$$\begin{aligned} E' &= \tfrac{1}{2}mv'^2 + mgy' + \tfrac{1}{2}kx'^2 \\ &= \tfrac{1}{2}mv'^2 + mg(-d) + \tfrac{1}{2}kd^2 \\ &= \tfrac{1}{2}mv'^2 - mgd + \tfrac{1}{2}kd^2 \end{aligned}$$

If the energy of the spring is equal to the kinetic energy of the block, then

$$\tfrac{1}{2}kd^2 = \tfrac{1}{2}mv'^2$$

Thus, the mechanical energy at instant 2 is

$$\begin{aligned} E' &= \tfrac{1}{2}mv'^2 - mgd + \tfrac{1}{2}kd^2 \\ &= \tfrac{1}{2}kd^2 - mgd + \tfrac{1}{2}kd^2 \\ &= kd^2 - mgd \end{aligned}$$

Mechanical Energy Conservation

Then, the law of energy conservation gives

$$\begin{aligned} E &= E' \\ 0 &= kd^2 - mgd \end{aligned}$$

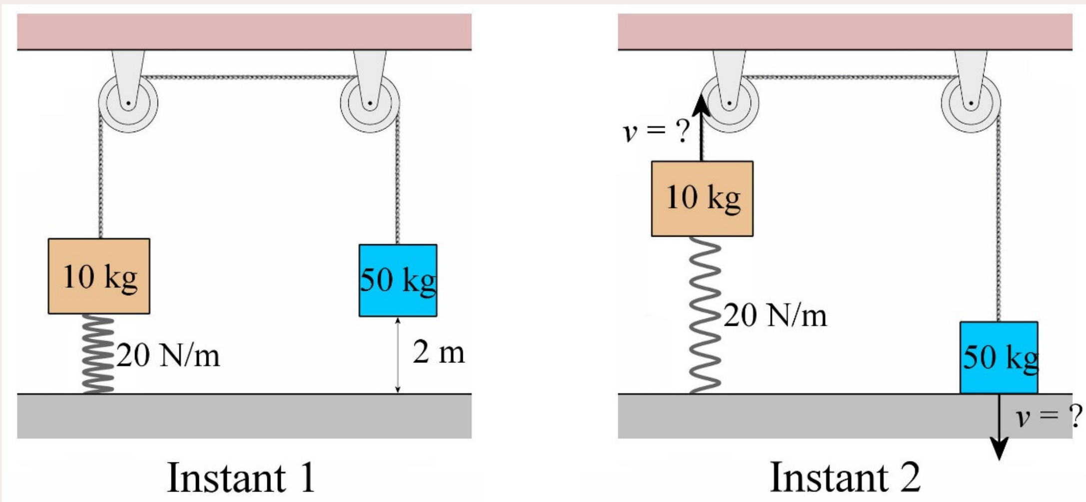
One of the solutions to this equation is $d = 0$. The kinetic energy and the energy of the spring are indeed equal at instant 1 because they are both equal to 0. The other solution is

$$\begin{aligned} 0 &= kd^2 - mgd \\ kd^2 &= mgd \\ kd &= mg \end{aligned}$$

$$d = \frac{mg}{k}$$
$$d = \frac{2kg \cdot 9.8\frac{N}{kg}}{10\frac{N}{m}}$$
$$d = 1.96m$$

Example 9.3.3

At instant 1, the system shown in the diagram is at rest and the spring is neither stretched nor compressed. The masses are then released. What will the speed of the 50 kg block be just before it hits the ground (instant 2)?



Mechanical Energy Formula

This system consists of 2 blocks and 1 spring. The mechanical energy formula is, therefore,

$$E_{mec} = \tfrac{1}{2}m_1v_1^2 + m_1gy_1 + \tfrac{1}{2}m_2v_2^2 + m_2gy_2 + \tfrac{1}{2}kx^2$$

To calculate the gravitational energy, the origin $y = 0$ must be chosen. It could be set on the ground, but then it would be difficult to determine the height of the 10 kg block because the initial height of this block is not known. However, the gravitational energy formula is more versatile than what you might think at first. Actually, **a different $y = 0$ can be chosen for each block**! Here, two different $y = 0$ will be used, each corresponding to the initial position of each block.

Energy at Instant 1

The mechanical energy at instant 1 is

$$\begin{aligned}E &= \tfrac{1}{2}m_1v_1^2 + m_1gy_1 + \tfrac{1}{2}m_2v_2^2 + m_2gy_2 + \tfrac{1}{2}kx^2\\ &= \tfrac{1}{2}m_1\left(0\tfrac{m}{s}\right)^2 + m_1g\left(0m\right) + \tfrac{1}{2}m_2\left(0\tfrac{m}{s}\right)^2 + m_2g\left(0m\right) + \tfrac{1}{2}k\left(0m\right)^2\\ &= 0\end{aligned}$$

Energy at Instant 2

The mechanical energy just before the 50 kg block hits the ground (instant 2) is

$$\begin{aligned}E' &= \tfrac{1}{2}m_1v_1'^2 + m_1gy_1' + \tfrac{1}{2}m_2v_2'^2 + m_2gy_2' + \tfrac{1}{2}kx'^2\\ &= \tfrac{1}{2}m_1v'^2 + m_1g\left(2m\right) + \tfrac{1}{2}m_2v'^2 + m_2g\left(-2m\right) + \tfrac{1}{2}k\left(2m\right)^2\end{aligned}$$

The height of the block 1 (the 10 kg block) is now 2 m because it ascends 2 m if the block 2 (the 50 kg block) descents 2 m. The stretching of the spring is also 2 m if the block 1 ascents 2 m. Both blocks have the same speed because a rope connects them. Therefore, the energy is

$$\begin{aligned}E' &= \tfrac{1}{2}m_1v'^2 + m_1g\left(2m\right) + \tfrac{1}{2}m_2v'^2 + m_2g\left(-2m\right) + \tfrac{1}{2}k\left(2m\right)^2\\ &= \tfrac{1}{2}\cdot 10kg\cdot v'^2 + 10kg\cdot 9.8\tfrac{N}{kg}\cdot\left(2m\right) + \tfrac{1}{2}\cdot 50kg\cdot v'^2\\ &\qquad + 50kg\cdot 9.8\tfrac{N}{kg}\cdot\left(-2m\right) + \tfrac{1}{2}\cdot 20\tfrac{N}{m}\cdot\left(2m\right)^2\\ &= 5kg\cdot v'^2 + 196J + 25kg\cdot v'^2 + -980J + 40J\\ &= 30kg\cdot v'^2 - 744J\end{aligned}$$

Mechanical Energy Conservation

The law of energy conservation then allows us to find the speed.

$$\begin{aligned}E &= E'\\ 0J &= 30kg\cdot v'^2 - 744J\\ v' &= 4.98\tfrac{m}{s}\end{aligned}$$

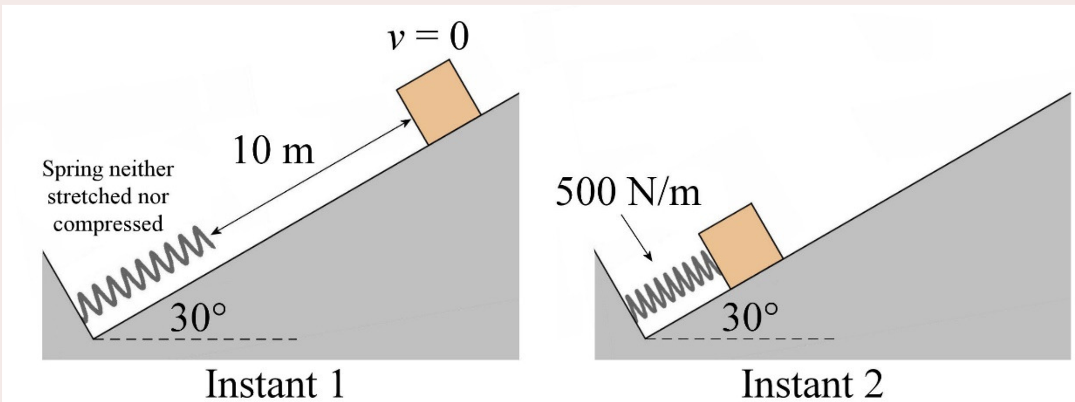
It is important to note that there are non-conservative forces acting here. These are the tension forces acting on the blocks. We then have to wonder whether it was legitimate to use the law of conservation of mechanical energy here.

It was legitimate because the total work done by the tension forces is zero. The tension pulls the 10 kg block upwards, and the block moves upwards over a distance of 2 m. The tension also pulls on the 50 kg block upwards while this block moves downwards over a distance of 2 m. As the tension forces and the distances are the same, the works have the same absolute value except that the work on the 10 kg block is positive and the work on the 50 kg block is negative. When these two works are added to get the net work done by the tension forces, the result is zero. The law of conservation of mechanical energy can, therefore, be applied here.

Actually, as soon as there is a rope connecting two objects in the system, the work done by the tension forces always cancel if the rope does not stretch. Therefore, the presence of a tension force in these cases does not invalidate the application of the law of conservation of mechanical energy.

Example 9.3.4

A 50 kg block initially at rest slides down a 30° slope. There is no friction between the block and the slope. 10 m downhill, there is a spring with a constant of 500 N/m. What will the maximum spring compression be when the block hits the spring?



Mechanical Energy Formula

As this system is composed of a mass and a spring, mechanical energy is

$$E_{mec} = \tfrac{1}{2}mv^2 + mgy + \tfrac{1}{2}kx^2$$

To calculate the gravitational energy, the origin $y = 0$ must be set. We choose here to set it at the initial position of the block.

Energy at Instant 1

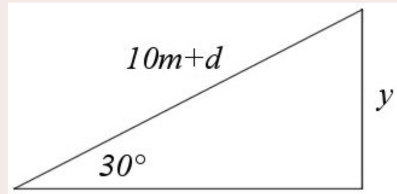
The mechanical energy at instant 1 is

$$\begin{aligned} E &= \tfrac{1}{2}mv^2 + mgy + \tfrac{1}{2}kx^2 \\ &= \frac{1}{2}m\left(0\tfrac{m}{s}\right)^2 + mg\left(0m\right) + \frac{1}{2}k\left(0m\right)^2 \\ &= 0 \end{aligned}$$

Energy at Instant 2

At instant 2, the spring is at maximum compression. The distance of maximum compression will be called d here. Also, the speed of the block is zero at maximum compression. It remains to find the height of the block at this time. The block has then travelled 10 m + *d* along the slope. The height can be found with the following triangle. Then, the height is found with

$$\sin 30° = \frac{y}{10m+d}$$
$$\frac{1}{2} = \frac{y}{10m+d}$$
$$y = \frac{10m+d}{2}$$



Thus, the energy when the spring is at maximum compression (instant 2) is

$$\begin{aligned} E' &= \tfrac{1}{2}mv'^2 + mgy' + \tfrac{1}{2}kx'^2 \\ &= \tfrac{1}{2}\cdot m\cdot\left(0\tfrac{m}{s}\right)^2 + 50kg\cdot 9.8\tfrac{N}{kg}\cdot\left(-\frac{10m+d}{2}\right) + \tfrac{1}{2}\cdot 500\tfrac{N}{m}\cdot\left(d\right)^2 \\ &= -50kg\cdot 9.8\tfrac{N}{kg}\cdot 5m - 50kg\cdot 9.8\tfrac{N}{kg}\cdot\frac{d}{2} + \tfrac{1}{2}\cdot 500\tfrac{N}{m}\cdot\left(d\right)^2 \\ &= -2450J - 245N\cdot d + 250\tfrac{N}{m}\cdot d^2 \end{aligned}$$

Mechanical Energy Conservation

The law of energy conservation gives

$$E = E'$$
$$0 = -2450J - 245N\cdot d + 250\tfrac{N}{m}\cdot d^2$$

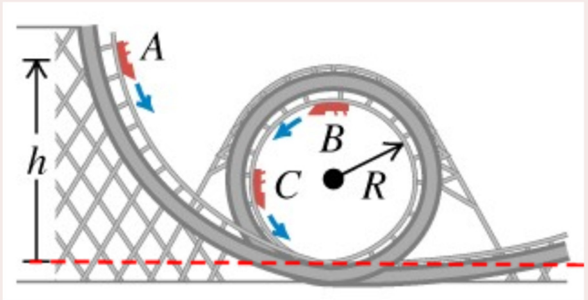
When this quadratic equation is solved, *d* = 3.659 m is obtained.

(There is another solution, which is negative and must be rejected because it corresponds to a stretching of the spring. However, this solution has a meaning: it gives us the maximum elongation of the spring if the mass bounces off the spring and remains attached to the spring. If the mass does not remain attached to the spring, it bounces back to its original position.)

Once again, there is a non-conservative force in this example. It is the normal force acting on the block. However, this normal force does no work because it is perpendicular to the slope while the block moves in a direction parallel to the slope. This means that there are 90° between the force and the displacement and that there is no work done by this force. The law of conservation of mechanical energy can, therefore, be applied since no non-conservative force does work.

Example 9.3.5

A roller coaster car initially at rest at the top of a slope (point A) descends the slope and then move into a loop as illustrated in the diagram. The radius of the loop is 25 m.



www.chegg.com/homework-help/questions-and-answers/physics-archive-2011-december-14

a) If the initial height h of the car is 100 m, what is the speed of the car at the highest point of the loop (point B)?

The two instants considered here are:

Instant 1: roller coaster car at point A.
Instant 2: roller coaster car at point B.

Mechanical Energy Formula

As there is a single object (the roller coaster car) in this example, the mechanical energy formula is

$$E_{mec} = \tfrac{1}{2}mv^2 + mgy$$

The $y = 0$ was set at the dotted line in the diagram.

Energy at Instant 1

When the roller coaster car is at the top of the slope, mechanical energy is

$$\begin{aligned} E &= \tfrac{1}{2}mv^2 + mgy \\ &= 0 + m \cdot 9.8\tfrac{N}{kg} \cdot 100m \end{aligned}$$

Energy at Instant 2

When the roller coaster car is at the top of the loop, the mechanical energy is

$$\begin{aligned} E' &= \tfrac{1}{2}mv'^2 + mgy' \\ &= \tfrac{1}{2}mv'^2 + m \cdot 9.8\tfrac{N}{kg} \cdot 50m \end{aligned}$$

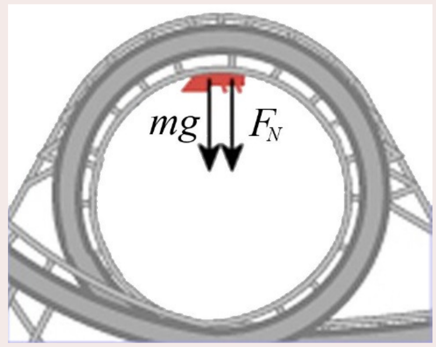
Mechanical Energy Conservation

The law of energy conservation allows us to find the speed.

$$E = E'$$
$$\cancel{m} \cdot 9.8 \tfrac{N}{kg} \cdot 100m = \tfrac{1}{2} \cancel{m} v'^2 + \cancel{m} \cdot 9.8 \tfrac{N}{kg} \cdot 50m$$
$$980 \tfrac{J}{kg} = \tfrac{1}{2} v'^2 + 490 \tfrac{J}{kg}$$
$$v' = 31.3 \tfrac{m}{s}$$

b) What should be the minimum height h for the roller coaster car to remain in contact with the rails at the highest point of the loop?

We'll assume that the car is in contact with the rails. Then the forces shown in the diagram acts on the car. The sum of the y-components of the forces is, therefore,



$$\sum F_y = ma$$
$$-mg - F_N = -\frac{mv'^2}{r}$$

The acceleration is equal to the centripetal acceleration. This acceleration is directed downwards here since the centre of the circle is downwards when you are at the top of the loop. The normal force is then

$$F_N = \frac{mv'^2}{r} - mg$$

There is a contact between the car and the rails as long as the normal force is positive. This means that

$$\frac{mv'^2}{r} > mg$$
$$v'^2 > rg$$

Returning now to the equation of conservation of mechanical energy obtained in a), but with h instead of 100 m for the height at instant 1, we have

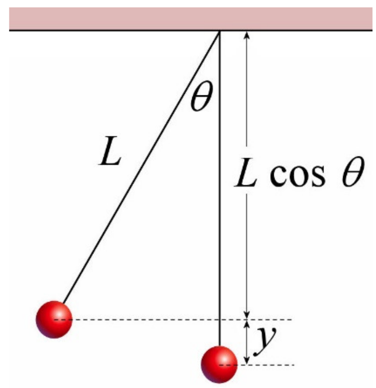
$$E = E'$$
$$\cancel{m} gh = \tfrac{1}{2} \cancel{m} v'^2 + \cancel{m} g \cdot 50m$$
$$g \cdot h = \tfrac{1}{2} v'^2 + g \cdot 50m$$
$$h = \frac{1}{2g} v'^2 + 50m$$

Since we must have $v'^2 > rg$, the equation becomes

$$h = \frac{1}{2g}v'^2 + 50m > \frac{1}{2g}rg + 50m$$
$$h > \frac{1}{2\cancel{g}}r\cancel{g} + 50m$$
$$h > \frac{1}{2}\cdot 25m + 50m$$
$$h > 62.5m$$

Application of the Law With a Pendulum

The law of conservation of energy will now be used to solve problems with a pendulum. However, before doing so, the relationship between the height of the pendulum (y) and the angle between the rope (of length L) and a vertical line must be found. The origin $y = 0$ is often set at the lowest position for a pendulum.



The length of the vertical line in the diagram corresponds to the length of the rope. Therefore

$$L = y + L\cos\theta$$

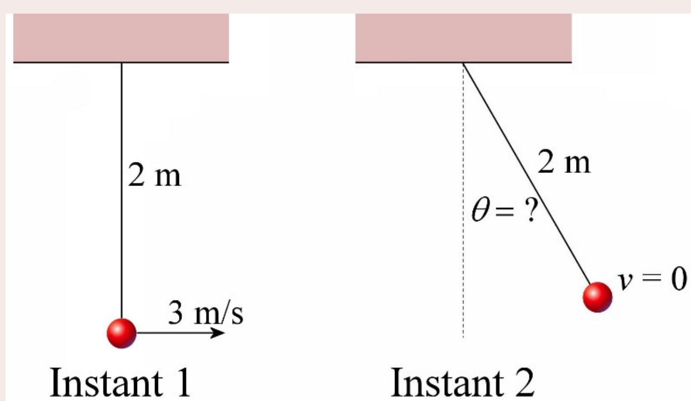
This can be solved for y or $\cos\theta$ to obtain

Link between Angle and Height for a Pendulum

$$y = L(1-\cos\theta) \qquad \cos\theta = \frac{L-y}{L}$$

Example 9.3.6

Up to what maximum angle will this pendulum swing if the speed of the pendulum's bob is 3 m/s when the rope is vertical?



Mechanical Energy Formula

There is only one massive object in this problem. The mechanical energy is, therefore,

$$E_{mec} = \tfrac{1}{2}mv^2 + mgy$$

Energy at Instant 1

At instant 1, this energy is

$$\begin{aligned} E &= \tfrac{1}{2}mv^2 + mgy \\ &= \tfrac{1}{2}mv^2 + mg\cdot(0m) \\ &= \tfrac{1}{2}mv^2 \end{aligned}$$

Energy at Instant 2

At instant 2, the energy is

$$\begin{aligned} E' &= \tfrac{1}{2}mv'^2 + mgy' \\ &= \tfrac{1}{2}m\cdot\left(0\tfrac{m}{s}\right)^2 + mgy' \\ &= mgy' \end{aligned}$$

Mechanical Energy Conservation

Thus, the law of conservation of energy allows us to calculate the height of the bob at instant 2.

$$E = E'$$

$$\tfrac{1}{2}\cancel{m}v^2 = \cancel{m}gy'$$

$$y' = \frac{v^2}{2g}$$

$$y' = \frac{\left(3\tfrac{m}{s}\right)^2}{2\cdot 9.8\tfrac{N}{kg}}$$

$$y' = 0.459m$$

Therefore, the angle is

$$\cos\theta = \frac{L - y'}{L}$$

$$\cos\theta = \frac{2m - 0.459m}{2m}$$

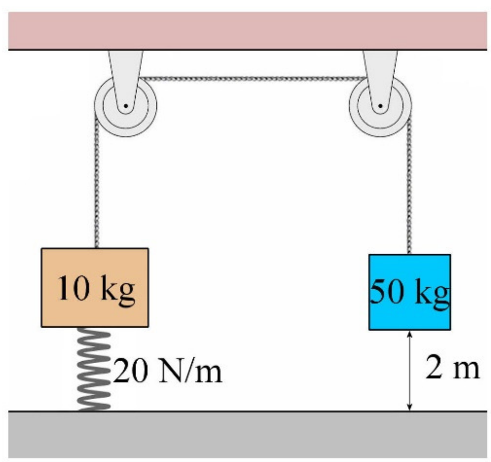
$$\theta = 39.6°$$

An Experimental Demonstration

Here's an experimental demonstration of the law of conservation of mechanical energy inspired by the previous example. As the gravitational energy of a free-falling object decreases, the kinetic energy must increase according to the law of conservation of mechanical energy. The trajectory of the object has, in fact, no influence on this conclusion. If a 1 kg object falls 1 m, its gravitational energy decreases by 9.8 J and its kinetic energy increases by 9.8 J, regardless of the trajectory. This means that an object initially at rest will gain the same speed if it descends from a certain distance, irrespective of the path.

In the first part of the experiment presented here, a ball falls straight down, and its speed is measured at the lowest point using a laser. In fact, the device measures the time it takes for the ball to cross the laser. The smaller the time, the faster the ball goes. In the second part, an identical ball falls along a circular path, like a pendulum. They make sure that the ball falls from the same height as the ball dropped straight down. With the same device, the speed at the lowest point of the trajectory is measured. Will the speed be the same as it must be according to the law of conservation of mechanical energy?
http://www.youtube.com/watch?v=L2mdAvdPhT4

A More Precise Formulation of the Law of Conservation of Mechanical Energy

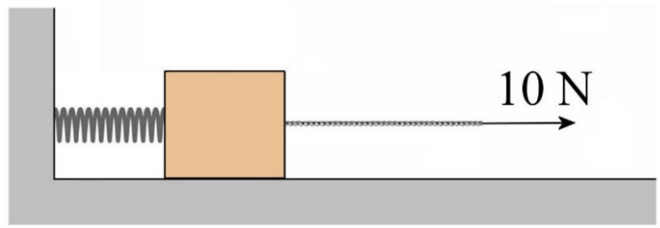


A **system** is the set of objects that are considered in the problem. Thus, in the example of the two blocks, the system was composed of two blocks and a spring. This choice was reflected in the energy formula in which there were two kinetic energies (one for each block), two gravitational energy (one for each block) and a spring energy. Note that the Earth is also part of our system since the gravitational energy of an object is actually the gravitational energy of the Earth and the object. The Earth is therefore automatically included in the system. However, the kinetic energy of the Earth is not written in the formula as it is always considered at rest.

The **mechanical energy of the system** is simply the sum of the mechanical energies of all objects in the system. With two blocks and a spring in the system, the mechanical energy was, therefore, the sum of the mechanical energy of the 10 kg block (kinetic and gravitational), the mechanical energy of the 50 kg block (kinetic and gravitational) and the energy of the spring.

An **internal force** is a force between two objects included the system. In the example of the two blocks and the spring, the force made by the spring on the 10 kg block is an internal force since the two objects (the 10 kg block and the spring) are both part of the system. The gravitational forces exerted on the blocks are also internal forces as the Earth and the blocks are all included in the system.

An **external force** is a force acting on an object in the system made by an object not included the system. If someone pulls on the rope in the following diagram with a force of 10 N and the system was chosen to include only the block and the spring, the 10 N force is an external force.



If there is no external force, then the system is an **isolated system**.

With this more accurate terminology, the law of conservation of mechanical energy can be stated more precisely.

Law of Conservation of Mechanical Energy

The energy of an isolated system is conserved, i.e.

$$\Delta E_{mec} = 0$$

or

$$E_{mec} = E'_{mec}$$

or

$$E_{mec} = \text{constant}$$

if the work done on the system is done exclusively by conservative forces.

There are, therefore, two conditions to respect if the law of conservation of mechanical energy is to be applied correctly.

1) There is no external force (it must be an isolated system).
2) There is no work done by non-conservative forces.

9.4 MECHANICAL ENERGY WHEN THERE ARE EXTERNAL FORCES AND NON-CONSERVATIVE FORCES

Proof that Non-Conservative Internal Forces and External Forces Change the Mechanical Energy of a System

First, let's show that external forces and internal non-conservative forces break the conservation of mechanical energy of a system.

Let's start with the work-energy theorem.

$$\Delta E_k = W_{net}$$

As the work is done by internal conservative forces or by non-conservative internal forces or by external forces, the work done can be divided into two parts the work done by the internal conservative forces (W_c), the work done by internal non-conservative forces (W_{nc}) and the work done by external forces (W_{ext}).

$$\Delta E_k = W_c + W_{nc} + W_{ext}$$

The work done by conservative internal forces can be calculated with the potential energy with $W_c = -\Delta U$. Therefore,

$$\begin{aligned} \Delta E_k &= -\Delta U + W_{nc} + W_{ext} \\ \Delta E_k + \Delta U &= W_{nc} + W_{ext} \\ \Delta\left(E_k + U\right) &= W_{nc} + W_{ext} \\ \Delta E_{mec} &= W_{nc} + W_{ext} \end{aligned}$$

In other words, the mechanical energy is not conserved anymore since mechanical energy can be added to or removed from a system by an external force or by a non-conservative internal force. The mechanical energy change corresponds to the work done by these forces.

Calculation Using Mechanical Energy When Non-Conservative Internal Forces or External Forces Are Doing Work

The formula obtained previously shows how mechanical energy can be used to solve a problem even if there is work done by a conservative internal force or an external force. Here are two equivalent versions of this formula.

Mechanical Energy with External Forces or Non-Conservative Internal Forces

$$\Delta E_{mec} = W_{nc} + W_{ext}$$

or

$$E_{mec} + W_{nc} + W_{ext} = E'_{mec}$$

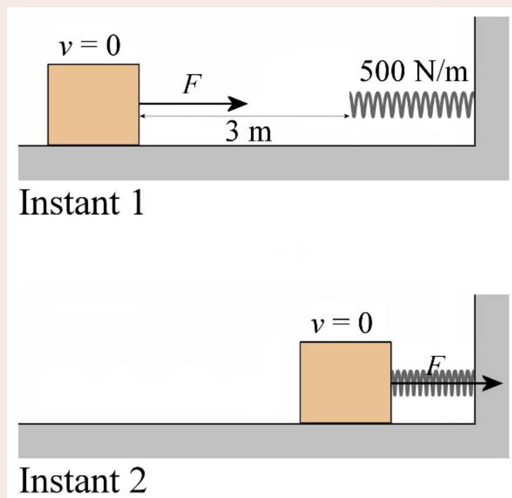
To resolve, the mechanical energy is still calculated at instants 1 and 2 but now the work done by non-conservative and external forces must also be calculated. Then these 3 results are used in the following equation, and the equation is solved.

$$E_{mec} + W_{nc} + W_{ext} = E'_{mec}$$

In this first example, there is an external force.

Example 9.4.1

A block initially at rest is pushed by a constant force. The block slides 3 m on a frictionless horizontal surface and then hits a spring with a constant of 500 N/m. If the maximum spring compression is 15 cm, what is the magnitude of force?



Mechanical Energy Formula

As there are a block and a spring in this system, the mechanical energy is

$$E_{mec} = \tfrac{1}{2}mv^2 + mgy + \tfrac{1}{2}kx^2$$

Energy at Instant 1

At instant 1 (block initially at rest). The energy is

$$\begin{aligned} E &= \tfrac{1}{2}mv^2 + mgy + \tfrac{1}{2}kx^2 \\ &= 0 + 0 + 0 \\ &= 0 \end{aligned}$$

The kinetic energy is zero since the block is at rest. The gravitational energy is zero because the block height is zero (the origin $y = 0$ is set at the initial height of the block). The spring energy is zero because it is neither compressed nor stretched initially.

Energy at Instant 2

At instant 2 (maximally compressed spring), the energy is

$$\begin{aligned} E' &= \tfrac{1}{2}mv'^2 + mgy' + \tfrac{1}{2}kx'^2 \\ &= 0 + 0 + \tfrac{1}{2}\cdot 500\tfrac{N}{m}\cdot\left(0.15m\right)^2 \\ &= 5.625J \end{aligned}$$

It is obvious that the mechanical energy has changed. This is not really surprising since an external force is acting on the system.

Work Done by the External Force

The work done by the external force is then

$$\begin{aligned} W_{ext} &= F_{ext}\Delta s\cos\theta \\ &= F_{ext}\cdot 3.15m\cdot\cos 0° \\ &= F_{ext}\cdot 3.15m \end{aligned}$$

Application of $E_{mec} + W_{ext} + W_{nc} = E'_{mec}$

With the energies and work found, the equation becomes

$$\begin{aligned} E_{mec} + W_{ext} &= E'_{mec} \\ 0 + F_{ext}\cdot 3.15m &= 5.625J \end{aligned}$$

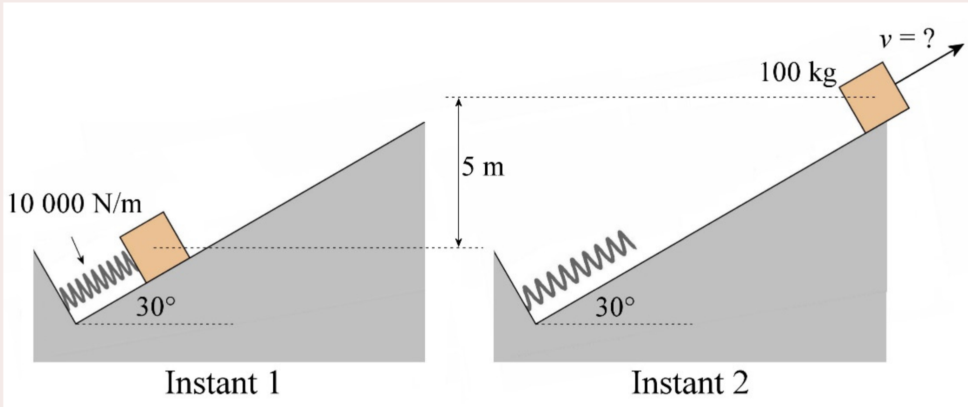
Thus, the external force is

$$F_{ext} = 1.7857N$$

In this second example, there is a non-conservative force.

Example 9.4.2

A 100 kg block is initially at rest on a 30° slope is resting against a spring (k = 10 000 N/m) which is compressed 2 m. If the top of the slope is 5 m higher than the starting point of the block and the coefficient of friction between the block and the slope is 0.2, what will the speed of the block be at the top of the slope?



Mechanical Energy Formula

As there are a block and a spring in this system, the mechanical energy is

$$E_{mec} = \tfrac{1}{2}mv^2 + mgy + \tfrac{1}{2}kx^2$$

Energy at Instant 1

At instant 1 (when the block is resting against the compressed spring), the energy is

$$\begin{aligned} E &= \tfrac{1}{2}mv^2 + mgy + \tfrac{1}{2}kx^2 \\ &= 0 + 0 + \tfrac{1}{2}\cdot 10\,000\tfrac{N}{m}\cdot(2m)^2 \\ &= 20\,000J \end{aligned}$$

The kinetic energy is zero since the block is initially at rest. The gravitational energy is zero because the block height is zero (the origin $y = 0$ is set at the initial height of the block).

Energy at Instant 2

At instant 2 (the mass is at the top of the slope), the mechanical energy is

$$\begin{aligned} E' &= \tfrac{1}{2}mv'^2 + mgy' + \tfrac{1}{2}kx'^2 \\ &= \tfrac{1}{2}\cdot 100kg\cdot v'^2 + 100kg\cdot 9.8\tfrac{N}{kg}\cdot 5m + 0 \\ &= 50kg\cdot v'^2 + 4900J \end{aligned}$$

The spring energy is now zero because the spring decompresses while pushing the block. It is now neither stretched nor compressed.

Work Done by the External Force

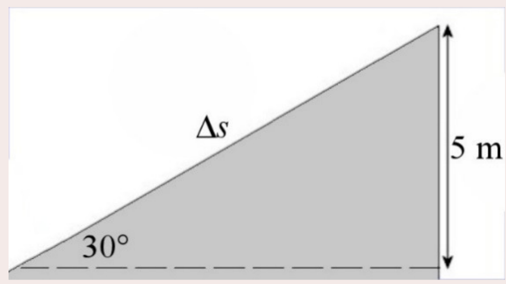
The work done by the non-conservative force is

$$\begin{aligned} W_{nc} &= F_f \Delta s \cos\theta \\ &= \mu_k F_N \Delta s \cos 180° \\ &= -\mu_k F_N \Delta s \end{aligned}$$

The angle is 180° because the force is directly opposed to the motion. The normal force is found with the sum of the y-component of the forces acting on the block. (An x-axis directed uphill is used.)

$$\sum F_y = ma_y$$
$$mg\sin(-120°) + F_N = 0$$
$$F_N = mg\sin(120°)$$
$$F_N = 848.7N$$

The displacement can be found with a little trigonometry.



$$\frac{5m}{\Delta s} = \sin 30°$$
$$\Delta s = 10m$$

Therefore, the work is

$$\begin{aligned} W_{nc} &= -\mu_k F_N \Delta s \\ &= -0.2 \cdot 848.7N \cdot 10m \\ &= -1697.4J \end{aligned}$$

Application of $E_{mec} + W_{ext} + W_{nc} = E'_{mec}$

The speed can then be found with the equation of mechanical energy.

$$E + W_{nc} = E'$$
$$20\,000J + -1697.4J = 50kg \cdot v'^2 + 4900J$$
$$13\,402.6J = 50kg \cdot v'^2$$
$$v' = 16.37 \tfrac{m}{s}$$

Often, the non-conservative force is a friction force opposed to the motion. Then, the work done by the friction force is

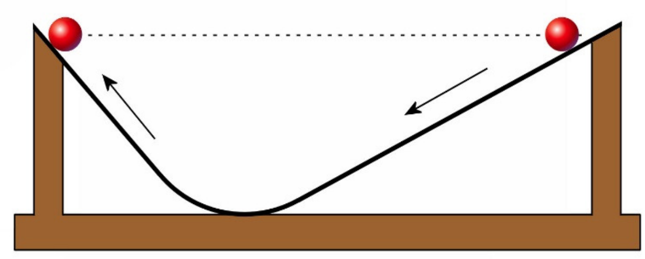
$$\begin{aligned} W_{nc} &= F_f \Delta s \cos 180° \\ &= -F_f \Delta s \end{aligned}$$

and the following result is obtained.

$$\Delta E_{mec} = -F_f \Delta s$$

This equation indicates that the mechanical energy decreases. It tells us that the variation of mechanical energy is negative, meaning that the energy decreases.

This conclusion corresponds better to our understanding of the world surrounding us. Even if the law of conservation of mechanical energy tells us that the ball in this experiment should go up to the same height it was released; we know that this is not what really happens.

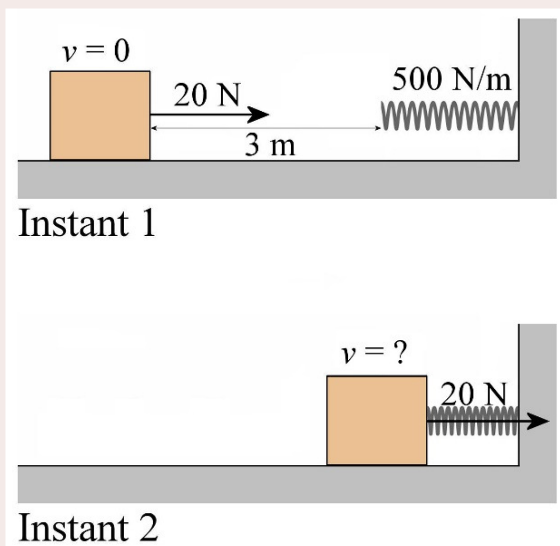


Instead, you should expect that the ball will go up and stop before reaching the same height. This occurs because friction slowly removes mechanical energy. Thus, the ball does not rise as high as it started because a decrease of mechanical energy means less height when all the mechanical energy is in the form of gravitational energy.

In this last example, we have an external force and a non-conservative force.

Example 9.4.3

A 6 kg block initially at rest is pushed by a constant force of F = 20 N. The block slides 3 m on a horizontal surface and then hits a spring with a constant of 500 N/m. The coefficient of friction between the ground and the block is 0.25. What is the speed of the block when the spring is compressed 20 cm?



Mechanical Energy Formula

As there are a block and a spring in this system, the mechanical energy is

$$E_{mec} = \tfrac{1}{2}mv^2 + mgy + \tfrac{1}{2}kx^2$$

Energy at Instant 1

At instant 1 (block initially at rest), the energy is

$$\begin{aligned} E &= \tfrac{1}{2}mv^2 + mgy + \tfrac{1}{2}kx^2 \\ &= 0 + 0 + 0 \\ &= 0 \end{aligned}$$

The kinetic energy is zero since the block is then at rest. The gravitational energy is zero because the height of the block is zero (the origin y = 0 was set at the initial height of the block). The energy of the spring is zero because it is neither compressed nor stretched.

Energy at Instant 2

At instant 2 (spring compressed 20 cm), the energy is

$$\begin{aligned} E' &= \tfrac{1}{2}mv'^2 + mgy' + \tfrac{1}{2}kx'^2 \\ &= \tfrac{1}{2}\cdot 6kg \cdot v'^2 + 0 + \tfrac{1}{2}\cdot 500\tfrac{N}{m}\cdot(0.2m)^2 \\ &= 3kg \cdot v'^2 + 10J \end{aligned}$$

Work Done by the External Force and by the Non-Conservative Force

As there are an external force and a non-conservative force (the friction force), the work done by these forces must be found.

The work done by the external force is

$$\begin{aligned} W_{ext} &= F\Delta s\cos\theta \\ &= 20N \cdot 3.2m \cdot \cos(0°) \\ &= 64J \end{aligned}$$

The work by the non-conservative force (the friction force) is

$$\begin{aligned} W_{nc} &= F_f \Delta s\cos\theta \\ &= (\mu_k F_N)\cdot 3.2m \cdot \cos(180°) \\ &= \left(0.25\cdot 6kg \cdot 9.8\tfrac{N}{kg}\right)\cdot 3.2m\cdot\cos(180°) \\ &= -47.04J \end{aligned}$$

Application of $E_{mec} + W_{ext} + W_{nc} = E'_{mec}$

The speed can then be found with the energy equation.

$$E + W_{ext} + W_{nc} = E'$$
$$0J + 64J + -47.04J = 3kg \cdot v'^2 + 10J$$

$$3kg \cdot v'^2 = 6.96J$$
$$v' = 1.523 \tfrac{m}{s}$$

9.5 SIMPLE MACHINES

Mechanical energy conservation makes it easy to understand why it is advantageous to use simple machines.

Suppose someone wants to raise a mass object *m* to a certain height *y*. No matter how it is done, the energy of the object increases by *mgy*. To give that energy, an external work must be done. Therefore, the force required is found with

$$W_{ext} = mgy$$
$$F \Delta s \cos\theta = mgy$$
$$F = \frac{mgy}{\Delta s \cos\theta}$$

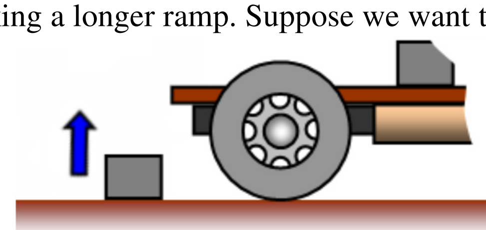
If the force exerted on the object is in the direction of movement (this is often the case with an inclined plane and pulleys), then the angle is zero. Thus, the force is

$$F = \frac{mgy}{\Delta s \cos 0°}$$
$$F = \frac{mgy}{\Delta s}$$

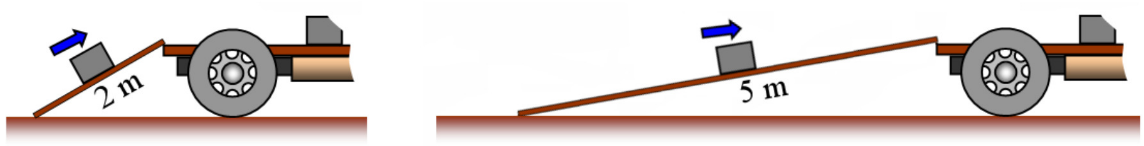
To reduce the force needed to bring the object to the height *y,* the distance must be increased. It is this concept that lies behind the use of the inclined plane and pulleys.

Inclined Plane

With an inclined plane, the force is reduced by taking a longer ramp. Suppose we want to put a 50 kg box on a 1 m high truck platform. To achieve this, 490 J (*mgy*) must be given to the object. Without an inclined plane (the box is lifted straight up), a 490 N force is required over a distance of 1 m.



With an inclined plane, the force required is decreased if the length of the plane in increased.



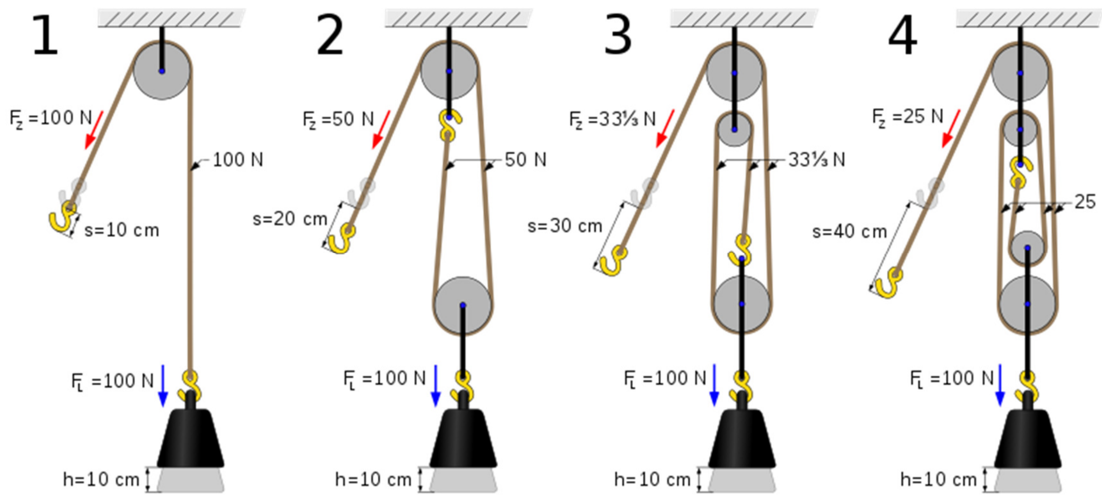
www.schoolphysics.co.uk/age11-14/Mechanics/Forces%20in%20motion/text/Machines_/index.html

With an inclined plane at 30 degrees (left), a force of 245 N is required suffice. On the other hand, it will be necessary to exert force over a distance of 2 m. With a plane at 11.5 degrees, a force of 98 N is sufficient, but the box will have to be pushed over a distance of 5 m.

Pulleys

Suppose now that an object weighing 100 N must be lifted over a distance of 10 cm. To achieve this, 10 J (*mgy*) must be given to the object.

Using the pulley system 1 shown on the diagram, the rope must be pulled 10 cm to lift the object 10 cm. As the distance is the same, the force is still 100 N.



en.wikipedia.org/wiki/File:Four_pulleys.svg

With the pulley system 2, the rope must be pulled 20 cm to lift the mass 10 cm. This is so because the rope supports the object attached to the pulley twice and it is necessary to reduce the length of the rope twice by 10 cm to lift the mass 10 cm. By multiplying the rope distance to be pulled by 2, the force is divided by 2.

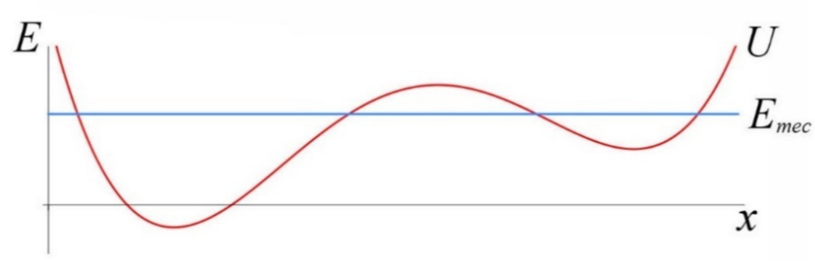
With the pulley system 3, the rope must be pulled 30 cm to lift the mass 10 cm. This is so because the rope supports the object attached to the pulley 3 times and it is necessary to reduce the length of the rope 3 times by 10 cm to lift the mass 10 cm. By multiplying the rope distance to be pulled by 3, the force is divided by 3.

With the pulley system 4, the rope must be pulled 40 cm to lift the mass 10 cm. This is so because the rope supports the object attached to the pulley 4 times and it is necessary to reduce the length of the rope 4 times by 10 cm to lift the mass 10 cm. By multiplying the rope distance to be pulled by 4, the force is divided by 4.

These results are consistent with the tension calculations made in Chapter 4.

9.6 GRAPH OF POTENTIAL ENERGY AS A FUNCTION OF POSITION

A lot of information can be obtained from a potential energy-versus-position graph if the mechanical energy is conserved. Only the one-dimensional case (motion only along the *x*-axis) will be explored here. As an example, the following graph will be used to illustrate the results here.



On this graph, the potential energy changes depending on the position of the object. The mechanical energy as a function of the position is obviously a straight horizontal line since energy is conserved and so always keeps the same value whatever the object's position. This E_{mec} horizontal line goes up if energy is added to the system and moves down if energy is removed from the system.

1) Force Acting on an Object

The force is

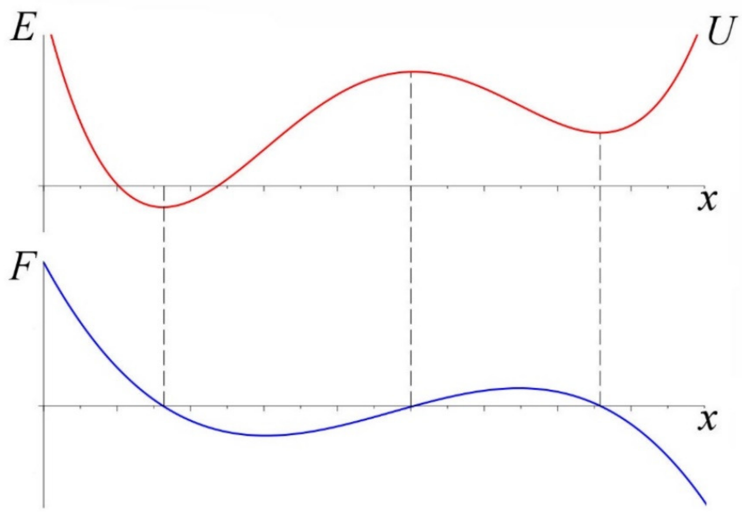
$$F_x = -\frac{dU}{dx}$$

This leads to this graphical interpretation.

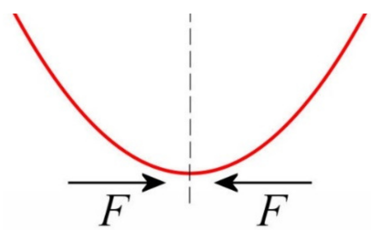
Force Acting on an Object

The force is minus the slope on a potential energy versus position graph.

Thus, the graph of the force exerted on the object can be obtained by plotting the graph of minus the derivative of the potential energy. Here an example.



At the start, the slope is negative, so the force is positive. Then, the slope is less and less negative, meaning that the force decreases. Then the slope becomes positive, so the force is now negative. At the maximum of *U,* the slope and the force are both zero. After that, the slope becomes negative (positive force) and then positive again (negative force) after the relative minimum.



This also means that if the slope is positive, the force is negative and if the slope is negative, the force is positive.

Thus,

Force Acting on an Object

The force on an object is always directed towards the place where the potential energy is lowest.

2) Areas Where the Object Can Be and Positions Where the Object Has its Maximum Speed

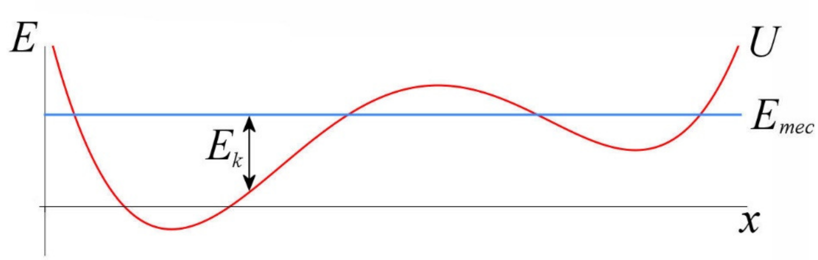
Since the mechanical energy is

$$E_{mec} = E_k + U$$

the kinetic energy is

$$E_k = E_{mec} - U$$

The kinetic energy is, therefore, the difference between mechanical energy and the potential energy. Graphically, the kinetic energy is equal to the gap between the E_{mec} line and the curve of *U*.

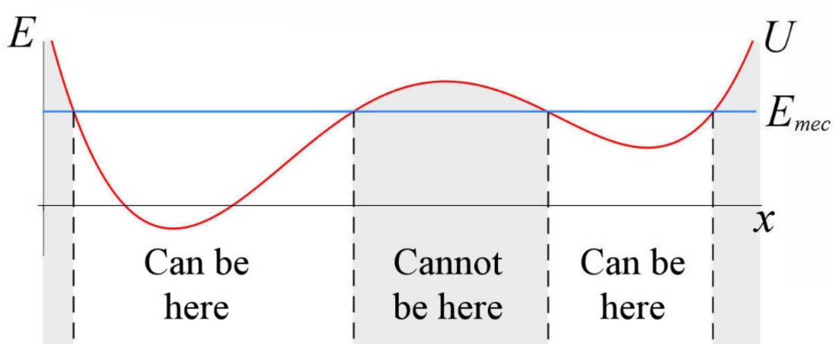


Obviously, the object cannot be at places where the curve of U is above the E_{mec} line because the kinetic energy would then be negative, which is impossible.

Areas Where an Object Cannot Be

An object cannot be at places where U is above E_{mec}.

In our example, this means that



In this case, the object must always remain in the permitted area on the right or always stay in the permitted area on the left. It is absolutely impossible for the object to passes from one area to the other, for it would then have to pass through the forbidden zone between two permitted areas.

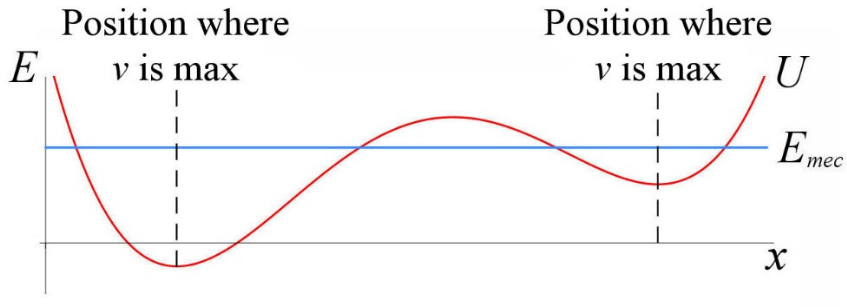
On the other hand, if the mechanical energy of the object is modified, the areas where the object can be also change. If the mechanical energy is increased (by giving energy to the object) until the E_{mec} line is above the maximum at the centre of the U curve, the object would then be able to pass from one permitted area to the other because there would be no more forbidden zone between the two.

The position where the speed is largest can also be found. This happens when the kinetic energy is largest. As this energy is the difference between E_{mec} and U, the position where the gap between the two curves is greatest is the position where the speed is largest. Obviously, this happens when U is at a minimum.

Position where an Object Has its Maximum Speed

The maximum of the speed of the object occurs when the difference between E_{mec} and U is the largest (and E_{mec} is above U).

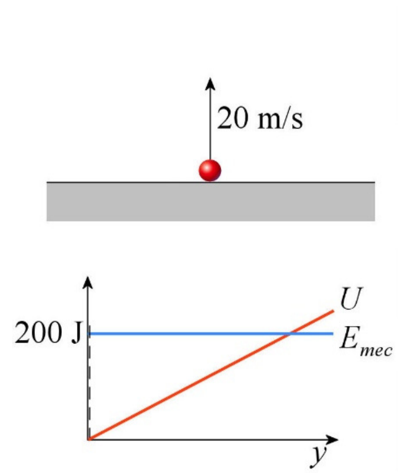
In our example, this means that



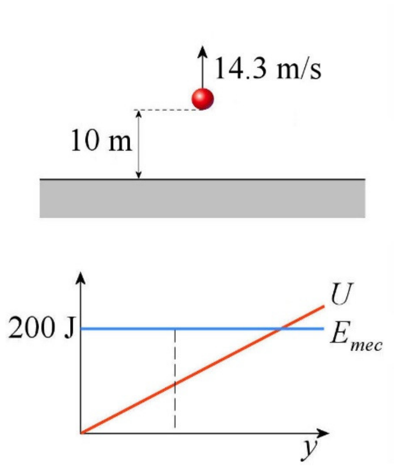
There are two positions because there are two areas where the object can be. If the object is in the permitted area on the left, its maximum speed occurs at the leftmost marked position for v_{max} on the graph. If the object is in the permitted area on the right, its maximum speed occurs when the object is at the rightmost marked position for v_{max} on the graph.

A Stone Thrown Upwards

For example, consider the graph of the energy of a 1 kg object that was thrown directly upwards with a kinetic energy of 200 J. With gravity, we have $U = mgy$, which corresponds to a straight line.



Initially, the object is at $y = 0$ m and its potential energy is zero. All the mechanical energy is in the form of kinetic energy. On the graph, it can be seen that the object is at $y = 0$ m, and the gap between the mechanical energy (E_{mec}) and the potential energy (U) at this position is 200 J. This gap is the kinetic energy of the object. This corresponds to a speed of 20 m/s.

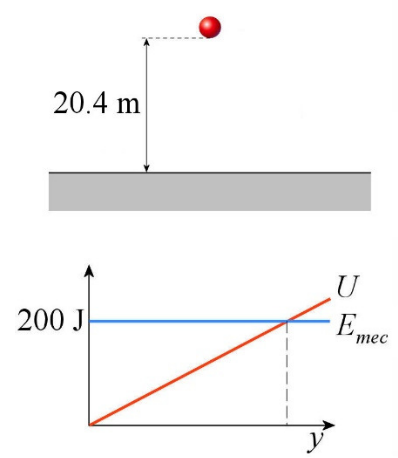


A little later the object is 10 m high. At this instant, the speed of the object is 14.3 m/s

On the graph of energy, the situation at $y = 10$ m can be seen. At this position, the potential energy is 98 J so that the gap between E_{mec} and U is 102 J. This corresponds to the kinetic energy of the object.

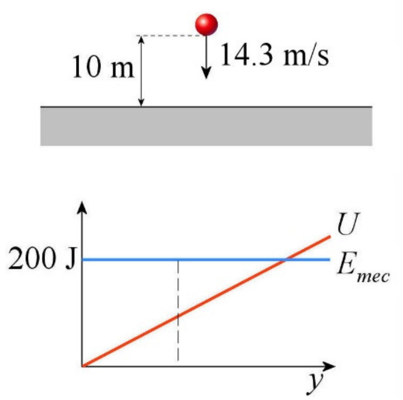
Note that as the value of y increases, the gap between E_{mec} and U decreases which means that the kinetic energy decreases as y increases. This is actually what is happening with an object thrown upwards.

Later, the object reaches its maximum height, which is 20.4 m.



On the graph, the maximum height corresponds to the point of intersection of the curves of U and E_{mec}. The object cannot go higher, because the object would then go where $U > E_{mec}$ and this area is a forbidden area.

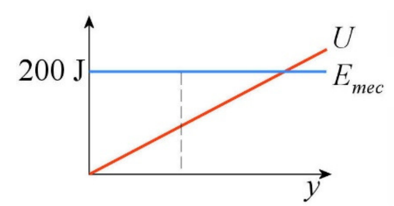
If the mechanical energy were increased, the forbidden area would move to the right on the graph, which means that the forbidden area would begin at a larger value of y. Thus, the object would climb higher if there were more mechanical energy at the start.



Finally, the object starts to move downwards. When it returns to a height of 10 m, its speed is 14.3 m/s again since the gap between E_{mec} and U is back to 102 J at this position.

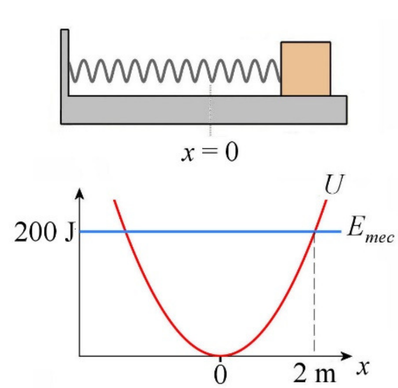
As y decreases, the gap between E_{mec} and U increases again, which means that the speed of the object increases as it moves towards the ground.

Note that the slope of the potential energy is constant and positive. This means that the force, which is equal to -(slope), is constant and negative (directed downwards). This conclusion agrees with what is known about the gravitational force.



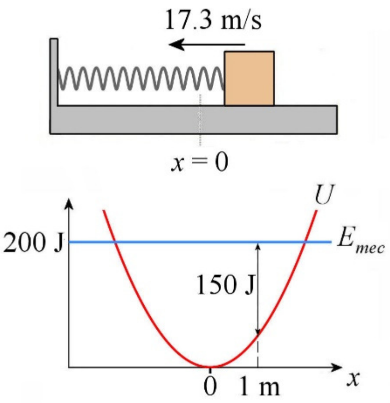
A Mass Attached at the End of a Spring

Now, the graph of the energy of a 1 kg object attached to a spring with a constant of 100 N/m will be considered. Initially, the spring is stretched 2 m, and the velocity of the mass is zero, which gives the system a mechanical energy of 200 J. With a spring, the potential energy is $U = ½kx^2$, which is a parabola.

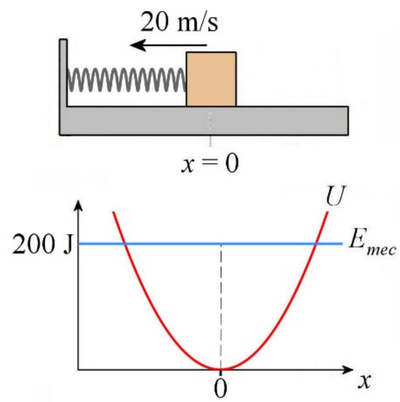


At the start of this motion, the object is in the position where E_{mec} and U intersect. As there is no gap between the two curves at this location, the kinetic energy is zero.

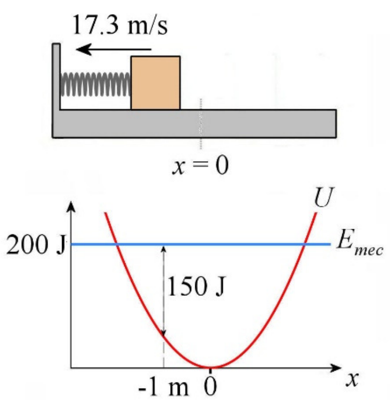
In addition, the object could not go farther than $x = 2$ m since the area where $x > 2$ m is a forbidden area. Actually, this 1 kg mass will always be between $x = -2$ m and $x = 2$ m. To go farther than $x = 2$ m, some energy must be added to the system so that the E_{mec} line moves up on the graph.



A little later, the object, pulled by the spring, arrives at the position $x = 1$ m. At this position, the potential energy of the spring is only 50 J and the gap between E_{mec} and U is then 150 J. Therefore, the kinetic energy of the object is 150 J, which means that its speed is 17.3 m/s.



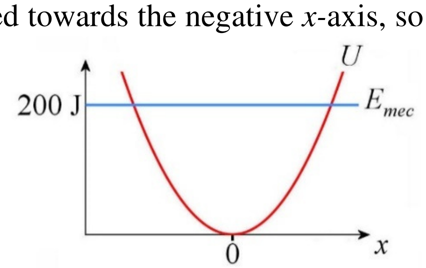
Later, the object arrives at $x = 0$. At this position, the potential energy reaches its minimum value (0 J) and kinetic energy reaches its maximum value of 200 J. The minimum of the potential energy is the position where the speed of the object is the largest.



Finally, the object continues its motion towards the negative x's. At $x = -1$ m, the speed of the object is again equal to 17.3 m/s because the gap between E_{mec} and U is back to 150 J. The speed will continue to decrease as the object moves towards the negative x's since the gap between E_{mec} and U will continue to decrease. The object stops at $x = -2$ m since the gap between E_{mec} and U vanishes at this place. The object cannot go farther than $x = -2$ m since this is a forbidden area when the mechanical energy is 200 J.

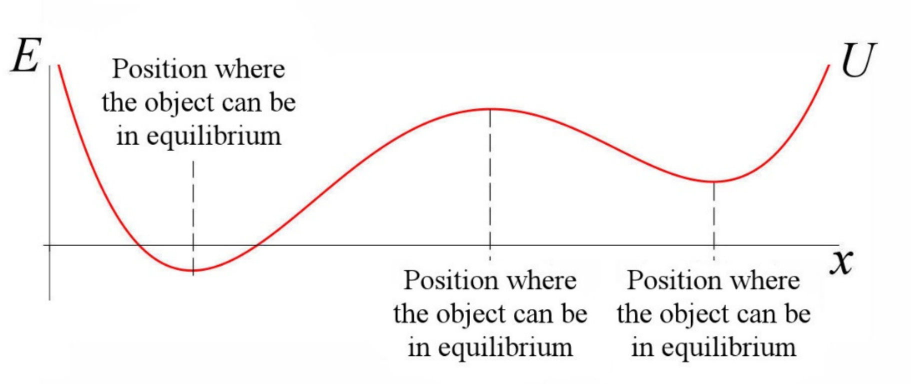
From $x = -2$ m, the object will start to move towards $x = 0$ again. It cannot stay at $x = -2$ m since there is a force acting on the object (the slope of U is not zero).

Finally, note how the slope of the potential energy changes. To the right of $x = 0$, the slope is positive and gets more positive as x increases. This indicates that the force exerted by the spring force is increasingly large and that it is directed towards the negative x-axis, so towards $x = 0$. To the left of $x = 0$, the slope is negative and gets more negative farther away from $x = 0$. This indicates that the force exerted by the spring force is increasingly large and it is directed towards the positive x-axis, so towards $x = 0$. It agrees well with what is known about the force exerted by a spring.



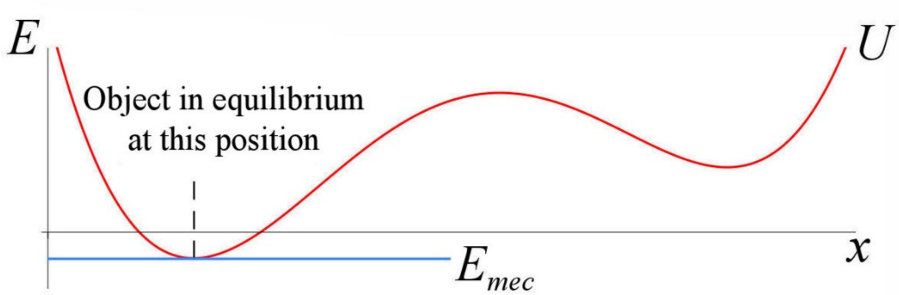
3) Equilibrium Position

If the object is in equilibrium, then there is no net force exerted on the object. If there is a force, the object accelerates and cannot remain in place. As the force is minus the slope of U, the force is zero when the slope is zero. Thus, in our example, there are three places where the object can be in equilibrium.

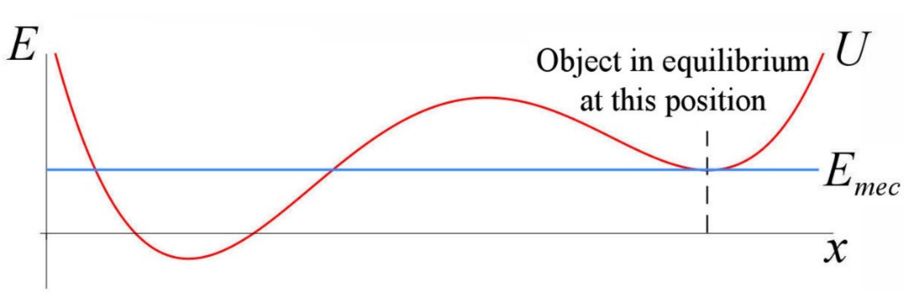


Obviously, the mechanical energy must be exactly equal to U for the object to remain at the equilibrium position. If the mechanical energy were to be larger than U, the object would have some kinetic energy and the object would move away from the equilibrium position.

The following graph is the graph on an object at equilibrium at the leftmost equilibrium position.

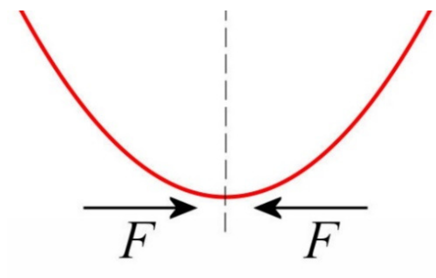


It is obvious in this situation that the object must remain at the equilibrium position since the object can be at places where U is larger than E_{mec}. It can then only be at the equilibrium position here since U is above E_{mec} everywhere else. The same thing happens if the object is at the rightmost equilibrium position, as shown in this next graph.

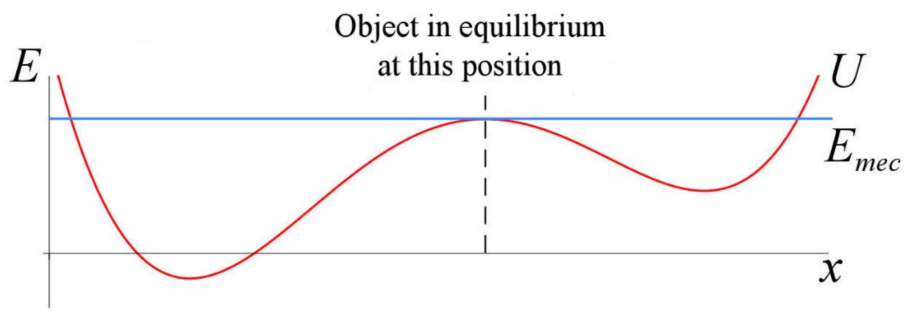


If the object is at the specified position, it must remain there because it cannot move without going into a prohibited area. With this mechanical energy, the object can also be in the trough on the left where the mechanical energy is larger than U. However, if the object is initially at the equilibrium position on the right, it is impossible for the object to move into the permitted area on the left because it would have to go through a prohibited area to get there.

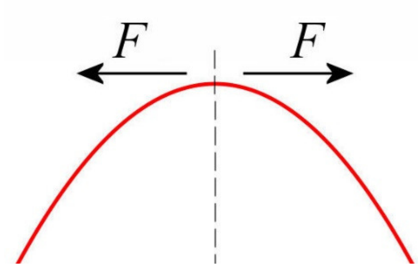
These two previous situations correspond to a stable equilibrium. Even if the mechanical energy is slightly increased, the object remains close to the equilibrium position, as it cannot move away much without entering into a forbidden area. Also, the forces on each side of the equilibrium position are always directed towards the equilibrium position. To the right of the equilibrium position, the slope is positive so that the force is negative. This force towards the left brings the object back to the equilibrium position. Left of the equilibrium position, the slope is negative so that the force is positive. This force towards the right also brings the object back to the equilibrium position.



There is a third position where the force is zero: at the relative maximum of the function.

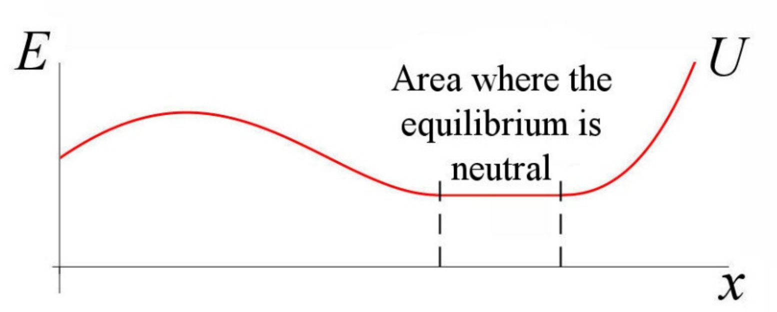


Even though the object is in equilibrium at this position, the object can leave this position without entering into a prohibited area. This corresponds to an unstable equilibrium. The forces on the object on each side the equilibrium position are always directed away from the equilibrium position. To the right of the equilibrium position, the slope is negative, so the force is positive. This means that the force is towards the right, away from the equilibrium position. To the left of the equilibrium position, the slope is positive, so the force is negative. This force is towards the left, away from the equilibrium position.



In fact, the situation is similar as if you attempt to place a ball in equilibrium on a surface whose shape is the same as the graph of U. You can place the ball in the trough, and it then remains stationary at this place, even if there is a small perturbation (like a light wind) acting on the ball. This is a stable equilibrium. The ball could also be placed on top of the crest. However, this equilibrium is unstable, and the ball falls on one side or the other of the equilibrium position at the slightest disturbance.

There is another possibility to have no force: when the slope and the concavity are both zero. This is what is happening in the area shown in the following graph.



If the object is somewhere in this area, there would be no force acting on it even if there were a small perturbation causing a small displacement, because the slope is always zero. As this equilibrium is neither stable nor unstable, it is called neutral equilibrium.

Here's a summary of all the types of equilibrium position.

Equilibrium Positions

The object can be in equilibrium at the positions where the slope of *U* is zero. The equilibrium is stable if the graph of *U* is concave upwards at this position, the equilibrium is unstable if the graph of *U* is concave downwards at this position and the equilibrium is neutral if the concavity is zero at this position.

9.7 TIME OF DISPLACEMENT FROM *U*

The mechanical energy is

$$E_{mec} = \frac{1}{2}mv^2 + U$$

If this equation is solved for the velocity, the result is

$$v = \pm\sqrt{\frac{2}{m}\left(E_{mec} - U\right)}$$

Since the velocity is the derivative of the position ($v = dx/dt$), the equation is

Equation to Solve to Obtain the Time of Displacement from *U*

$$\frac{dx}{dt}=\pm\sqrt{\frac{2}{m}\left(E_{mec}-U\right)}$$

This formula makes it possible to calculate the time that an object will take to move from one place to another if the formula of potential energy is known. Here's how.

Example 9.7.1

The potential energy of a 2 kg object is given by the following formula.

$$U=25\tfrac{N}{m}x^2$$

How long will it take for the object to go from $x = 0$ to $x = 2$ m if the mechanical energy is 200 J?

With the values given, the equation

$$\frac{dx}{dt}=\pm\sqrt{\frac{2}{m}\left(E_{mec}-U\right)}$$

becomes

$$\begin{aligned}\frac{dx}{dt}&=\sqrt{\frac{2}{2kg}\left(200J-25\tfrac{N}{m}\cdot x^2\right)}\\&=\sqrt{200\tfrac{m^2}{s^2}-25\tfrac{1}{s^2}\cdot x^2}\\&=5\tfrac{1}{s}\cdot\sqrt{8m^2-x^2}\end{aligned}$$

(The positive sign is used because the sign in front of the root is the sign of the velocity. Since the object moves from $x = 0$ to $x = 2$ m, the velocity is positive.)

Time is found in finding the solution to this equation. The equation can first be written in the following form.

$$5\tfrac{1}{s}\cdot dt=\frac{dx}{\sqrt{8m^2-x^2}}$$

The equation is then integrated by taking the beginning and the end as integration limits. This means that on the left side, the limits will be the time at the position $x = 0$ (which will be called t_1) and the time at the position $x = 2$ m (which will be called t_2). On the right side, the limits will be the initial position ($x = 0$) and the final position ($x = 2$ m). Thus, the equation becomes

$$\int_{t_1}^{t_2}5\tfrac{1}{s}\cdot dt=\int_0^{2m}\frac{dx}{\sqrt{8m^2-x^2}}$$

$$\left[5\tfrac{1}{s}\cdot t\right]_{t_1}^{t_2}=\left[\arcsin\left(\frac{x}{\sqrt{8}m}\right)\right]_0^{2m}$$

$$5\tfrac{1}{s}\cdot t_2-5\tfrac{1}{s}\cdot t_1=\arcsin\left(\frac{2}{\sqrt{8}}\right)-0$$

$$5\tfrac{1}{s}\cdot\left(t_2-t_1\right)=\arcsin\left(\frac{2}{\sqrt{8}}\right)$$

$$t_2-t_1=0.2s\cdot\arcsin\left(\frac{2}{\sqrt{8}}\right)$$

$$t_2-t_1=0.3927s$$

The position as a function of time can also be obtained from this equation.

Example 9.7.2

The potential energy of a 2 kg object is given by the following formula.

$$U=25\tfrac{N}{m}x^2$$

Find the formula that gives the position of the object as a function of time if the mechanical energy of the object is 200 J, knowing that the object is at $x = 0$ and that its speed is positive at $t = 0$.

With the values given, the equation

$$\frac{dx}{dt}=\pm\sqrt{\frac{2}{m}\left(E_{mec}-U\right)}$$

becomes

$$\begin{aligned}\frac{dx}{dt}&=\pm\sqrt{\frac{2}{2kg}\left(200J-25\tfrac{N}{m}\cdot x^2\right)}\\&=\pm\sqrt{200\tfrac{m^2}{s^2}-25\tfrac{1}{s^2}\cdot x^2}\\&=\pm5\tfrac{1}{s}\cdot\sqrt{8m^2-x^2}\end{aligned}$$

The equation can be written in the following form.

$$5\tfrac{1}{s}\cdot dt=\frac{dx}{\sqrt{8m^2-x^2}}$$

To obtain the equation of motion, the indefinite integral is performed on each side.

$$\int 5\tfrac{1}{s}\cdot dt=\pm\int\frac{dx}{\sqrt{8m^2-x^2}}$$

$$5\tfrac{1}{s}\cdot t + C = \pm\arcsin\left(\frac{x}{\sqrt{8m}}\right)$$

If the object is at $x = 0$ at $t = 0$, then the integration constant is

$$0 + C = \arcsin(0)$$
$$C = 0$$

Thus

$$5\tfrac{1}{s}\cdot t = \pm\arcsin\left(\frac{x}{\sqrt{8m}}\right)$$

Solving for x, the result is

$$x = \sqrt{8}m\cdot\sin\left(\pm 5\tfrac{rad}{s}\cdot t\right)$$
$$x = \pm\sqrt{8}m\cdot\sin\left(5\tfrac{rad}{s}\cdot t\right)$$

Finally, the sign can be found since we know that the initial speed is positive. The velocity is

$$\begin{aligned} v &= \frac{dx}{dt} \\ &= \pm\sqrt{8}m\cdot 5\tfrac{1}{s}\cdot\cos\left(5\tfrac{rad}{s}\cdot t\right) \end{aligned}$$

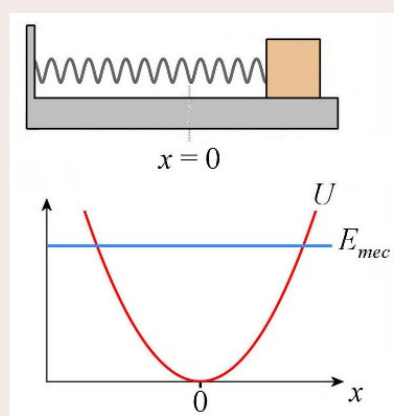
At $t = 0$, we have

$$v = \pm\sqrt{8}m\cdot 5\tfrac{1}{s}$$

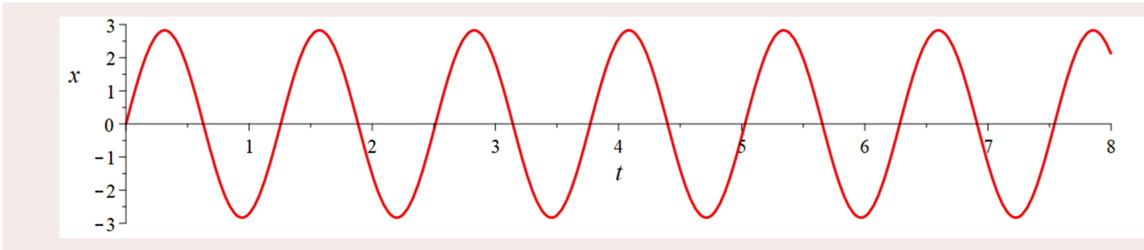
Since the velocity is positive, the positive sign must be used. Thus, the position as a function of time is

$$x = 0.2m\cdot\sin\left(5\tfrac{rad}{s}\cdot t\right)$$

This answer is consistent with what was found previously. It had been found, from the energy graph, that for such a potential energy, which corresponds to the energy of a spring, the object would oscillate between two extreme positions and have a maximum speed at $x = 0$.



This is exactly what our formula of the position as a function of time shows. It describes an oscillation between two positions (here -$\sqrt{8}$ m and $\sqrt{8}$ m) with a maximum speed at $x = 0$ (the slope on the graph, which is the speed, is the largest at $x = 0$).

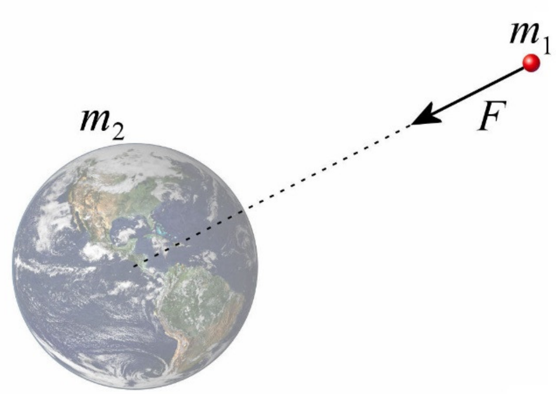


Everything seemed relatively easy here, but most of the time it is not that easy. Sometimes the integral is really difficult to do, and some other times it is not easy to solve the equation for x once the integral is made to arrive at the formula of the position as a function of time.

9.8 GENERAL FORMULA OF GRAVITATIONAL ENERGY

The Formula

The gravitational energy was previously obtained from the gravitational force. However, the formula for gravity near Earth (mg) was used, so the result obtained is valid only for positions close to the surface of the Earth. It is, therefore, not correct in general because the gravitational force between a planet and an object decreases with the square of the distance. A more general formula of gravitational energy will, therefore, be obtained here from the law of gravitation.



$$F = \frac{Gm_1m_2}{r^2}$$

With a central force like gravitation, the formula of the potential energy is obtained with the following formula.

$$U_g = -\int Fdr$$

As the axis is directed away from the Earth (since r increases as we move away from the Earth), the force is negative since it is directed towards the Earth. The equation is then

$$\begin{aligned} U_g &= -\int Fdr \\ &= -\int \left(-\frac{Gm_1m_2}{r^2}\right)dr \\ &= \int \frac{Gm_1m_2}{r^2}dr \end{aligned}$$

This integral is

$$U_g = -\frac{Gm_1m_2}{r} + \text{constant}$$

With the gravitational energy mgy, the constant of integration allowed us to change the position of the origin $y = 0$. The constant allows us to do that only if the energy is a linear function of the position. Here, the constant does not enable us to change the position of the origin $r = 0$ because r is the denominator. The constant is actually useless, so it is set to zero. Thus, the energy is

Gravitational Energy (General Formula)

$$U_g = -\frac{Gm_1m_2}{r}$$

In this equation, r is the distance between the object and the centre of the planet or the star and it is not possible to change that. The gravitational energy is always negative and approaches zero as the object moves away from the planet or star. Thus, the gravitational energy increases as the object moves away from the planet or star. It's convenient that the gravitational energy tends to zero when an object is far from a planet or a star because, then, you do not need to worry about all the other planets and stars of the universe when you calculate the gravitational energy of an object near Earth. All these planets and stars being so far away, the gravitational energy due to these heavenly bodies is quite negligible.

Note that when the gravitational energy of an object nearby a planet is calculated, the result obtained is not the energy of the object only, but rather the energy of the planet-object system. If the gravitational energy is converted into kinetic energy, the kinetic energy can go into the object or the planet, not just in the object. The distribution depends on the masses and on the constraints on the system. However, if the planet is much more massive than the object, the gravitational energy will often be transformed only into the kinetic energy of the object.

Example 9.8.1

A ball is thrown upwards at 5000 m/s from the surface of the Earth. How high will the ball go if the friction made by the atmosphere is neglected? (The mass of the Earth is 5.9722×10^{24} kg and the radius of the Earth is 6371 km.)

Mechanical Energy Formula

As there is only one object in this system, the mechanical energy is (where M_E is the mass of the Earth)

$$E_{mec} = \frac{1}{2}mv^2 + \frac{-GM_Em}{r}$$

Energy at Instant 1

The energy at instant 1 (object at the surface of the Earth) is

$$\begin{aligned} E &= \frac{1}{2}mv^2 + \frac{-GM_E m}{r} \\ &= \frac{1}{2}mv^2 + \frac{-GM_E m}{R_E} \end{aligned}$$

At this moment, r is the radius of the Earth (R_E) because the distance between the object and the centre of the Earth is equal to the radius of the Earth.

Energy at Instant 2

At instant 2 (object farthest from the Earth) the speed is zero. The energy is then

$$\begin{aligned} E' &= \frac{1}{2}mv'^2 + \frac{-GM_E m}{r'} \\ &= \frac{-GM_E m}{r'} \end{aligned}$$

Mechanical Energy Conservation

Since the mechanical energy is conserved, the distance at instant 2 is found with

$$E = E'$$

$$\frac{1}{2}\cancel{m}v^2 + \frac{-GM_E \cancel{m}}{R_E} = \frac{-GM_E \cancel{m}}{r'}$$

$$\frac{1}{2}\cdot\left(5000\tfrac{m}{s}\right)^2 + \frac{-6.6743\times10^{-11}\,\tfrac{Nm^2}{s^2}\cdot 5.9722\times10^{24}\,kg}{6.371\times10^{6}\,m} = \frac{-6.6743\times10^{-11}\,\tfrac{Nm^2}{s^2}\cdot 5.9722\times10^{24}\,kg}{r'}$$

$$r' = 7.962\times10^{6}\,m$$

$$r' = 7962 km$$

This is the distance between the object and the centre of the Earth. As the distance from the surface was sought, the radius of the Earth must be subtracted.

$$\begin{aligned} \text{distance} &= 7962 km - 6371 km \\ &= 1591 km \end{aligned}$$

If the distance had been calculated with mgy, 1250 km would have been obtained. This is 341 km less than the result obtained here. So, the formula $U_g = mgy$ must not be used if the object is moving too far away from the surface of the Earth. The calculation is slightly more difficult with the general formula, but the answer is exact.

Mechanical Energy of an Object in Orbit

From the gravitational energy general formula, a simple formula can be obtained for the mechanical energy of an object in a circular orbit around a planet or a star. The mechanical energy of the object of mass m in orbit around a planet or a star of mass M_c is

$$E_{mec} = \frac{1}{2}mv^2 + \frac{-GM_c m}{r}$$

In orbit, the centripetal force on the object is equal to the gravitational force. This means that

$$\frac{mv^2}{r} = \frac{GM_c m}{r^2}$$

and that

$$mv^2 = \frac{GM_c m}{r}$$

This value is then substituted into the energy equation to get

$$E_{mec} = \frac{1}{2}mv^2 + \frac{-GM_c m}{r}$$
$$E_{mec} = \frac{1}{2}\left(\frac{GM_c m}{r}\right) + \frac{-GM_c m}{r}$$
$$E_{mec} = \left(\frac{1}{2} - 1\right)\left(\frac{GM_c m}{r}\right)$$

Finally, the energy is

Mechanical Energy of an Object of Mass *m* in Orbit Around a Planet or a Star of Mass M_c

$$E_{mec} = -\frac{GM_c m}{2r}$$

The fact that the energy is negative indicates that the object is bound to the planet, that it cannot leave the planet. Indeed, at a great distance from the planet, the gravitational energy is zero, and the kinetic energy must be positive. This means that the mechanical energy is, at least, zero when the object is far from the planet. If the object has a negative total mechanical energy, it cannot go to places where energy is zero, and so the orbiting object cannot go away from the planet.

In order for the orbiting object to leave the planet, energy must be given to the object until its mechanical energy becomes zero (at least). Thus, if the mechanical energy of an object in orbit around the Earth is -1000 J, at least 1000 J must be given to the object so that it can leave the Earth.

Example 9.8.2

How much energy must be given to a 100 kg satellite to place it into a circular orbit 200 km above the Earth's surface? (The mass of the Earth is 5.9722 x 10^{24} kg, and the radius of the Earth is 6371 km.)

The mechanical energy of this system is

$$E_{mec} = \frac{1}{2}mv^2 + \frac{-GM_E m}{r}$$

where M_E is the mass of the Earth. (There are two objects in the system: The Earth and the satellite. As the Earth is at rest, the kinetic energy of the Earth was not included in the formula. However, the Earth is included in the gravitational energy since this energy is the gravitational energy of both the satellite <u>and</u> the Earth.)

At instant 1, the satellite is at rest on the Earth's surface. The energy is then

$$\begin{aligned}
E &= \frac{1}{2}mv^2 + \frac{-GM_E m}{r} \\
&= 0 + \frac{-6.6743\times10^{-11} \frac{Nm^2}{s^2} \cdot 5.9722\times10^{24} kg \cdot 100kg}{6371\times10^3 m} \\
&= -6.2565\times10^9 J
\end{aligned}$$

(Do not use the formula for the mechanical energy of an object in orbit since the satellite is not in orbit at this time. Note also that we have neglected the initial velocity of the satellite due to the rotation of the Earth.)

At instant 2, the satellite is in orbit. The energy is

$$\begin{aligned}
E' &= \frac{-GM_E m}{2r'} \\
&= \frac{-6.6743\times10^{-11} \frac{Nm^2}{s^2} \cdot 5.9722\times10^{24} kg \cdot 100kg}{2\left(6371\times10^3 m + 200\times10^3 m\right)} \\
&= -3.0330\times10^9 J
\end{aligned}$$

Therefore, the energy variation is

$$\begin{aligned}
\Delta E_{mec} &= E' - E \\
&= -3.0330\times10^9 J - -6.2565\times10^9 J \\
&= 3.2235\times10^9 J
\end{aligned}$$

This is the energy that must be given to this satellite to put it into orbit. A more exact formulation of this problem is (since the force made by a rocket is non-conservative)

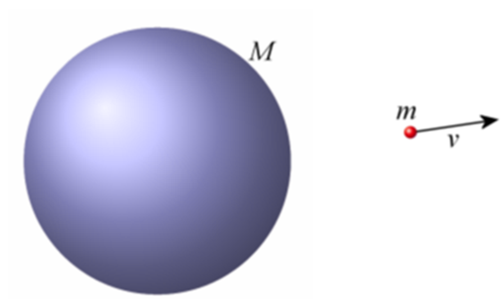
$$E + W_{nc} = E'$$
$$-6.2565\times10^{9}J + W_{nc} = -3.0330\times10^{9}J$$
$$W_{nc} = 3.2235\times10^{9}J$$

This means that a work of 3.2235 x 10^9 J must be done on this satellite by the engine to put it into orbit.

(Note that this energy is the energy that can be obtained by burning about 100 litres of gasoline. This is not a lot. Actually, it takes much more fuel that this because some mechanical energy must also be given to the rocket and the fuel. Moreover, only 100 litres are needed if it is assumed that all the energy released by the burning gasoline goes into mechanical energy whereas much of it actually becomes heat. It takes much fuel to compensate for this loss of energy as heat.)

Escape Velocity

If an object is thrown from a position near a planet or star with enough velocity, it is possible for the object to reach a position very far from the planet without falling back on it. The minimum speed needed to achieve this is called the *escape velocity*.



Initially (instant 1), the object has some speed and is at a distance r from a planet of mass M. The mechanical energy is

$$E = \frac{1}{2}mv^2 + \frac{-GMm}{r}$$

When the object is far away from the planet (instant 2), the energy is

$$E' = \frac{1}{2}mv'^2 + \frac{-GMm}{r'}$$
$$= \frac{1}{2}mv'^2$$

The gravitational energy becomes negligible since r' is very large. If the minimum velocity to get away from the planet is sought, then the object must have its minimum mechanical energy. Looking at mechanical energy at instant 2, it is clear that this minimum energy is obtained if $v' = 0$. Thus, the energy at instant 2 becomes

$$E' = 0$$

The mechanical energy conservation law then gives

$$E = E'$$

$$\frac{1}{2}\cancel{m}v^2 + \frac{-GM\cancel{m}}{r} = 0$$

If this equation is solved for *v,* the escape velocity is obtained.

Escape Velocity Near a Planet with Mass *M*

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

Example 9.8.3

What is the escape velocity of an object initially 1000 km from the Earth's surface knowing that the Earth has a mass of 5.9722 x 10^{24} kg and a radius of 6371 km?

Initially, the distance between the centre of the Earth and the object is

6371 km + 1000 km = 7371 km

Therefore, the escape velocity is

$$\begin{aligned} v_{esc} &= \sqrt{\frac{2 \cdot 6.6743 \times 10^{-11} \frac{Nm^2}{kg^2} \cdot 5.9722 \times 10^{24} kg}{7371 \times 10^3 m}} \\ &= 10.40 \frac{km}{s} \end{aligned}$$

If the object has a speed greater than or equal to 10.40 km/s, it will move very far from Earth and never return. It also means that an object with a speed smaller than 10.40 km/s when it is 1000 km from the Earth's surface has negative mechanical energy. Therefore, the object cannot go at a very great distance where the gravitational energy is zero. (Remember, the object cannot be at the places where the mechanical energy is smaller than *U*.) If it has a speed greater than 10.40 km/s, its mechanical energy is positive and therefore it can go far from Earth where gravitational energy is zero.

Example 9.8.4

What is the escape velocity of an object initially on the surface of the Earth knowing that the Earth has a mass of 5.9722 x 10^{24} kg and a radius of 6371 km? (Neglecting the effect of the atmosphere.)

Initially, the distance between the centre of the Earth and the object is 6371 km.

Therefore, the escape velocity is

$$\begin{aligned} v_{esc} &= \sqrt{\frac{2\cdot 6.6743\times 10^{-11}\frac{Nm^2}{kg^2}\cdot 5.9722\times 10^{24}kg}{6371\times 10^3 m}} \\ &= 11.19\tfrac{km}{s} \end{aligned}$$

This means that an object is launched from the Earth's surface with a speed larger than 11.19 km/s, it will not fall back to Earth. If it is launched with a speed of smaller than 11.19 km/s, it will eventually return to Earth (obviously, air friction is neglected in this calculation).

9.9 ENERGY CONSERVATION

The principle of Energy Conservation

We have seen that mechanical energy is not always conserved. For example, mechanical energy decreases when friction acts on an object that slides on a horizontal surface. Energy is conserved only if there are no non-conservative forces. This is a significant restriction since there are a lot of non-conservative forces.

However, there is an even more general principle of conservation in which there is no problem of non-conservative forces. This is the principle of energy conservation. According to this principle, energy, which encompasses many more forms of energy than mechanical energy, is always a constant.

Principle of Energy Conservation

$$\begin{aligned} &E_k + E_{gravitational} + E_{spring} + E_{thermal} + E_{sound} + E_{light} \\ &\qquad + E_{electric} + E_{chemical} + E_{nuclear} + E_{mass} + \ldots = \text{constant} \end{aligned}$$

Non-conservative internal forces do not change the quantity of energy. When an object slides on a horizontal surface and there is friction, the mechanical energy (here in the form of kinetic energy) decreases. However, the energy remains constant as kinetic energy transforms into other forms of energy, such as heat and sound.

Only one restriction remains. The energy of a system is conserved if the system is isolated. External forces can still change the energy of a system. This means that

$$\Delta E_{tot} = W_{ext}$$

If the energy that appears in the form of heat and light when this car brakes (towards the end of the clip, we see the brakes become all red) were to be measured

https://www.youtube.com/watch?v=NBoMTYZAyJc
it would be equal to the kinetic energy lost by the car.

If the energy that appears in the form of heat, sound, light and kinetic energy of the breath in this explosion were to be measured
https://www.youtube.com/watch?v=9vFmSOQ93as
it would be equal to the chemical energy lost by molecules.

The formulas needed for calculating these other forms of energy will not be given in this mechanics course, but some of them will be seen in future physics courses.

Example 9.9.1

Harold lands on an airfield with his Cessna 172 at a speed of 30 m/s. The Cessna, with everything inside, has a mass of 1000 kg. How much energy goes in heat in the brakes (if the energy losses due to air friction are neglected)?

> When a Cessna lands, the kinetic energy of the aircraft must be transformed into another form of energy. The friction in the brakes will turn much of that energy into heat in the brakes and air friction will turn a small portion of the kinetic energy of the plane into kinetic energy of the air and heat in the air. If the losses due to the air friction are neglected, only the heat in the brakes remains. Thus, all the kinetic energy of the aircraft will go into heat in the brakes.
>
> The kinetic energy of the aircraft is
>
> $$\begin{aligned} E_k &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}\cdot 1000kg\cdot\left(30\tfrac{m}{s}\right)^2 \\ &= 450\,000J \end{aligned}$$
>
> As this energy turns into heat in the brakes, the thermal energy of the brakes after braking will be 450 000 J.

Note that, according to this calculation, the brakes will receive this energy regardless of the braking distance. In reality, the thermal energy in the brakes decreases if the braking distance is extended because the air circulating in the brakes will have more time to remove some of that heat and cool them. In addition, a longer braking time will increase the amount of energy removed by the drag force acting on the aircraft.

The brakes of a Cessna can easily receive these 450 000 J. This thermal energy will increase the temperature of the brakes, but this temperature increase will not be catastrophic. However, there is a limit to the energy that the brakes can receive. A 320-ton Boeing 777-300ER landing at a speed of 73 m/s has a kinetic energy of 850 million joules

(850 000 000 J = 850 MJ). If the brakes were to take all this energy, then each of the 12 braking systems would receive about 70 MJ of thermal energy. That's a lot of energy. See what happens in this test where a brake system receive 125 MJ.
https://www.youtube.com/watch?v=qew09gao3S8
Can each of the Boeing's braking systems receive 70 MJs? No. Such an amount of energy would lead to permanent damage to brakes, tires and wheels (the acceptable limit is about 45 MJ). Fortunately, the passage of air through the brakes removes some of the heat. Under normal braking, only 40% of the energy received in each braking system will remain at the end of braking. Thus, only 28 MJ of thermal energy (40% of 70 MJ) will remain in each of the Boeing 777 braking system at the end of braking. This energy level is now acceptable, but it will take about 1 hour for the brakes to cool down enough to take off again. To further reduce the energy received, and thus reduce cooling time and brake wear, a whole series of means are added to slow down the aircraft. Air brakes and thrust reversers in the engines transfer some of the aircraft's energy to the air.

Sometimes amounts of energy received are a little too great. This happens during an aborted take-off. A Boeing 777 takes off at 80 m/s and its weight is greater than on landing. Thus, its kinetic energy can reach 1200 MJ just before leaving the track. If take-off is aborted close to take-off speed, each of the 12 braking systems would then receive 100 MJ of energy. The passage of air through the brakes again removes some of this heat, but only 35% is eliminated in this case (since they are at maximum braking, and this reduces the braking time). At the end of braking, each braking system could therefore have up to 65 MJ of thermal energy (air brakes and thrust reversers are not used to help brake the aircraft during an aborted take-off). At this energy level, there will be most likely permanent damage to brakes, tires and wheels. This video shows the landing gear of a Boeing 777 after an aborted take-off test. Clearly, the energy received is beyond the acceptable maximum.
https://www.youtube.com/watch?v=Mr4V680UQ-k

History of the Principle

The discovery of the principle of conservation of energy is interesting because it is a wonderful example of simultaneous discovery.

By the middle of the 19th century, a significant number of observations had been accumulated showing the equivalences between different phenomena. For example, a link between currents and the heat produced by them had been found and a link between mechanical work and the heat produced by this work had been found. To arrive at the principle of conservation of energy, someone had to take a step back and consider all the transformations observed to realize that there was a more general principle. Around 1840, a sufficient number of experiments showing these links were available, and someone just had to make a synthesis of all these experiments.

Thus, at least 12 scientists (Sadi Carnot, Marc Séguin, Karl Holtzmann, Gustave-Adolphe Hirn, William Grove, Michael Faraday, Ludwig Colding, Karl Friedrich Mohr, Justus von

Liebig, James Prescott Joule, Julius Robert von Mayer and Hermann von Helmholtz) stated, with varying degrees of perfection, the principle of conservation of energy between 1839 and 1850. Of these, Helmholtz probably presented the most complete version of this principle in 1847.

If you want to know more about the history of the discovery of the law of energy conservation, click on this link.
http://physique.merici.ca/mechanics/discoveryE.pdf

However, it was difficult to prove theoretically that the principle was true. The proof of energy conservation was given by Emily Noether in 1918. She then showed that if the laws of physics do not change over time, then the energy must be conserved. In fact, the principle discovered by Noether is broader than that since it shows that for each symmetry (the time invariance here); there is a conserved quantity (the energy here).

Why Is the Term “Potential Energy” Used?

Bernoulli uses the term “energy” in 1717 for work. In 1807, Thomas Young proposes to use it instead for the quantity mv^2 (which bore the name *vis viva* then). In Greek, “energy” refers to an activity, a motion, and it seems appropriate to use it for a quantity which depends on the speed. The term was, however, little used then. Its use spread only in 1847 when Helmholtz proposes to use it for the conserved quantity he had just discovered.

In 1853, Rankine classified the energies into 2 categories.

1) *Actual energy*

   All forms of energy related to motion fell into this category: kinetic energy, light energy, the energy associated with electric currents…

2) *Potential energy*

   All forms of energy that do not depend on motion fall in this category: gravitational energy, energy of a gas and energy of a compressed spring…

As the term “energy” was associated with motion, only the forms of energy in which there is a motion were considered as actual energies. The other forms of energy were, therefore, called “potential energy” because they had the potential to be transformed into actual energy, such as gravitational energy which becomes kinetic energy for a free-falling object.

This classification, quite useless, ceased to be used relatively quickly, and no one uses the term “actual energy” now. Curiously, the term “potential energy” has survived until today even though it has lost its meaning. Many students try to understand what is so potential in “gravitational potential energy”. They have good reason to be puzzled because there is actually nothing “potential” about gravitational energy. It is as much a form of energy as is kinetic energy and other forms of energy.

In fact, potential energy is a form of energy that depends on the forces acting on the objects in the system and so on the position of the objects relative to each other. For example, the gravitational energy depends on the height of the mass and the spring energy depends on the stretching of the spring. The potential energies are, therefore, dependent on the system configuration, which is why it would be more appropriate to call U "configuration energy" rather than "potential energy". However, since everyone uses the name potential energy for U, we will keep this terminology to avoid any confusion. However, we will sometimes slightly simplify by omitting the word "potential" in the expressions "gravitational potential energy" and "spring potential energy".

What Is Energy?

Even for a physicist, this is a hard question to answer. In 1961, Richard Feynman, Nobel Prize in Physics, summarizes what is known about energy.

> *There is a fact, or if you wish, a law, governing all natural phenomena that are known to date. There is no known exception to this law—it is exact so far as we know. The law is called the conservation of energy. It states that there is a certain quantity, which we call energy, that does not change in the manifold changes which nature undergoes. That is a most abstract idea, because it is a mathematical principle; it says that there is a numerical quantity which does not change when something happens. It is not a description of a mechanism, or anything concrete; it is just a strange fact that we can calculate some number and when we finish watching nature go through her tricks and calculate the number again, it is the same.*

No one really knows what energy is, but we know that its value remains constant and that this can help us to predict the result of certain transformations.

SUMMARY OF EQUATIONS

Work Done from Potential Energy

$$W = -\Delta U$$

Condition for the Potential Energy of 1-Dimensional Force to Exist

The force must depend only on the position.

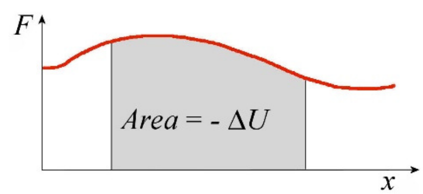
Relationship between Potential Energy and Force in One Dimension

$$F = -\frac{dU}{dx} \qquad \text{and} \qquad U = -\int F dx$$

Potential Energy Variation from Force in One Dimension

$$\Delta U = -\int_{x}^{x'} F_x dx$$

The change in potential energy of an object is minus the area under the curve of the force acting on the object as a function of the position.



Components of *F* From *U* in Two Dimensions

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y}$$

***U* From the Components of *F* in Two Dimensions**

$$U = -\int F_x dx \qquad U = -\int F_y dy$$

Condition for the Existence of the Potential Energy of a 2-Dimensional Force

The force must depend only on the position

and

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

Link between the Potential Energy and the Force for a Central Force

$$F = -\frac{dU}{dr} \qquad \text{and} \qquad U = -\int F dr$$

Link between *U* and the Components of a 3-Dimensional Force

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z}$$

$$U = -\int F_x dx \qquad U = -\int F_y dy \qquad U = -\int F_z dz$$

Potential Energy of a System

$$U = U_g + U_{sp}$$

Gravitational Energy Near the Surface of the Earth

$$U_g = mgy$$

Gravitational Energy (General Formula)

$$U_g = -\frac{Gm_1m_2}{r}$$

Spring Energy

$$U_{sp} = \frac{1}{2}kx^2$$

Mechanical Energy

$$E_{mec} = E_k + U$$

Law of Conservation of Mechanical Energy

$$\Delta E_{mec} = 0$$

$$E_{mec} = E'_{mec}$$

$$E_{mec} = \text{constant}$$

If there is no work done by external forces
if the internal work is done exclusively by conservative forces.

Link between Angle and Height for a Pendulum

$$y = L(1-\cos\theta) \qquad \cos\theta = \frac{L-y}{L}$$

The Mechanical Energy with External Forces or Non-Conservative External Forces

$$\Delta E_{mec} = W_{nc} + W_{ext}$$

$$E_{mec} + W_{nc} + W_{ext} = E'_{mec}$$

Equation to Solve to Obtain the Time of Displacement from *U*

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\left(E_{mec} - U\right)}$$

Mechanical Energy of an Object of Mass *m* in Orbit Around a Planet or a Star of Mass M_c

$$E_{mec} = -\frac{GM_c m}{2r}$$

Escape Velocity Near a Planet with Mass *M*

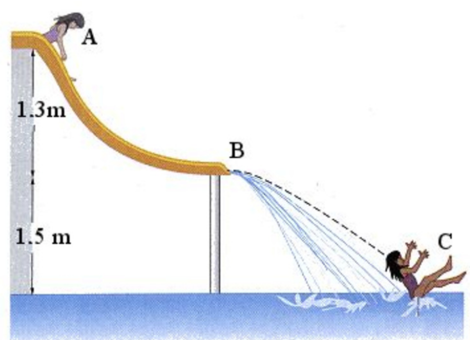
$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

EXERCISES

9.1 Gravitational Energy

1. Huguette, whose mass is 30 kg, goes down the slide shown in the diagram.

    a) What is Huguette's gravitational energy change when she moves from point A to point B?
    b) What is Huguette's gravitational energy change when she moves from point A to point C?



www.chegg.com/homework-help/questions-and-answers/height-water-slide-figure-8-26-h-30-m-person-s-initial-speedat-point-055-m-s-thenew-horizo-q109994

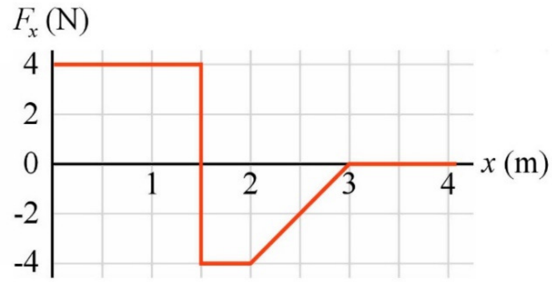
9.2 Conservative Forces

2. The following force acts on an object moving along the x-axis.

$$F_x = 2\tfrac{N}{m^3} \cdot x^3 + 2N$$

    a) What is the formula of the potential energy associated with this force?
    b) What is the change in the potential energy associated with this force if the object moves from x = -2 m to x = 5 m?
    c) What is the work done by this force if the object moves from x = -2 m to x = 5 m?

3. What is the variation of potential energy of an object moving from x = 0.5 m to x = 3 m if the force on the object is given by this graph?



4. Determine whether the following forces are conservative or not.

a) $\vec{F} = 3\frac{N}{m^2}\cdot x^2\vec{i} + 3\frac{N}{m^2}\cdot y^2\vec{j}$

b) $\vec{F} = \left(3\frac{N}{m^3}\cdot x^2 y + 1\frac{N}{m}\cdot x\right)\vec{i} + \left(3\frac{N}{m^3}\cdot xy^2 + 1\frac{N}{m}\cdot y\right)\vec{j}$

c) $\vec{F} = \left(3\frac{N}{m^3}\cdot xy^2 + 1\frac{N}{m}\cdot y\right)\vec{i} + \left(3\frac{N}{m^3}\cdot x^2 y + 1\frac{N}{m}\cdot x\right)\vec{j}$

5. The potential energy of an object is given by the following formula.

$$U = 4\tfrac{J}{m^2}\cdot x^2 + 2\tfrac{J}{m}\cdot(x+y) - 3\tfrac{J}{m^2}\cdot xy$$

What is the force exerted on the object when it is at the position x = 1 m and y = 2 m?

6. The conservative force acting on an object is given by

$$\vec{F} = \left(-4\tfrac{N}{m^3}\cdot xy^2 + 2\tfrac{N}{m}\cdot x\right)\vec{i} + \left(-4\tfrac{N}{m^3}\cdot x^2 y + 4\tfrac{N}{m}\cdot y\right)\vec{j}$$

a) What is the formula of the potential energy associated with this force?
b) What is the potential energy of the object when it is at the position (2 m, 1 m) (Use 0 for the constant in U)?
c) What is the potential energy of the object when it is at the position (5 m, 2 m) (Use 0 for the constant in U)?
d) What work is done by this force on an object that moves from the point (2 m, 1 m) to the point (5 m, 2 m)?

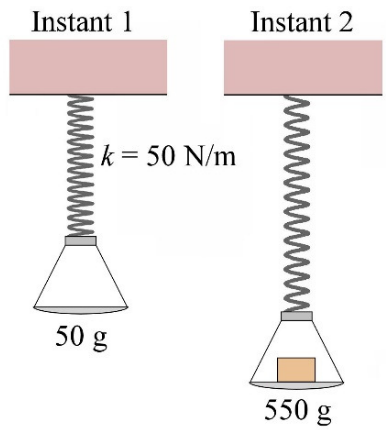
7. The potential energy of an object is given by the following formula.

$$U = 4\tfrac{J}{m^2}\cdot x^2 + 2\tfrac{J}{m}\cdot(x+y+z) - 3\tfrac{J}{m^3}\cdot xyz$$

What is the force exerted on the object when it is at the position x = 1 m, y = 2 m and z = - 4 m?

8. A spring with a constant of 50 N/m is used to suspend a 50 g tray. A 500 g mass is put on the tray, which extends the spring, as shown in the diagram.

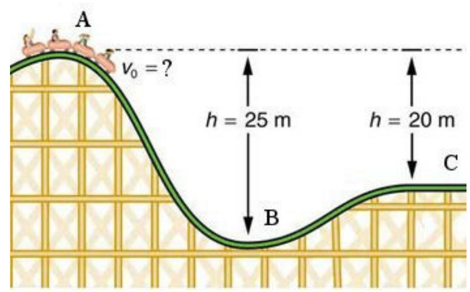
   a) What is the spring energy at instant 1?
   b) What is the spring energy at instant 2?
   c) What is the work done by the spring between instants 1 and 2?



9.3 Mechanical Energy Conservation

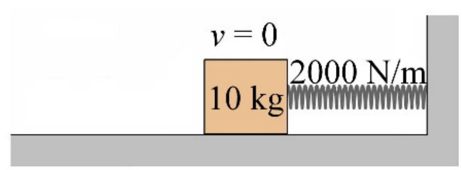
9. A 2000 kg (including passengers) roller coaster car moves on the track shown in the diagram. At point B, the speed of the car is 25 m/s. There is no friction.

   a) What is the speed of the car at point A?
   b) What is the speed of the car at point C?



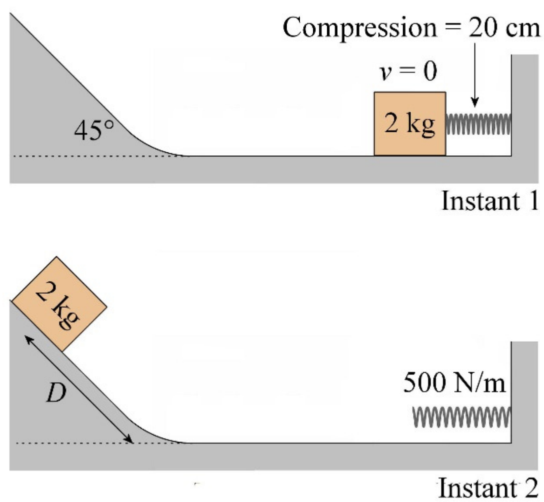
cnx.org/content/m42148/latest/?collection=col11406/latest

10. In the following situation, the spring is initially compressed 20 cm, and the block is initially at rest. There is no friction between the block and the ground. The block is then released.

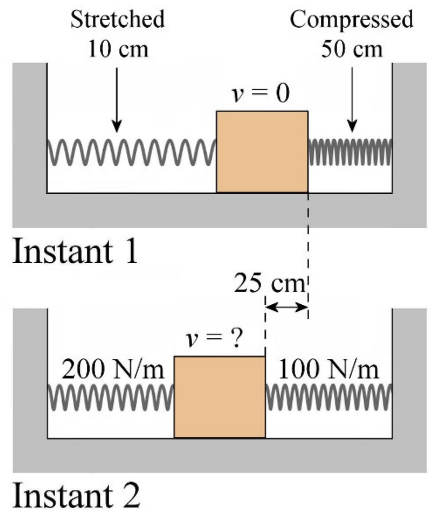


   a) What is the speed of the block when the spring is only compressed 5 cm?
   b) What is the speed of the block once it left the spring?

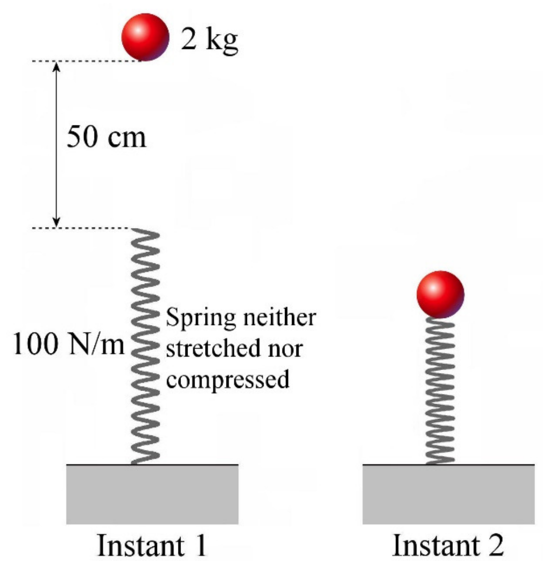
11. In the following situation, how far will the 2 kg block travel on the slope before stopping (D in the diagram) if there is no friction between the block and the surface?



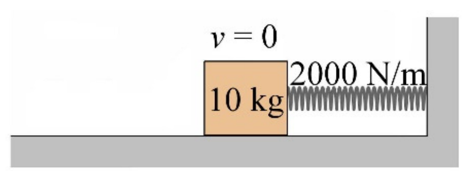
12. In the following situation, what is the speed of the 5 kg block at instant 2 if there is no friction between the ground and the block?



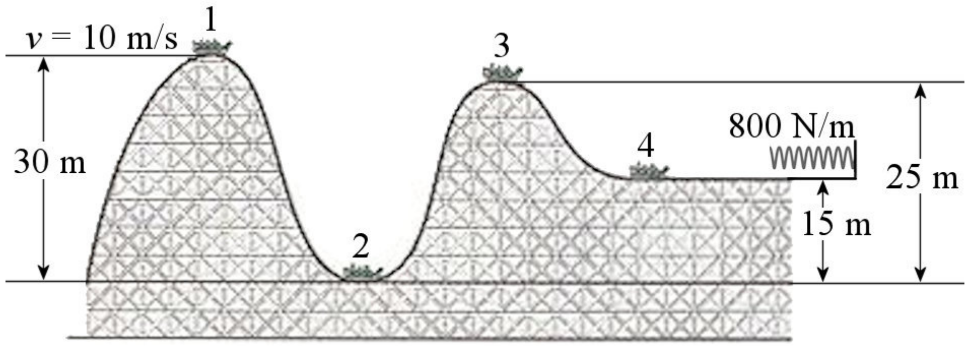
13. A 2 kg ball initially at rest falls on a spring as shown in the diagram. What will the maximum compression of the spring be (*d* in the diagram)? There is no friction.



14. In the following situation, the spring is initially compressed 20 cm, and the block is initially at rest. There is no friction between the block and the floor. The block is then released. What is the compression of the spring when the kinetic energy of the block is equal to the spring energy?



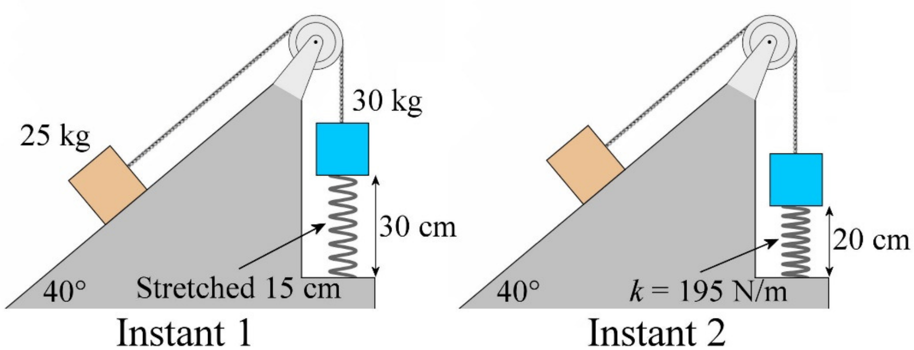
15. A 2000 kg (including passengers) roller coaster car follows the track shown in the diagram. At the end of the track, the car is stopped by a spring. There's no friction here.



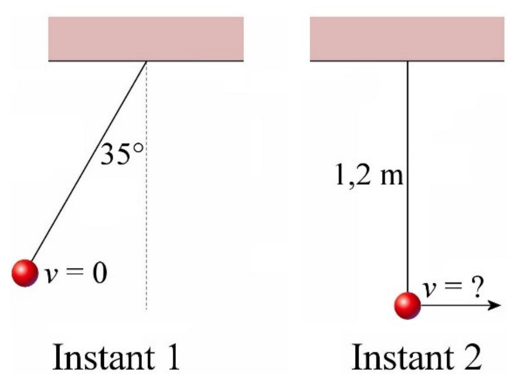
miretia.tistory.com/?page=2

a) What will the maximum speed reached by the car be?
b) What will the maximum spring compression be when the car hits the spring?
c) What will the speed of the car be when the kinetic energy of the car is equal to twice the energy of the spring?

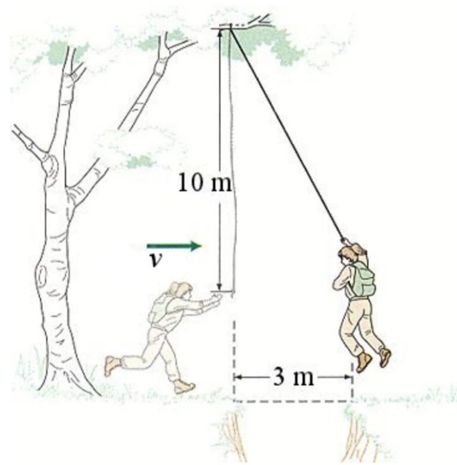
16. Two blocks are attached to a spring stretched 15 cm as shown in the diagram on the left (instant 1). The blocks are then released. What will the speed of the blocks be when the 30 kg block is 10 cm lower (instant 2)? There's no friction here.



17. At instant 1, the speed of a 4 kg pendulum bob is zero. What will its speed be at instant 2? There is no friction.



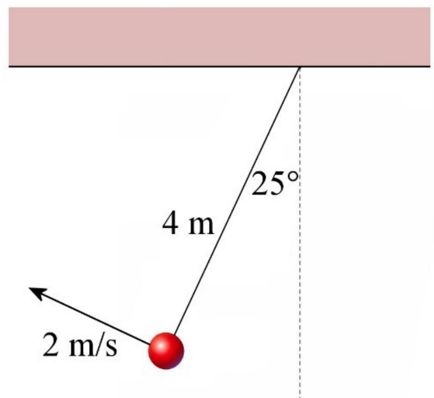
18. Radu wants to cross a ravine by holding on to a rope hanging from a tree. When the rope reaches its maximum angle with the vertical, Radu will simply let himself drop vertically. With what minimum speed must Radu run to avoid falling to the bottom of the ravine? There is no friction.



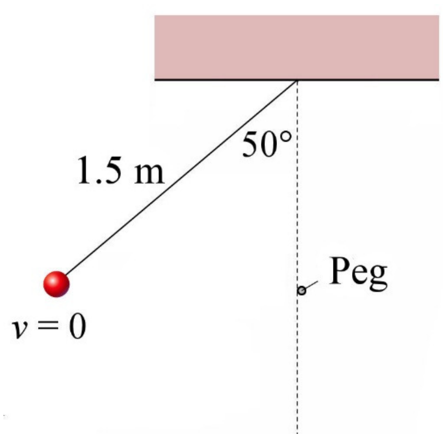
www.chegg.com/homework-help/questions-and-answers/physics-archive-2009-march-12

19. A 2 kg pendulum bob has a speed of 2 m/s when the pendulum makes an angle of 25° with the vertical. There is no friction.

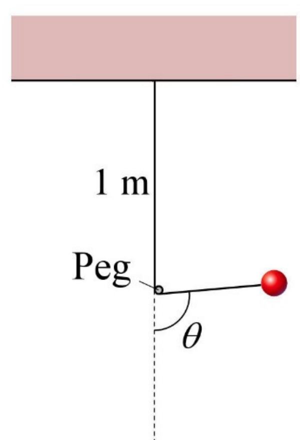
a) What will the maximum speed of this pendulum bob be?
b) What will the maximum angle between the rope and the vertical line be?



20. A pendulum starts its motion at the position shown in the diagram on the left (instant 1). When the rope becomes vertical, it hits a small peg, which gives the motion shown in the illustration to the right (instant 2). What is the maximum value of the angle θ in the diagram to the right? There is no friction.



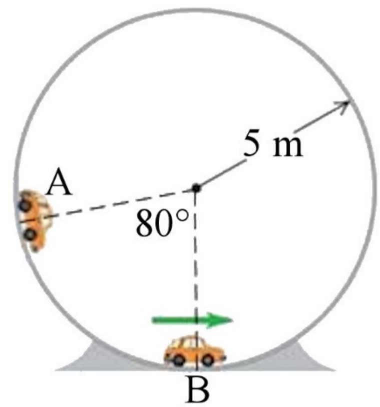
Instant 1



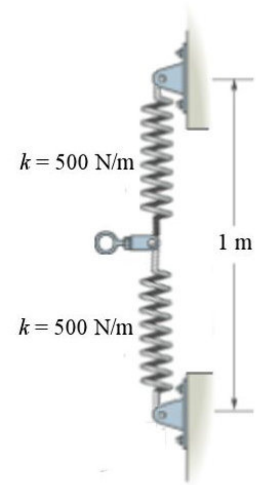
Instant 2

21. At point A, the car is at rest. What is the number of g's experienced by someone in the car at point B? (There's no engine in the car. Only the gravitational force acts on it and there is no friction.)

www.chegg.com/homework-help/questions-and-answers/physics-archive-2011-october-07

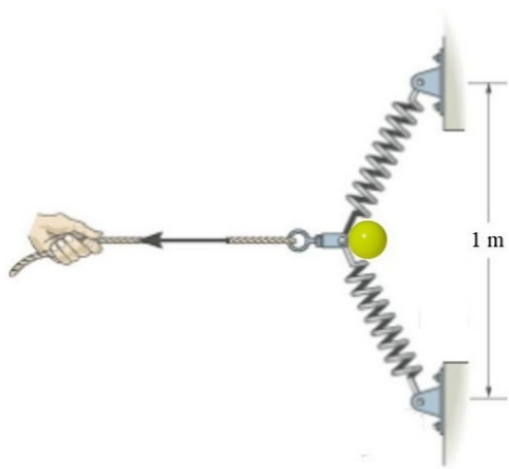


22. In this configuration, the springs are neither stretched nor compressed. (This situation is seen from above.)



www.chegg.com/homework-help/questions-and-answers/springs-ab-bc-stiffness-k-unstretched-length-l-determine-displacement-d-cord-wall-force-f--q5863675

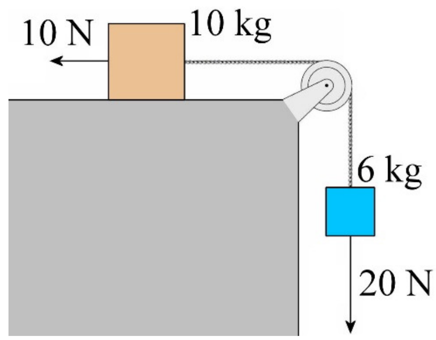
The ring is then pulled back 30 cm with a rope, and 100 g ball is placed between the two springs.



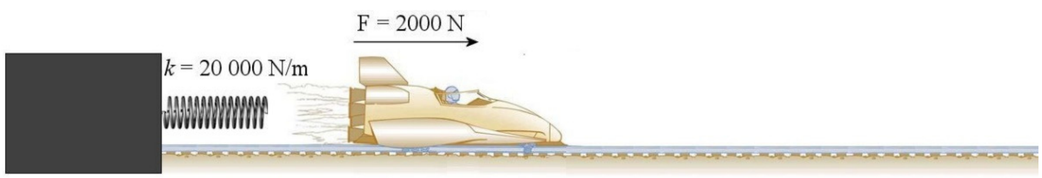
The rope is then released, and the ball is projected. If there is no friction, what is the final speed of the ball?

9.4 Mechanical Energy when There Are External Forces or Non-Conservative Forces

23. In the situation shown in the diagram, the blocks are initially at rest and there is no friction. There are, however, two forces acting on the blocks. What will the speed of the blocks be after they have moved 2 metres?



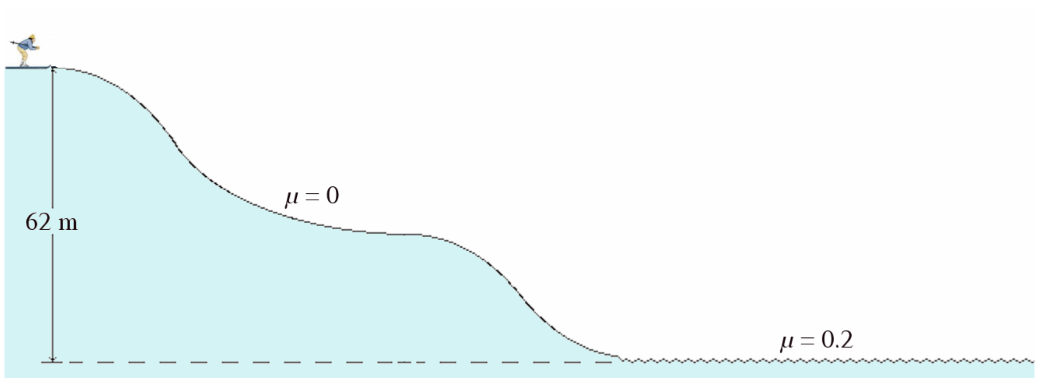
24. To help a 500 kg rocket sled achieve faster speeds, a compressed spring is used to propel it at the beginning of its motion. Initially, the spring was compressed 3 m. The engines of the rocket sled (exerting a 2000 N force) are started at the same time the sled is released.



cnx.org/content/m42073/latest/?collection=col11406/latest

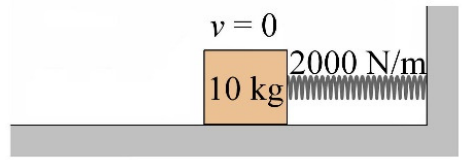
What will the speed of the sled be after a 50 m displacement if there is no friction?

25. Archibald skis downhill on a slope. His initial velocity is zero. After a 62 m drop of altitude, he arrives on a flat section. There is no friction on the slope whereas the coefficient of kinetic friction between the ground and Archibald's skis is 0.2 on the flat section. How far will Archibald travel on the flat section before stopping?

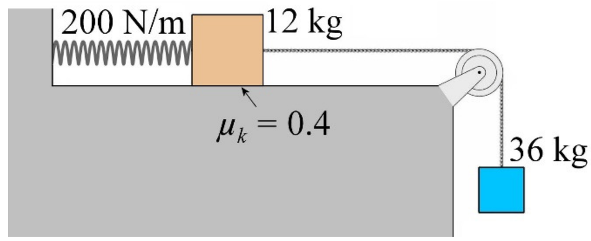


www.physics.ucdavis.edu/Classes/Physics7/QPS/QP2.HTML

26. A 10 kg block is pushed by a spring with a constant of 2000 N/m compressed 50 cm. When the block is released, it travels 42 m before stopping. What is the coefficient of friction between the block and the floor?

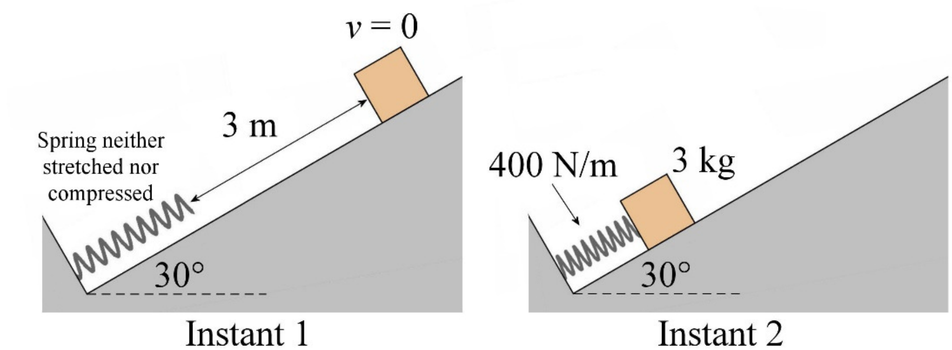


27. In the situation shown in the diagram, the spring is neither stretched nor compressed initially.

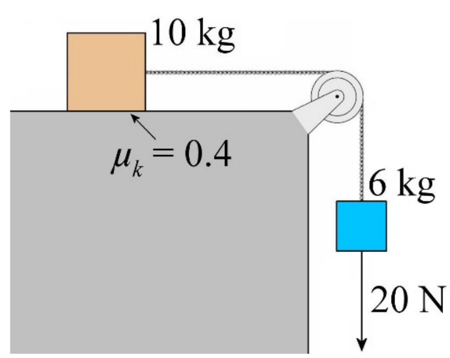


a) What will the speed of the blocks be after a 1 m displacement if they were initially at rest?
b) What will the speed of the blocks be after a 3.2 m displacement if they were initially at rest?

28. In the following situation, the block is initially at rest and the coefficient of kinetic friction between the block and the slope is 0.2. What will the maximum compression of the spring be?

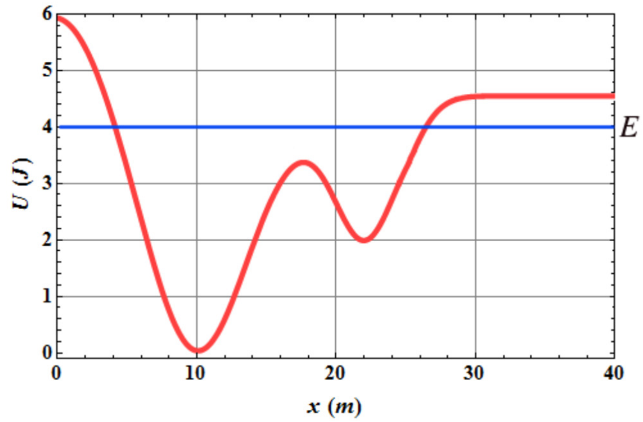


29. In the situation shown in the diagram, the blocks are initially at rest. There is a force acting on the 6 kg block, and there is friction acting on the 10 kg block. What will the speed of the blocks be after the 6 kg block has moved 2 metres downwards?



9.6 Graph of Potential Energy as a Function of Position

30. Here is a graph showing the potential energy of a 2 kg object with a mechanical energy of 4 J.



www.rakeshkapoor.us/ClassNotes/PotentialEnergy.html

Approximate the answers to the following questions.

a) Where cannot the object be?
b) Where is the object when its velocity is the largest?
c) What is the speed of the object when it is at $x = 20$ m?
d) Where could the object be in stable equilibrium and what should be the mechanical energy of the object to be in equilibrium at these locations?
e) Where could the object be in unstable equilibrium and what should be the mechanical energy of the object to be in equilibrium at this location?
f) What is the work done by the force associated with this potential energy when the object moves from $x = 10$ m to $x = 20$ m?
g) Sketch the graph of the force acting on this object as a function of the position?

31. The potential energy of an object is given by the following formula.

$$U = 2\tfrac{J}{m^3}\cdot x^3 + 3\tfrac{J}{m^2}\cdot x^2 - 72\tfrac{J}{m}\cdot x + 36J$$

Find out where the object can be in equilibrium and determine if the equilibrium is stable, neutral or unstable.

9.7 Time of displacement from *U*

32. The potential energy of a 2 kg object is given by the following formula.

$$U = 9\tfrac{N}{m^3}x^4$$

How long will it take for the object to go from $x = 0$ to $x = 1$ m if the mechanical energy is 144 J?

(Use Wolfram to make the integral: https://www.wolframalpha.com/)

33. The potential energy of a 2 kg object is given by the following formula.

$$U = 2N \cdot x$$

Find the formula that gives the position of the object as a function of time if the mechanical energy of the object is 16 J, knowing that the object is at $x = 0$ and that its speed is positive at $t = 0$.

9.8 General Formula of the Gravitational Energy

In this section, use the following data.

Earth

Mass of the Earth = 5.9722×10^{24} kg
Radius of the Earth = 6371 km
Distance between the Earth and the Sun = 149 600 000 km

Moon

Mass of the Moon = 7.346×10^{22} kg
Radius of the Moon = 1737 km
Distance between the Earth and the Moon = 384 400 km

34. Neil, whose mass is 100 kg, is 400 000 km from the Earth's surface. He falls towards Earth starting from rest. (For this problem, suppose there is no atmosphere.)

a) How far will Neil be from the surface of the Earth when its speed is 5000 m/s?
b) How fast will he hit the surface of the Earth?

35. What is the escape velocity on the surface of the Moon?

36. An object launched upwards from the surface of the Moon rises to an altitude of 3000 km. How fast was this object launched?

37. A 350 kg satellite is in a circular orbit at an altitude of 500 km above the Earth's surface. What work must be done by a rocket on this satellite initially at rest on the Earth's surface to put it into this orbit?

38. To place a satellite in orbit, energy must be given to it. Part of this energy is in the form of kinetic energy (since the satellite must move quickly in orbit) and a part is in the form of gravitational energy (since the satellite is farther away from the Earth).

a) What percentage of the given energy is in the form of kinetic energy if the satellite is placed in low orbit (150 km altitude)?
b) What must be the radius of the orbit so that 50% of the given energy is in the form of kinetic energy and 50% in the form of gravitational energy?

39. In an old television series called *Cosmos 1999*, a nuclear explosion on the surface of the Moon allowed the Moon to escape from its orbit around the Earth and move freely in the universe (which enabled them to meet different kinds of aliens in each episode).

a) Assuming that all the energy of the explosion was transferred to the Moon, what should be the minimum energy released by the blast so that the Moon can escape from the Earth?
b) Knowing that the explosion of Hiroshima's atomic bomb has released 6.3×10^{13} J of energy, determine how many Hiroshima's atomic bombs have to be detonated to make such an explosion that frees the Moon from the Earth.

www.oocities.org/area51/jupiter/1630/

40. How much work must be done by the engine of a 100 kg probe to move it from a position at rest on the Earth's surface (on the side where the Moon is) to a position at rest on the surface of the Moon (on the side where is the Earth is)?

41. A 500 kg satellite is orbiting at an altitude of 1000 km above the surface of the Earth.

a) What is its mechanical energy?
b) What is its gravitational energy?
c) What is its kinetic energy?
d) How much energy must be given to the satellite so that it can escape from the Earth?

Challenges

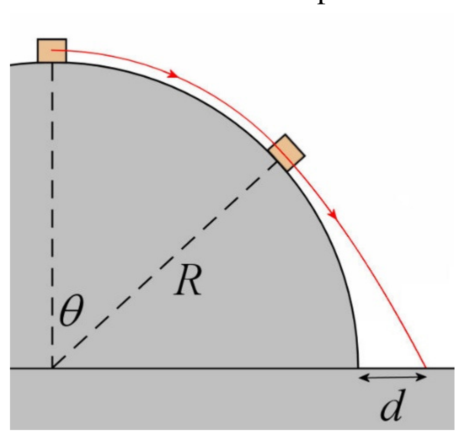
(Questions more difficult than the exam questions.)

42. The force acting on an object is given by

$$\vec{F} = \left(5\tfrac{N}{m^2}\cdot yz - 2\tfrac{N}{m}\cdot x + 2\tfrac{N}{m}\cdot y\right)\vec{i} + \left(5\tfrac{N}{m^2}\cdot xz - 4\tfrac{N}{m}\cdot y + 2\tfrac{N}{m}\cdot x\right)\vec{j} + \left(5\tfrac{N}{m^2}\cdot xy + 6\tfrac{N}{m}\cdot z\right)\vec{k}$$

a) Verify that this force is conservative.
b) What work is done by this force on an object that moves from the point (1 m, 1 m, 0 m) to the point (4 m, -2 m, 3 m)?

43. A person slides from the top of a hemisphere with a radius of 5 m. The speed at the top of the hemisphere is very small (but not zero otherwise the person would remain at the top) and there is no friction between the person and the surface. He will slide along the hemisphere for a while, but at some point, he will fly off of the hemisphere.



a) At what angle θ does the person leave the sphere?
b) At what distance d does the person land away from the edge of the sphere?

If you want to visualize what's happening, here's a quick simulation.
http://physique.merici.ca/mecanique/slideoffsphere.mpg

ANSWERS

9.1 Gravitational Energy

1. a) -382.2 J b) 823.2 J

9.2 Conservative Forces

2. a) $U = -\frac{1}{2}\frac{N}{m^3}x^4 - 2N\cdot x + Cst$ b) -318.5 J c) 318.5 J

3. -2 J
4. a) yes b) no c) yes
5. $\vec{F} = \left(-4\vec{i} + 1\vec{j}\right)N$
6. a) $U = 2\frac{N}{m^2}x^2y^2 - 1\frac{N}{m}x^2 - 2\frac{N}{m}y^2 + Cst$ b) 2 J c) 167 J d) -165 J
7. $\vec{F} = \left(-34\vec{i} - 14\vec{j} + 4\vec{k}\right)N$
8. a) 0.0024 J b) 0.2905 J c) -02881 J

9.3 Mechanical Energy Conservation

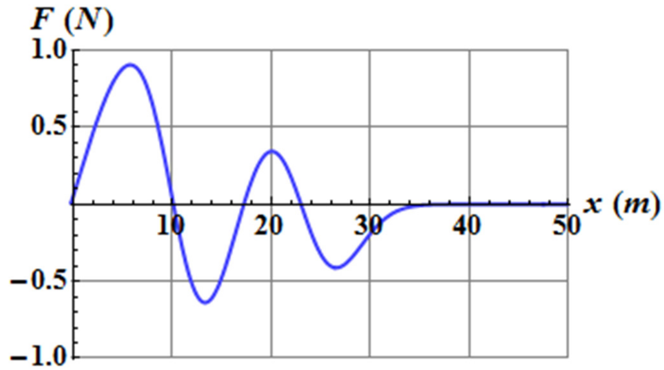
9. a) 11.62 m/s b) 22.96 m/s
10. a) 2.739 m/s b) 2.828 m/s
11. 72.15 cm
12. 1.803 m/s
13. 68.01 cm
14. 14.14 cm
15. a) 26.23 m/s b) 31.38 m c) 16.21 m/s
16. 0.7532 m/s
17. 2.062 m/s
18. 3.005 m/s
19. a) 3.368 m/s b) 31.2°
20. 94.1°
21. 2.653
22. 8.31 m/s

9.4 Mechanical Energy when There Are External Forces and Non-Conservative Forces

23. 4.147 m/s
24. 27.568 m/s
25. 310 m
26. 0.06074
27. a) 2.928 m/s b) impossible
28. 40.44 cm
29. 3.146 m/s

9.6 Graph of Potential Energy as a Function of Position

30. a) $x < 4$ m et $x > 26$ m b) $x = 10$ m c) 1.2 m/s
    d) Stable equilibrium at $x = 10$ m. The mechanical energy must be 0.2 J
    Stable equilibrium at $x = 22$ m. The mechanical energy must be 2 J
    e) Unstable equilibrium at $x = 17$ m. The mechanical energy must be 3.4 J
    f) -2.4 J
    g)



www.rakeshkapoor.us/ClassNotes/PotentialEnergy.html

31. Stable equilibrium at x = 3 m and unstable equilibrium at x = – 4 m.

9.7 Time of displacement from *U*

32. 0.08387 s
33. $x = 8m - \frac{1}{2}\frac{m}{s^2}(t-4s)^2$ or $x = -\frac{1}{2}\frac{m}{s^2}t^2 + 4\frac{m}{s}t$ (They are equivalent)

9.8 General Formula of Gravitational Energy

34. a) 23 197 km b) 11 098 m/s
35. 2376 m/s
36. 1891 m/s
37. 1.175 x 10^{10} J
38. a) 95.5% b) 9556,5 km
39. a) 3.809 x 10^{28} J b) 6.046 x 10^{14} Hiroshima bombs!
40. 5.871 x 10^{9} J
41. a) -1.352 x 10^{10} J b) -2.704 x 10^{10} J c) 1.352 x 10^{10} J d) 1.352 x 10^{10} J

Challenges

42. a) The force is conservative because it respects all 3 conditions. b) 186 J
43. a) 48.19° b) 3.023 m