2 KINEMATICS IN TWO OR THREE DIMENSIONS

On May 24, 1941, the German battleship Bismarck sank the British battlecruiser HMS Hood by a direct hit to the ammunition chamber with an 800 kg shell. Surprisingly, the Bismarck was 14 km from the Hood at that time. What was the launch angle of the shell knowing that its initial velocity was 820 m/s?

Discover the answer to this question in this chapter.

2.1 COMPONENTS OF A VECTOR

In 2 or 3 dimensions, several quantities are represented by vectors. A quantity is represented by a vector if its direction is important. This is the case for displacement, velocity and acceleration.

We will essentially work with the components of the vectors. It is therefore necessary to be able to find the components of a vector from its magnitude and direction or to be able to find the magnitude and direction of the vector from the components.

Components from the Vector

The diagram on the right illustrates what are the x and y components of a two-dimensional vector F.

Although the displacement, velocity and acceleration vectors that we will see in this chapter are 3-dimensional vectors, we will almost always work in two dimensions. This is why we will only look at the components of 2-dimensional vectors.
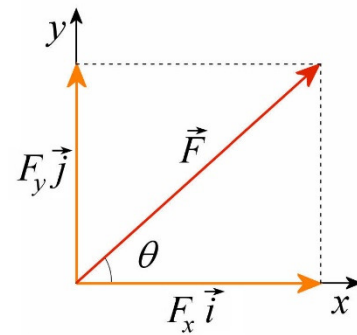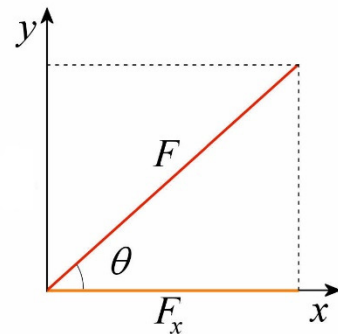
The components can easily be found with trigonometric functions.

For the x-component, a triangle is formed, as shown in the diagram.

The x-component is, therefore, found with
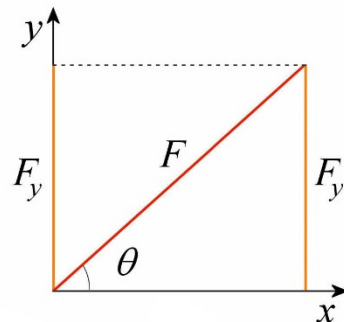
$$\cos\theta = \frac{F_x}{F}$$

$$F_x = F\cos\theta$$

For the y-component, a triangle is also formed, as shown in the diagram.

The y-component is thus found with

$$\sin\theta = \frac{F_y}{F}$$

$$F_y = F\sin\theta$$

To resolve into components, the angle between the force and the **positive x-axis** must be found. With this angle, those formulas always give the good signs, and it is never necessary to add a negative sign to get the right component. Of course, there are other ways to find the components, and you can use them if you are comfortable with these methods. However, a large proportion of errors in problem-solving with Newton's laws comes from a faulty resolution of forces into components. By always using the angle between the force and the positive x-axis, you will surely avoid these errors. Here's a good trick to find the angle if you're not sure: draw the force vector and the positive x-axis starting from the same point and find the angle between these two vectors.

There is only one danger; the angle may be negative! The rule is: the positive direction is always in the direction of rotation from the positive x-axis towards the positive y-axis. The diagram on the right shows you the positive and negative directions of rotation in two situations.
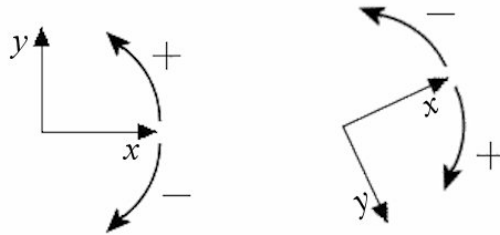
The sign of the angle is found by finding the direction of the rotation when <u>going from the positive x-axis towards the force</u>. If the rotation is in the positive direction, the angle is positive and if the rotation is in the negative direction, the angle is negative. Here are some examples for the values of the angles (with the correct sign).
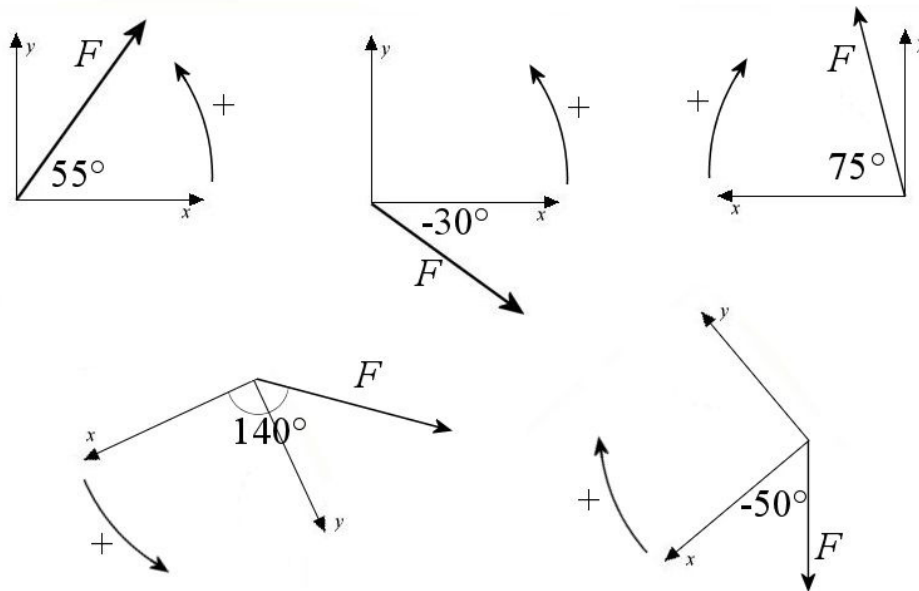
Example 2.1.1

What are the components of this vector of length 8?

To find the components, we need the angle between the vector and the positive *x*-axis.
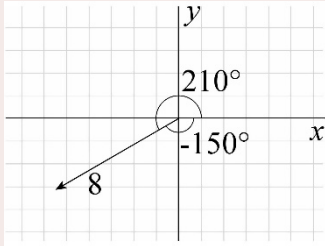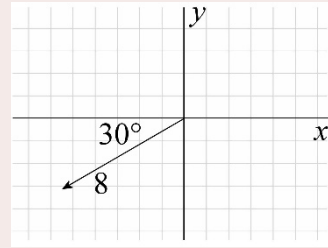
The diagram on the left shows this angle. (Actually, There are 2 angle possibilities for this angle. There is an angle in the positive direction and an angle in the negative direction.)

The *x*-component is

$$F_x = F \cos \theta$$
$$= 8 \cdot \cos 210°$$
$$= -6.698$$

The same value would have obtained with $8 \cdot \cos(-150°)$.

The *y*-component is

$$F_y = F \sin \theta$$
$$= 8 \cdot \sin 210°$$
$$= -4$$

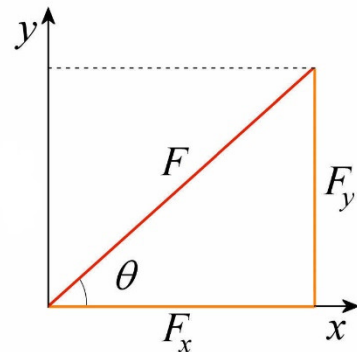The same value would have obtained with $8 \cdot \sin(-150°)$.

The Vector From its Components

The magnitude of a vector can also be found from the components. The diagram on the right shows that the magnitude of the vector is the hypotenuse of the triangle. So, we have

$$F = \sqrt{F_x^2 + F_y^2}$$

From the components of the vector, the direction of the vector can also be found with

$$\theta = \arctan \frac{F_y}{F_x}$$

You probably saw in your youth that this formula is valid only in the first quadrant. In fact, it can be used for vectors in any direction provided you follow the following rule: add 180° to the value given by the calculator if the value of F_x is negative.

Example 2.1.2

The components of a vector are A_x = -6 and A_y = 8. What are the magnitude and the direction of this vector?
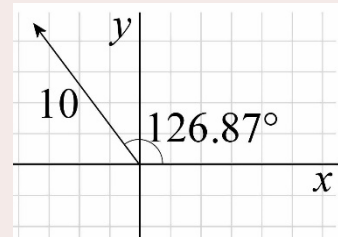
The magnitude of the vector is

$$A = \sqrt{A_x^2 + A_y^2}$$
$$= \sqrt{(-6)^2 + 8^2}$$
$$= 10$$

The direction of the vector is

$$\theta = \arctan \frac{A_y}{A_x}$$
$$= \arctan \frac{8}{-6}$$
$$= 126.87°$$

(180° were added to the answer given by the calculator since the value of A_x is negative.)

The angle obtained with this formula is always the angle between the vector and the positive *x*-axis. The diagram on the right shows the direction of this vector.

# 2.2 THE LAWS OF KINEMATICS IN TWO OR THREE DIMENSIONS

Here we will determine how to treat two- or three-dimensional motion by defining what the displacement, the velocity and the acceleration are in this case. The demonstrations in this first section may not be exciting, but the conclusions are very useful. We will prove the equations for a two-dimensional motion and then generalize for three-dimensional motions (although we will not study much three-dimensional motion).

Position

Axes

In 2 dimensions, 2 axes must be used to specify to position of an object.

Classical axes, with a horizontal *x*-axis positive towards the right and a vertical *y*-axis positive upwards, are often used.
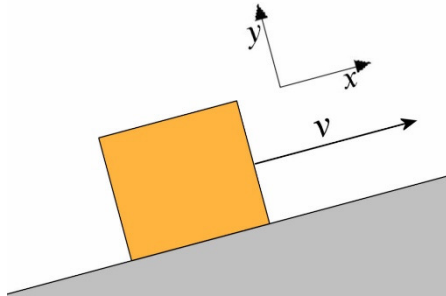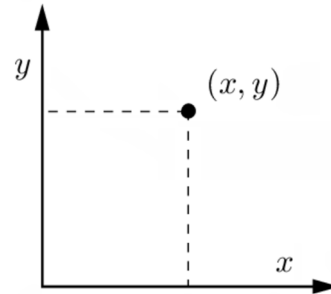
However, the axes must sometimes be rotated. Actually, there is only one restriction: the two axes must be perpendicular. For example, the axes shown on this diagram will be used for an object travelling along an incline (we will see in chapter 4 that an axis pointing in the direction of the velocity or in a direction opposite to the velocity must be used for a moving object).

In 3 dimensions, 3 mutually perpendicular axes are used.
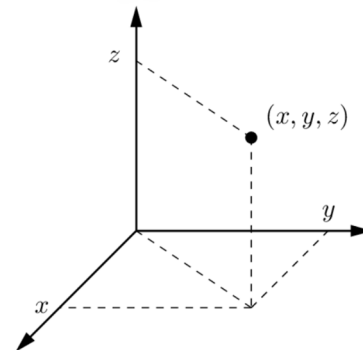
www.researchgate.net/figure/Systeme-de-coordonnees-cartesiennes_fig2_278632180

Position

The position of an object can then be given by simple coordinates (*x, y*) or (*x, y, z*) depending on the number of dimensions.

The position can also be given by a vector. Here is a position vector in two dimensions.

This vector can be resolved into components, allowing the vector to be written in the following form.

$$\vec{r} = x\vec{i} + y\vec{j}$$
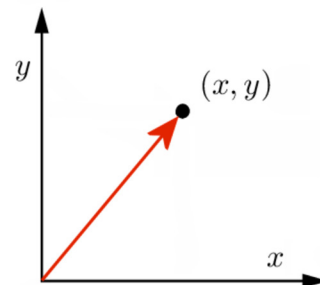
(The components of the vector *r* could also have been written as r_x and r_y, But we decided to simply write *x* and *y* since these components are equal to positions *x* and *y*.)

In three dimensions, this vector would have one more component.

Position of an Object
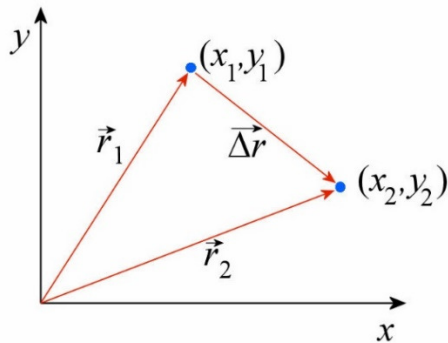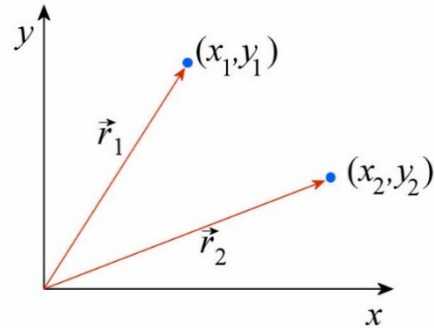
$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Displacement

An object moves from position 1 to position 2, represented by these two vectors.

Each of these vectors can be resolved into its components. Thus

$$\vec{r_1} = x_1\vec{i} + y_1\vec{j}$$
$$\vec{r_2} = x_2\vec{i} + y_2\vec{j}$$

The displacement $\overrightarrow{\Delta r}$, i.e. the change in position, is a vector going from position 1 to position 2.

Since the following addition is true

$$\vec{r_1} + \overrightarrow{\Delta r} = \vec{r_2}$$

it follows that

Displacement

$$\overrightarrow{\Delta r} = \vec{r_2} - \vec{r_1}$$

Velocity

Average velocity

Average velocity is defined as the displacement divided by the time required to accomplish this displacement.

$$\vec{\bar{v}} = \frac{\overrightarrow{\Delta r}}{\Delta t}$$

Since any vector can be resolved into its components, this vector is

$$\vec{\bar{v}} = \bar{v}_x\vec{i} + \bar{v}_y\vec{j}$$

Using what is known about the displacement vector, the components of this vector are also

$$\vec{\bar{v}} = \frac{\overrightarrow{\Delta r}}{\Delta t}$$

$$= \frac{\vec{r_2} - \vec{r_1}}{\Delta t}$$

$$= \frac{\left(x_2 \vec{i} + y_2 \vec{j}\right) - \left(x_1 \vec{i} + y_1 \vec{j}\right)}{\Delta t}$$

$$= \frac{x_2 - x_1}{\Delta t} \vec{i} + \frac{y_2 - y_1}{\Delta t} \vec{j}$$

$$= \frac{\Delta x}{\Delta t} \vec{i} + \frac{\Delta y}{\Delta t} \vec{j}$$

As the components must be equal, the following equations are obtained

$$\bar{v}_x = \frac{\Delta x}{\Delta t} \qquad \bar{v}_y = \frac{\Delta y}{\Delta t}$$

Generalizing for three dimensions, the following equations are obtained.

Average Velocity

$$\vec{\bar{v}} = \frac{\overrightarrow{\Delta r}}{\Delta t}$$

In components:

$$\bar{v}_x = \frac{\Delta x}{\Delta t} \qquad \bar{v}_y = \frac{\Delta y}{\Delta t} \qquad \bar{v}_z = \frac{\Delta z}{\Delta t}$$

These formulas are used to calculate the average velocity components. Note that these formulas are identical to the formula for one-dimensional motion, except that there are now three formulas, one for x, one for y and one for z. Note also that the x-component of the motion has no influence whatsoever on the y and z components of the average velocity, that the y-component of the motion has no influence on the x and z components of the average velocity, and that the z-component of the motion has no effect on the x and y components of the average velocity. This will be true for every formula in kinematics in two or three dimensions.

Instantaneous Velocity

As in the first chapter, instantaneous velocity is calculated by dividing the displacement by the shortest possible time so that the velocity does not have time to change. With vectors, this gives

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

Since any vector can be resolved into its components, this vector is

$$\vec{v} = v_x \vec{i} + v_y \vec{j}$$

This vector can also be resolved into its components. From what is known about the position, the components are

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$= \frac{d\left(x\vec{i} + y\vec{j}\right)}{dt}$$

$$= \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$$

As these components must be equal, the following equations are obtained

$$v_x = \frac{dx}{dt} \qquad\qquad v_y = \frac{dy}{dt}$$

In three dimensions, the following equations are obtained.

Instantaneous Velocity

$$\vec{v} = \frac{d\vec{r}}{dt}$$

In components:

$$v_x = \frac{dx}{dt} \qquad\qquad v_y = \frac{dy}{dt} \qquad\qquad v_z = \frac{dz}{dt}$$
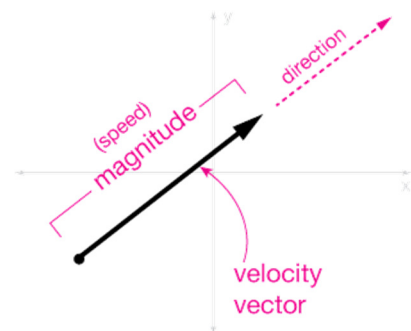
Again, these formulas are similar to the formula of the instantaneous velocity obtained in the first chapter.

Speed

Note that <u>the magnitude of the velocity vector is the speed of the object</u>.

$$speed = \sqrt{v_x^2 + v_y^2} \qquad \text{(2 dimensions)}$$
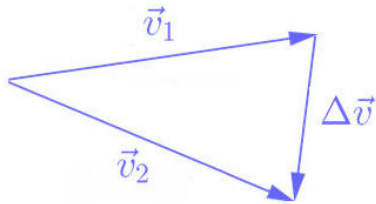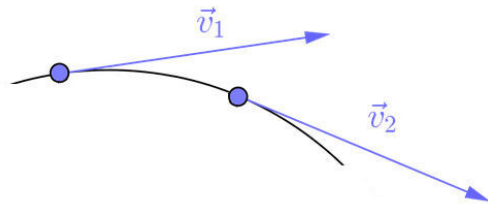
$$speed = \sqrt{v_x^2 + v_y^2 + v_z^2} \quad \text{(3 dimensions)}$$

xaktly.com/Velocity.html

Variation of Velocity

When the velocity of an object changes, there is an acceleration. The diagram shows the velocity of an object at 2 places along its trajectory. (The velocity changes if the magnitude or the direction of the velocity changes.)

The variation of velocity $\overrightarrow{\Delta v}$ is a vector going from $\vec{v}_1$ to $\vec{v}_2$ (figure). It can easily be seen that

$$\vec{v}_1 + \Delta\vec{v} = \vec{v}_2$$

This means that

Variation of Velocity

$$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$$

Acceleration

Average Acceleration

The average acceleration is the variation of velocity divided by the time required to make such a change.

$$\bar{\vec{a}} = \frac{\overrightarrow{\Delta v}}{\Delta t}$$

In components, this vector is

$$\bar{\vec{a}} = \bar{a}_x \vec{i} + \bar{a}_y \vec{j}$$

However, the components of this vector are also, using what is known about velocity,

$$\bar{\vec{a}} = \frac{\overrightarrow{\Delta v}}{\Delta t}$$

$$= \frac{\vec{v}_2 - \vec{v}_1}{\Delta t}$$

$$= \frac{\left(v_{2x}\vec{i} + v_{2y}\vec{j}\right) - \left(v_{1x}\vec{i} + v_{1y}\vec{j}\right)}{\Delta t}$$

$$= \frac{v_{2x} - v_{1x}}{\Delta t}\vec{i} + \frac{v_{2y} - v_{1y}}{\Delta t}\vec{j}$$

$$= \frac{\Delta v_x}{\Delta t}\vec{i} + \frac{\Delta v_y}{\Delta t}\vec{j}$$

As the components must be equal, the following equations are obtained

$$\overline{a}_x = \frac{\Delta v_x}{\Delta t} \qquad \overline{a}_y = \frac{\Delta v_y}{\Delta t}$$

In three dimensions, the following equations are obtained.

Average Acceleration

$$\vec{\overline{a}} = \frac{\overrightarrow{\Delta v}}{\Delta t}$$

In components:

$$\overline{a}_x = \frac{\Delta v_x}{\Delta t} \qquad \overline{a}_y = \frac{\Delta v_y}{\Delta t} \qquad \overline{a}_z = \frac{\Delta v_z}{\Delta t}$$

Instantaneous Acceleration

Instantaneous acceleration is calculated by dividing the change of velocity by the shortest possible time so that the acceleration does not have time to change. With vectors, this is

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

Using the components, this vector is

$$\vec{a} = a_x \vec{i} + a_y \vec{j}$$

However, the components of this vector are also, using what is known about the velocity,

$$\vec{a} = \frac{d\vec{v}}{dt}$$
$$= \frac{d\left(v_x \vec{i} + v_y \vec{j}\right)}{dt}$$
$$= \frac{dv_x}{dt}\vec{i} + \frac{dv_y}{dt}\vec{j}$$

As these components must be equal, the following equations are obtained

$$a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt}$$

In three dimensions, the following equations are obtained.

Instantaneous Acceleration

$$\vec{a} = \frac{d\vec{v}}{dt}$$

In components:

$$a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt} \qquad a_z = \frac{dv_z}{dt}$$

Again, these formulas are similar to the formula obtained in the first chapter.

Example 2.2.1

The position of an object is given by the formula

$$\vec{r} = \left(2\tfrac{m}{s}\cdot t - 1\tfrac{m}{s^2}\cdot t^2\right)\vec{i} + \left(2\tfrac{m}{s}\cdot t + 1m\right)\vec{j}$$

(which is the same as saying that $x = 2\tfrac{m}{s}\cdot t - 1\tfrac{m}{s^2}\cdot t^2$ and $y = 2\tfrac{m}{s}\cdot t + 1m$).

a) What are the positions at $t = 1$ s and $t = 4$ s?

At $t = 1$ s, the position is

$$x = 2\tfrac{m}{s}\cdot t - 1\tfrac{m}{s^2}\cdot t^2 \qquad y = 2\tfrac{m}{s}\cdot t + 1m$$
$$x = 2\tfrac{m}{s}\cdot 1s - 1\tfrac{m}{s^2}\cdot (1s)^2 \qquad y = 2\tfrac{m}{s}\cdot 1s + 1m$$
$$x = 1m \qquad y = 3m$$

The position is, therefore, (1 m, 3 m)

At $t = 4$ s, the position is

$$x = 2\tfrac{m}{s}\cdot t - 1\tfrac{m}{s^2}\cdot t^2 \qquad y = 2\tfrac{m}{s}\cdot t + 1m$$
$$x = 2\tfrac{m}{s}\cdot 4s - 1\tfrac{m}{s^2}\cdot (4s)^2 \qquad y = 2\tfrac{m}{s}\cdot 4s + 1m$$
$$x = -8m \qquad y = 9m$$

The position is, therefore, (-8 m, 9 m)

b) What is the average velocity between $t = 1$ s and $t = 4$ s?

The x-component of the average velocity is

$$\bar{v}_x = \frac{\Delta x}{\Delta t}$$

$$= \frac{-8m - 1m}{3s}$$

$$= -3 \tfrac{m}{s}$$

The *y*-component of the average velocity is

$$\bar{v}_y = \frac{\Delta y}{\Delta t}$$

$$= \frac{9m - 3m}{3s}$$

$$= 2 \tfrac{m}{s}$$

The average velocity is, therefore, $\vec{\bar{v}} = \left(-3\vec{i} + 2\vec{j}\right)\tfrac{m}{s}$

c) What is the instantaneous velocity at *t* = 4 s?

The *x*-component of the velocity is

$$v_x = \frac{dx}{dt}$$

$$= \frac{d\left(2\tfrac{m}{s}\cdot t - 1\tfrac{m}{s^2}\cdot t^2\right)}{dt}$$

$$= 2\tfrac{m}{s} - 2\tfrac{m}{s^2}\cdot t$$

Thus, at *t* = 4 s, its value is

$$v_x = 2\tfrac{m}{s} - 2\tfrac{m}{s^2}\cdot 4s$$

$$= -6\tfrac{m}{s}$$

The *y*-component of the velocity is

$$v_y = \frac{dy}{dt}$$

$$= \frac{d\left(2\tfrac{m}{s}\cdot t + 1m\right)}{dt}$$

$$= 2\tfrac{m}{s}$$

Thus, at *t* = 4 s, its value is
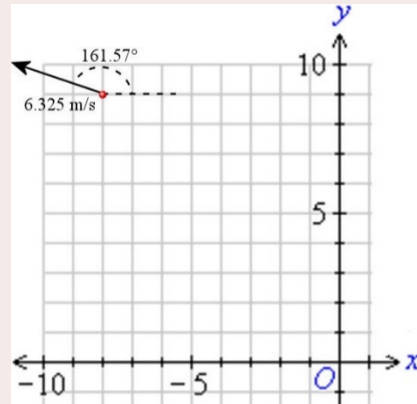
$$v_y = 2\tfrac{m}{s}$$

The velocity is, therefore, $\vec{v} = \left(-6\vec{i} + 2\vec{j}\right)\tfrac{m}{s}$

d) What are the speed and the direction of the velocity at $t = 4$ s?

The speed and the direction are

$$v = \sqrt{v_x^2 + v_y^2}$$
$$= \sqrt{\left(-6\tfrac{m}{s}\right)^2 + \left(2\tfrac{m}{s}\right)^2}$$
$$= 6.325\tfrac{m}{s}$$

$$\theta = \arctan\frac{v_y}{v_x}$$
$$= \arctan\left(\frac{2\tfrac{m}{s}}{-6\tfrac{m}{s}}\right)$$
$$= 161.57°$$

(Note that 180° has been added to the value given by the calculator because v_x is negative.)

e) What is the acceleration at $t = 4$ s?

The x-component of the acceleration is

$$a_x = \frac{dv_x}{dt}$$
$$= \frac{d\left(2\tfrac{m}{s} - 2\tfrac{m}{s^2}\cdot t\right)}{dt}$$
$$= -2\tfrac{m}{s^2}$$

Thus, at $t = 4$ s, its value is $a_x = -2\tfrac{m}{s^2}$.

The y-component of the acceleration is

$$a_y = \frac{dv_y}{dt}$$
$$= \frac{d\left(2\tfrac{m}{s}\right)}{dt}$$
$$= 0$$

Thus, at $t = 4$ s, its value is $a_y = 0\tfrac{m}{s^2}$.

The acceleration is, therefore, $\vec{a} = -2\vec{i}\ \tfrac{m}{s^2}$.

Three-Dimensional Motion with Constant Acceleration

Here are the equations for three-dimensional motion.

$$a_x = \frac{dv_x}{dt} \quad \text{and} \quad a_y = \frac{dv_y}{dt} \quad \text{and} \quad a_z = \frac{dv_z}{dt}$$

$$v_x = \frac{dx}{dt} \quad \text{and} \quad v_y = \frac{dy}{dt} \quad \text{and} \quad v_z = \frac{dz}{dt}$$

If the acceleration is constant, the three components of the acceleration (a_x, a_y and a_z) are also constant. These equations can then be solved to obtain the equations for the velocity and the position as a function of time. There's no need to do everything in detail here because those are the same equations as those found in Chapter 1, except that there is now one set of equations for x, another set for y and another set for z. Then, the solutions are the same as those obtained in chapter 1, but with three sets of equations: one for the x-component of the motion, one for the y-component and another one for the z-component.

Three-Dimensional Motion with Constant Acceleration

$$v_x = v_{0x} + a_x t \qquad\qquad v_y = v_{0y} + a_y t \qquad\qquad v_z = v_{0z} + a_z t$$

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \qquad y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \qquad z = z_0 + v_{0z}t + \frac{1}{2}a_z t^2$$

$$2a_x(x - x_0) = v_x^2 - v_{x0}^2 \qquad 2a_y(y - y_0) = v_y^2 - v_{y0}^2 \qquad 2a_z(z - z_0) = v_z^2 - v_{z0}^2$$

$$x = x_0 + \frac{1}{2}(v_{x0} + v_x)t \qquad y = y_0 + \frac{1}{2}(v_{y0} + v_y)t \qquad z = z_0 + \frac{1}{2}(v_{z0} + v_z)t$$

Note once again the independence of the 3 components of the motions. As there is no y or z in the left column, the x-components of the motion are not influenced by what is happening with the y or z components. As there is no x or z in the middle column, the y-components of the motion are not influenced by what is going on with the x or z components. As there is no x or y in the last column, the z-components of the motion also are not influenced by what is happening with the x or y components.

Example 2.2.2

An object has the following acceleration, velocity and position at $t = 0$ s.

$$\vec{a} = \left(2\vec{i} + 3\vec{j} - \vec{k}\right)\frac{m}{s^2}$$

$$\vec{v}_0 = \left(\vec{i} - 2\vec{j} + \vec{k}\right)\frac{m}{s}$$

$$\vec{r}_0 = \left(3\vec{i} + 2\vec{k}\right)m$$

What is the position of the object at $t = 2$ s if the acceleration is constant?

The x-coordinate of the position is

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$

$$= 3m + 1\tfrac{m}{s} \cdot 2s + \frac{1}{2} \cdot 2\tfrac{m}{s^2} \cdot (2s)^2$$

$$= 9m$$

The *y*-coordinate of the position is

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$$

$$= 0m + \left(-2\tfrac{m}{s}\right) \cdot 2s + \frac{1}{2} \cdot 3\tfrac{m}{s^2} \cdot (2s)^2$$

$$= 2m$$

The *z*-coordinate of the position is

$$z = z_0 + v_{0z}t + \frac{1}{2}a_z t^2$$

$$= 2m + 1\tfrac{m}{s} \cdot 2s + \frac{1}{2} \cdot \left(-1\tfrac{m}{s^2}\right) \cdot (2s)^2$$

$$= 2m$$

The position of the object is, therefore, (9 m, 2 m, 2 m).
(This answer can also be written $\vec{r} = \left(9\vec{i} + 2\vec{j} + 2\vec{k}\right)m$.)

2.3 PROJECTILE MOTION

Historical Background

The projectile was an important subject of study throughout history because of its importance in military science. Based on a false physics (as we shall see in the next chapter), these studies mainly arrive at false conclusions. For example, it was believed that the trajectory of the projectile was straight at the beginning, that there was, thereafter, a brief period during which the trajectory curves downwards to finally have a vertically falling object. This is what the following image, dating from 1577, shows (by Paulus Puchner).
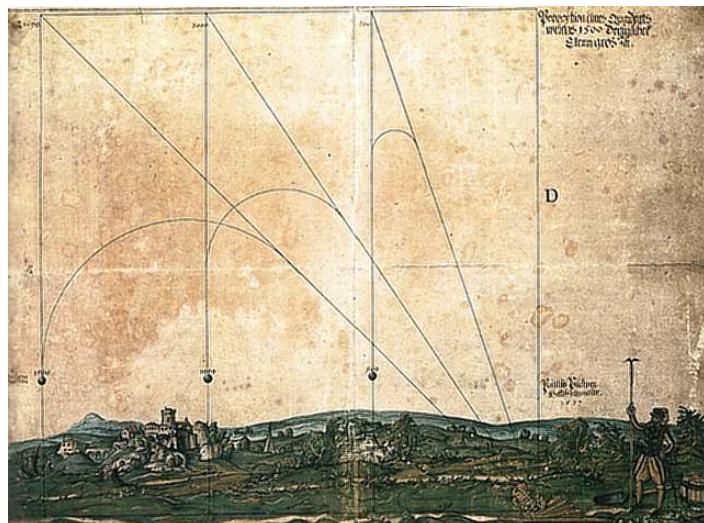
www.mlahanas.de/Greeks/AristotlePhysics.htm

It was Galileo who realized that it was all wrong. He was the first to come to the conclusion that the *x* and *y* components of the motion were independent from each other. That is what the formulas in the previous section were telling us: what is happening with the *x*-component has no influence whatsoever on the *y*-component of the motion and what is happening with the *y*-component has no influence on the *x*-component of the motion. As Galileo had already discovered that falling objects fall downwards with a constant acceleration, he realized that the projectile motion must be a combination of two motions:

1) A horizontal motion at constant velocity.
2) A vertical uniform accelerated motion (at 9.8 m/s² downwards).
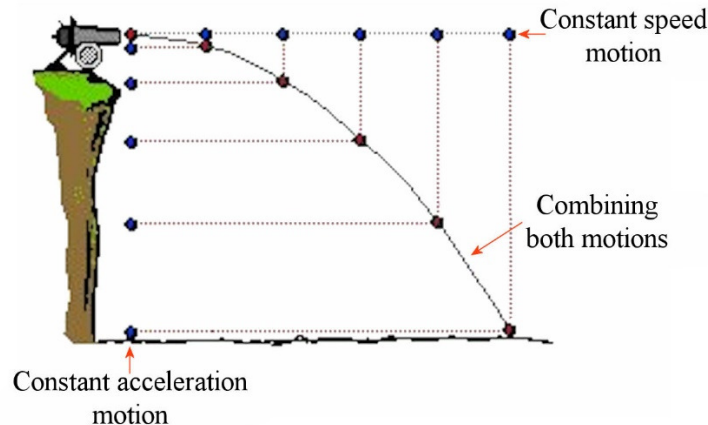
www.physicsclassroom.com/class/vectors/u3l2a.cfm

(Galileo published in 1632, but he almost had the priority of discovery stolen by Cavalieri, who also published in 1632.)

The picture on the right illustrates this fact. Both balls were released simultaneously. The ball on the left falls straight down while the ball on the right moves to the right. Clearly, both balls fall at the same rate so that they will hit the ground simultaneously even if the ball to the right moves horizontally. The horizontal motion of the ball directed towards the right has absolutely no influence on its vertical motion. A careful look at the trajectory of the ball to the right shows that the ball always moves by the same distance to the right at each time, which indicates that the *x*-component of the motion is a motion at a constant speed.

This means that if a bullet is shot precisely horizontally, it will hit the ground simultaneously as a ball released from the same height as the gun, at the same time the gun was fired. The horizontal speed of the bullet does not change the fall time. That is what they tried to check in this Mythbusters video.
https://www.youtube.com/watch?v=D9wQVIEdKh8

fphoto.photoshelter.com/image/I0000rRLe6wDNbBY
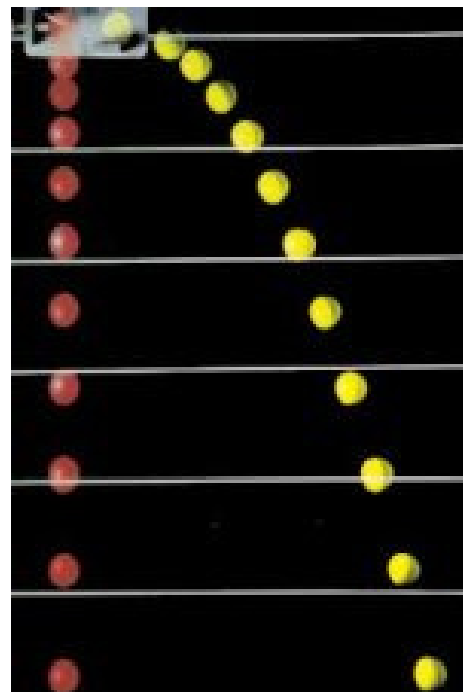
(Actually, air resistance changes this conclusion a little. With more advanced calculations taking into account air friction, it can be shown that the fired bullet would arrive at the ground slightly after the vertically falling bullet because of the enormous air resistance acting on the fired bullet in a direction opposite to the motion, which generate a greater vertical component than the total friction acting on the vertically falling bullet. For a caliber 22 bullet (which is nearly spherical with a 5.67 mm radius and a 3 g mass) shot horizontally at 1000 m/s from a height of 10 m, the fall time of the vertically falling bullet is 1.433 s while the fall time of the fired bullet is 1.762 s. This is the correct explanation of the delay observed in the video.)

Basic Formulas

Thus, it follows that the motion of a projectile is a constant acceleration motion. Using a horizontal x-axis and a pointing upwards y-axis, the acceleration is

Acceleration of a projectile

$$a_x = 0$$
$$a_y = -9.8 \tfrac{m}{s^2} = -g$$

The equations of the motion of a projectile are, therefore,

Basic equations for the motion of a projectile

$$x = x_0 + v_{0x}t$$
$$v_y = v_{0y} - gt$$
$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$
$$-2g(y - y_0) = v_y^2 - v_{y0}^2$$

These are the equations seen in chapter 1. The first equation is the equation for a constant velocity motion for the x-component. The three following equations are the equations of a motion with a constant acceleration of $-g$ for the y-component. (The fourth equation for the y-component of the motion, which is $y = y_0 + \frac{1}{2}(v_{0y} + v_y)t$, is not useful for a projectile since we know that the acceleration is 9.8 m/s².) To use these equations, you must have a perfectly horizontal x-axis and a perfectly vertical y-axis pointing upwards since we use a negative acceleration.

Other Formulas that Can Be Useful

1) Flight Time if the Projectile Returns to the Ground at the Same Height

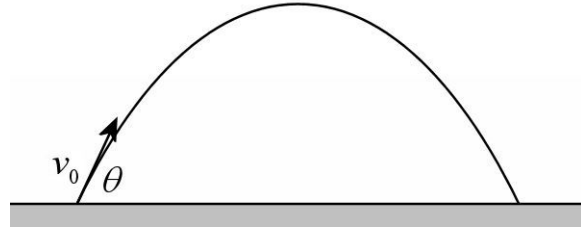If the projectile returns to the same height it was launched, we have

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

$$y_0 = y_0 + v_{0y}t_{flight} - \frac{1}{2}gt^2_{flight}$$

$$0 = v_{0y}t_{flight} - \frac{1}{2}gt^2_{flight}$$

$$0 = v_{0y} - \frac{1}{2}gt_{flight}$$

$$\frac{1}{2}gt_{flight} = v_{0y}$$

The flight time is, therefore,

$$t_{flight} = \frac{2v_{0y}}{g} = \frac{2v_0 \sin\theta}{g}$$

This formula is not really useful since the flight time can easily be found with the equation

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

which is one of the basic equations for the projectile motion. Moreover, the flight time formula just obtained is quite restrictive since the object must hit the ground at the same height it was launched.

2) Range if the Projectile Returns to the Ground at the Same Height

The range R is the horizontal distance travelled by the projectile during its flight. It is the distance between launching and landing, which means that $R = x - x_0$. If the object returns to the ground at the same height it was launched, it follows, using the formula for the flight time previously found that

$$x = x_0 + v_{0x}t$$

$$x - x_0 = v_{0x}t$$

$$R = v_{0x}t_{flight}$$

$$R = v_{0x}\frac{2v_0 \sin\theta}{g}$$

$$R = v_0 \cos\theta \frac{2v_0 \sin\theta}{g}$$
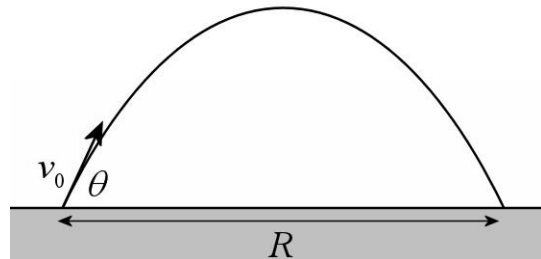
Using the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$, this equation can be simplified to finally get

Range of a Projectile Returning to the Ground at the Same Height it Was Launched

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

This formula is more useful than the flight time formula. As a trigonometric identity was used to simplify it, it greatly simplifies the solutions of problems where the launch angle of a projectile has to be calculated. Be careful to use this formula only if the projectile falls back to the ground at the same height it was launched.

The launch angle to have the maximum range when the projectile always has the same initial speed can easily be found. Since v_0 and g are constants, the range is maximum when the sine has its maximum value. The largest value that a sine can have is 1 and this occurs for $\sin 90° = 1$. The range is therefore maximized when $2\theta = 90$ °, so when $\theta = 45$ °.

The range of a projectile is maximized if the launch angle is 45°

Again, Galileo was the first to prove that the range of a projectile is greatest if the launch angle is 45 °, but it appears that gunners already knew this information. Besides, Tartaglia mentioned this fact in 1537, almost a century before Galileo's proof.

The diagram shows the path of different projectiles which were all launched at the same speed, but at different launch angles.

Note that the range is the same for 30° and 60° and is also the same for 75° and 15 °. In fact, the range is always the same for angles θ and 90° - θ. The range is the same because the $\sin 2\theta$ has the same value for these two angles.
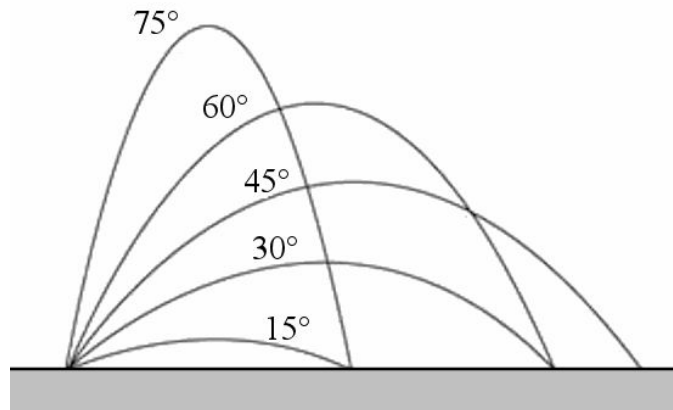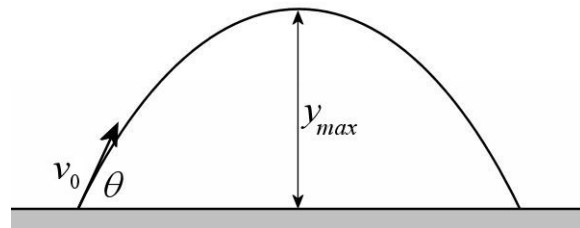
3) Maximum Height Reached by a Projectile

As in the previous chapter, the maximum height of a projectile is reached when it stops moving upwards. It is thus the position when $v_y = 0$. It is found with

$$-2g(y - y_0) = v_y^2 - v_{0y}^2$$
$$-2g(y_{max} - 0) = -v_{0y}^2$$

As $y_0 = 0$ was used, the maximum height given by this formula is always measured from the starting point of the projectile. Solving for the maximum height, the result is

$$y_{max} = \frac{v_{0y}^2}{2g}$$

$$y_{max} = \frac{v_0^2 \sin^2 \theta}{2g}$$

As this equation is obtained directly from one of the basic equations, it has no real usefulness in problem-solving.

The flight time and the maximum height formulas

$$y_{max} = \frac{v_{0y}^2}{2g} \qquad\qquad t_{flight} = \frac{2v_{0y}}{g}$$

both only depend on the y-component of the initial velocity. The horizontal motion has no influence on both these quantities.

It is possible to solve for this initial y-component velocity in the flight time formula and substitute the result in the maximum height equation to obtain

$$y_{max} = \frac{gt_{flight}^2}{8}$$

It is clear now that the flight time is directly related to the maximum height of the projectile.

The three projectiles on the next diagram, therefore, all have the same flight time as all have the same maximum height. However, they do not have the same range.
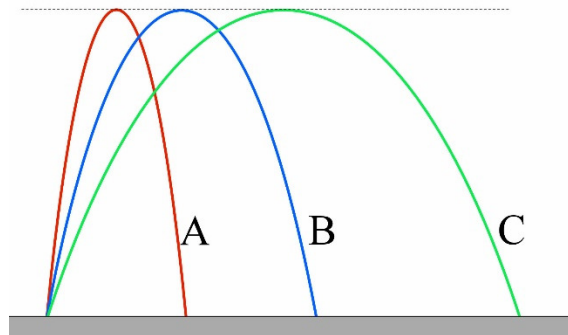
4) Link between x and y Positions

There is a formula giving the x-position as a function of time and another giving the y-position as a function of time. These two equations will now be combined to obtain a formula for the y-position as a function of the x-position of the projectile. First, let's solve for the t in the x-position formula.

$$x = x_0 + v_{0x}t$$

$$t = \frac{x}{v_{0x}}$$

As $x_0 = 0$ was used, <u>the starting point will always be at $x = 0$ here</u>. If t is now substituted by this formula in the y-position equation, it becomes

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

$$y = 0 + v_{0y}\left(\frac{x}{v_{0x}}\right) - \frac{1}{2}g\left(\frac{x}{v_{0x}}\right)^2$$

$$y = 0 + v_0\sin\theta\left(\frac{x}{v_0\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2$$

As $y_0 = 0$ was used, <u>the starting point will always be at $y = 0$ here</u>. After some simplifications, this formula is

Link between x and y Positions for a Projectile

$$y = (\tan\theta)\,x - \frac{g}{2v_0^2\cos^2\theta}x^2$$

The starting point must be (0,0)

This equation is particularly useful for solving problems in which the positions (x and y) of a projectile on its trajectory are known or if the equation of the path of the projectile is given.
This equation also shows that the shape of the path is a downwards concave parabola since it is of the form $y = Ax - Bx^2$ where A and B are constants. Again, Galileo proved this fact, although dal Monte, a friend of Galileo, stated some years before that the path looks like a parabola.

The jets of water in this image form parabolas because they are made of free-falling jets of water. This fountain is at the Detroit Metropolitan Wayne Airport.

mathtourist.blogspot.ca/2011/05/fountain-parabolas.html

Different positions of a projectile (a ball) can be observed in the following video. The parabolic shape of the path can then be seen.
https://www.youtube.com/watch?v=EUqpyia45PM

Examples

Most problems can be solved only from the basic formulas of the projectile.

$$x = x_0 + v_{0x}t$$

$$v_y = v_{0y} - gt$$

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

$$-2g(y - y_0) = v_y^2 - v_{y0}^2$$

⚠️ **Common mistake: Using a = 9.8 m/s² for the x-component of the motion.**

There is no horizontal acceleration. The downwards acceleration of 9.8 m/s² is only in the y direction.

⚠️ **Common Mistake: Using g = -9.8 m/s²**

When there is a g in a formula, replace g by 9.8 m/s² and not by -9.8 m/s². The minus sign is not a part of g. The minus sign of the acceleration was already taken care of while working on the formula for the range.

Example 2.3.1

A car driver tries to cross a very deep canyon. The car moves at 90 km/h, and the canyon has a width of 15 m. The ground on the other side of the ravine is 2 m below the ground on the side where the car is coming. Will the crossing be successful, or will the car fall at the bottom of the ravine?
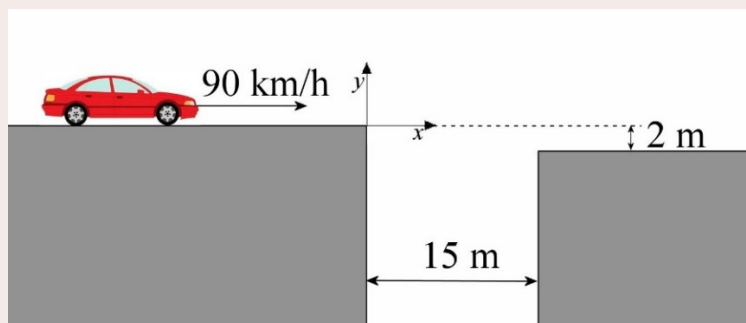
fr.depositphotos.com/2577683/stock-illustration-Car.html

To solve this problem, we will calculate by how much the car has fallen when it arrives at the other side of the ravine. If it falls by more than 2 m, it will hit the side of the ravine and fall to the bottom. If it falls by less than 2 m, it will fall to the ground, and the crossing will have succeeded.

The components of the initial velocity are v_{0x} = 25 m/s and v_{0y} = 0. As in many projectile problems, the origin of the coordinate axes can be set anywhere we want. We put it right at the beginning of the ravine, at the top of the cliff (see diagram). The initial position of the car, when the free-fall motion starts, is then $x_0 = 0$ and $y_0 = 0$. It remains to find y when the car is at $x = 15$ m. If we look at the equations of motion of the projectile

$$x = x_0 + v_{0x}t$$
$$v_y = v_{0y} - gt$$
$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$
$$-2g(y - y_0) = v_y^2 - v_{y0}^2$$

we see that no equation can help us to immediately find y when x is 15 m. We note, however, that the first equation allows us to find t. Once we have t, we can find y with the third equation because y will then be the only unknown in this equation.

1) t calculation

$$x = x_0 + v_{0x}t$$
$$15m = 0m + 25\tfrac{m}{s} \cdot t$$
$$t = 0.6s$$

2) y calculation

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$
$$= 0m + 0\tfrac{m}{s} \cdot 0.6s - \frac{1}{2} \cdot 9.8\tfrac{m}{s^2} \cdot (0.6s)^2$$
$$= -1.764m$$

The crossing is, therefore, successful.

Obviously, crossing the ravine cannot succeed if the other side is at the same height. As soon as the vehicle starts crossing the ravine, it begins to fall because of gravity and will be lower than the other side. This is true unless the vehicle is a bus in a movie …
https://www.youtube.com/watch?v=9tEAMLOupKs

Example 2.3.2

A projectile is launched from a roof with a speed of 6 m/s and a launch angle of 35°. The roof is 10 m above the ground.
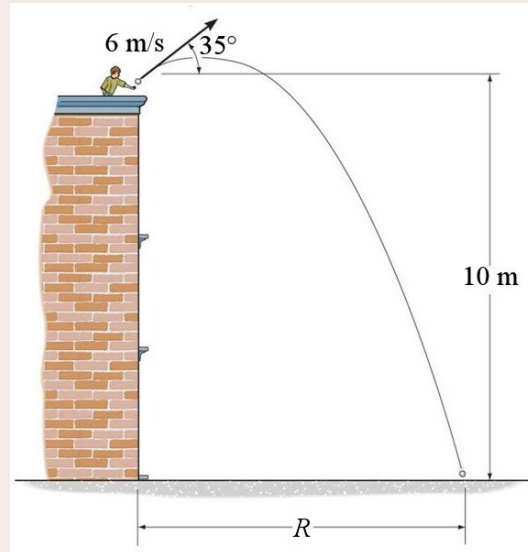
a) What is the flight time of the projectile?

The time of flight can be found with

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

We need the *y*-component of the initial velocity. Since we're at it, we will calculate the two components. They are almost always useful.

$$v_{0x} = v_0 \cos\theta \qquad\qquad v_{0y} = v_0 \sin\theta$$
$$= 6\tfrac{m}{s} \cdot \cos 35° \qquad\qquad = 6\tfrac{m}{s} \cdot \sin 35°$$
$$= 4.915\tfrac{m}{s} \qquad\qquad = 3.441\tfrac{m}{s}$$

We must also decide where the origin of our coordinates is. We chose to set $x = 0$ m and $y = 0$ m at the starting point of the projectile.

The flight time is, therefore, given by

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

$$-10m = 0m + 3.441\tfrac{m}{s} \cdot t - \frac{1}{2} \cdot 9.8\tfrac{m}{s^2} \cdot t^2$$

This quadratic equation can be solved to obtain two solutions: $t = -1.120$ s and $t = 1.822$ s. Obviously, the first value cannot be accepted since the motion of the projectile begins at $t = 0$ s. The flight time is therefore 1.822 s

b) What is the range (*R*) of this projectile?

The range must be calculated with the equation for the *x*-component of the motion.

$$x = x_0 + v_{0x}t$$
$$R = 0m + 4.915\tfrac{m}{s} \cdot 1.822s$$
$$R = 8.955m$$

c) What is the maximum height of this projectile (measured from the ground)?

At the maximum height $v_y = 0$, so that

$$-2g\left(y - y_0\right) = v_y^2 - v_{y0}^2$$
$$-2 \cdot 9.8\tfrac{m}{s^2} \cdot \left(y_{max} - 0m\right) = \left(0\tfrac{m}{s}\right)^2 - \left(3.441\tfrac{m}{s}\right)^2$$
$$y_{max} = 0.604m$$

As this value is always the height from our $y = 0$ m, a correction must sometimes be made. As the $y = 0$ m is at the roof, 10 m must be added to the height obtained to have the height from the ground. The maximum height from the ground is therefore 10.604 m.

d) What are the speed and the direction of the velocity of the projectile just before it touches the ground?

The magnitude and direction of the velocity cannot be found directly. The components of the velocity must be found first. First, the x-component of the velocity is constant.

$$v_x = v_{0x} = 4.915\tfrac{m}{s}$$

The y-component of the velocity changes according to the following formula.

$$v_y = v_{0y} - gt$$
$$= 3,441\tfrac{m}{s} - 9,8\tfrac{m}{s^2} \cdot 1,822s$$
$$= -14,42\tfrac{m}{s}$$

With the components, the speed and the direction of the velocity can be found with

$$v = \sqrt{v_x^2 + v_y^2}$$
$$= \sqrt{\left(4.915\tfrac{m}{s}\right)^2 + \left(14.42\tfrac{m}{s}\right)^2}$$
$$= 15.23\tfrac{m}{s}$$

$$\theta = \arctan\frac{v_y}{v_x}$$
$$= \arctan\frac{-14.42\tfrac{m}{s}}{4.915\tfrac{m}{s}}$$
$$= -71.17°$$

The diagram on the right shows what this negative angle means according to the rules given in section 2.1.
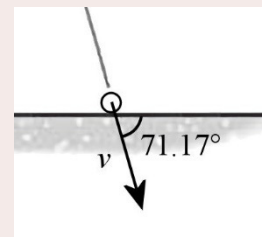
The basic formulas make it possible to find the solution of almost all projectile problems. Even if there is a formula for range, maximum height or flight time, these quantities can easily be found from the basic formulas.

However, there are cases where it is difficult to obtain a solution from the basic formulas. Finding the starting angle of a projectile is one of those difficult cases. In this case, the formula of the range of a projectile can sometimes be used to find the angle.

Example 2.3.3

On May 24, 1941, the German battleship Bismarck sank the British battlecruiser HMS Hood by a direct hit to the ammunition chamber with an 800 kg shell. Surprisingly, the Bismarck was 14 km from the Hood at that time. What was the launch angle of the shell knowing that its initial velocity was 820 m/s?

The angle can be found with

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

$$14\,000m = \frac{\left(820\frac{m}{s}\right)^2 \cdot \sin 2\theta}{9.8\frac{m}{s^2}}$$

$$\sin 2\theta = 0.204$$

Note that it is possible, with this equation, to arrive at a sine with a value greater than 1. This would have happened here if the speed had been less than 370 m/s. If this happens, there is no solution, and it is then impossible to reach the target, regardless of the launch angle.

Fortunately, the sine is less than 1 here, and there is a solution. In fact, there are two solutions to an arcsine. Your calculator gives you the first answer (that we will call θ_1), and the other answer is $\theta_2 = 180° - \theta_1$. Here, the solutions are

$$\sin 2\theta = 0.204$$

$$2\theta = 11.77° \quad \text{and} \quad 168.23°$$

$$\theta = 5.89° \quad \text{and} \quad 84.11°$$

Both answers are correct. However, since the guns of the Bismarck could not fire at an angle larger than 30°, the solution must be 5.89°.

Remember that the starting angle can only be found with this range formula if the projectile falls back to the same height as it was launched.

There is another case where the basic formulas are not used. These are cases where the positions of the projectile on its trajectory is known. In this case, we use

$$y = (\tan \theta)\, x - \frac{g}{2v_0^2 \cos^2 \theta} x^2$$

Example 2.3.4

To destroy another cannon on a castle, a cannon fires a projectile with a launch angle of 40 °. The distances of the enemy cannon are shown in the diagram. How fast should you launch the projectile?

www.kineticbooks.com/physics/librarycheck.php?src=triallabs

As 2 points of the trajectory are known ((0,0) and (60 m, 20 m)), the equation for the path of a projectile can be used to solve this problem.

$$y = (\tan \theta)\, x - \frac{g}{2v_0^2 \cos^2 \theta} x^2$$

$$20m = (\tan 40°) \cdot 60m - \frac{9.8\,\frac{m}{s^2}}{2 \cdot v_0^2 \cdot \cos^2 40°} \cdot (60m)^2$$

$$v_0 = 31.47\,\tfrac{m}{s}$$

Example 2.3.5

A cannon fires a projectile with a speed of 40 m/s. The shell hits an enemy cannon at the position indicated in the diagram. What was the launch angle of the projectile?

As 2 points of the trajectory are known ((0,0) and (60 m, 20 m)), the equation for the path of a projectile can be used to solve this problem.
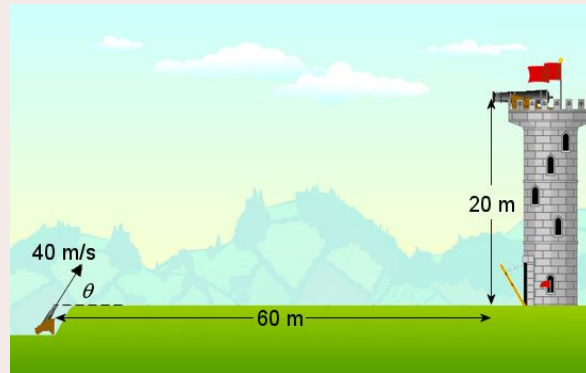
$$y = (\tan \theta)\, x - \frac{g}{2v_0^2 \cos^2 \theta} x^2$$

$$20m = \left(\tan\theta\right)\cdot 60m - \frac{9.8\frac{m}{s^2}}{2\cdot\left(40\frac{m}{s}\right)^2\cdot\cos^2\theta}\cdot\left(60m\right)^2$$

$$20m = \tan\theta\cdot 60m - 11.025m\cdot\frac{1}{\cos^2\theta}$$

$$20 = 60\cdot\tan\theta - 11.025\cdot\sec^2\theta$$

It only remains to solve this equation for θ. In order to do this, $\sec^2\theta = 1+\tan^2\theta$ must be used. Therefore,

$$20 = 60\cdot\tan\theta - 11.025\cdot\left(1+\tan^2\theta\right)$$

$$20 = 60\cdot\tan\theta - 11.025 - 11.025\cdot\tan^2\theta$$

$$31.025 - 60\cdot\tan\theta + 11.025\cdot\tan^2\theta = 0$$

This is a quadratic equation of $\tan\theta$. The solutions of this equation are

$$\tan\theta = \frac{60\pm\sqrt{60^2 - 4\cdot 31.025\cdot 11.025}}{2\cdot 11.025}$$

$$= 4.8636 \text{ and } 0.5786$$

The two solutions are

$$\theta = \arctan 4.8636 = 78.4° \quad\text{and}\quad \theta = \arctan 0.5786 = 30.05°$$

Example 2.3.6

The equation for the path of a projectile is

$$y = 3x - 5m^{-1}x^2$$

a) What is the launch angle of this projectile?
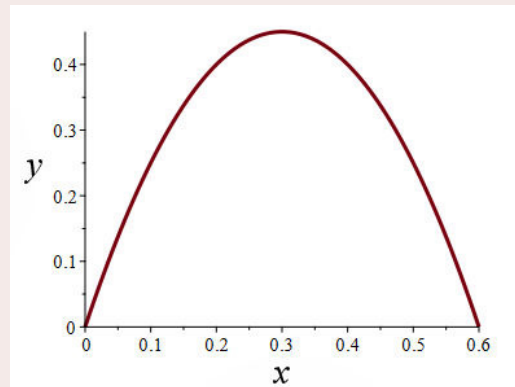
Since the path equation is

$$y = \left(\tan\theta\right)x - \left(\frac{g}{2v_0^2\cos^2\theta}\right)x^2$$

some information is obtained by comparing the two equations.

$$y = (\tan\theta)x - \left(\frac{g}{2v_0^2 \cos^2\theta}\right)x^2$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$y = 3x - 5m^{-1}x^2$$

It is then obvious that $\tan\theta = 3$ and, therefore, that $\theta = 71.56°$.

b) What is the initial speed of this projectile?

Still using the comparison, we find that

$$\frac{g}{2v_0^2 \cos^2\theta} = 5m^{-1}$$

The speed is then

$$\frac{9.8\frac{m}{s^2}}{2 \cdot v_0^2 \cdot \cos^2(71.56°)} = 5m^{-1}$$

$$v_0 = 3.13\frac{m}{s}$$

2.4 UNIFORM CIRCULAR MOTION

Centripetal Acceleration Formula

In uniform circular motion, an object moves along a circular path at a constant speed. Since the speed is constant, it is possible to think that there's no acceleration at first, but that would be false. Here's why.

In this diagram, the velocity vectors at three positions on the circular path are shown. Although all vectors have the same length, they do not have the same orientation. This changing orientation implies that the velocity changes, which implies that there is an acceleration. The following short video shows you how the velocity vector orientation changes during a circular motion.
https://www.youtube.com/watch?v=XeOI6YTwFpE
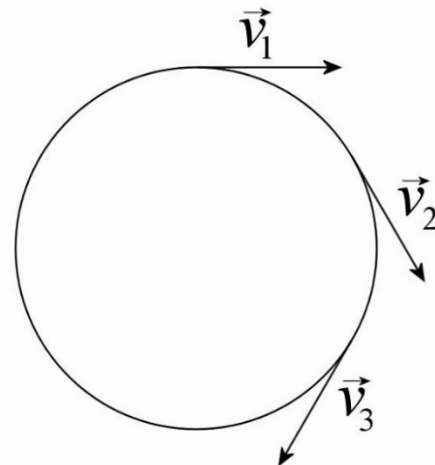
To find this acceleration, the x and y positions as a function of time for an object that makes a counterclockwise circular motion at a constant speed will be found.

According to the diagram on the right, those positions are given by

$$x = r\cos\theta$$
$$y = r\sin\theta$$

The angle (in radians) is, by definition, equal to

$$\theta = \frac{\text{lenght of an arc}}{\text{radius}}$$
$$= \frac{s}{r}$$

As the motion is a constant speed motion the length of the arc is $s = vt$. Therefore, the angle is

$$\theta = \frac{vt}{r}$$

The x and y position as a function of time are thus given by the following formulas.

$$x = r\cos\tfrac{vt}{r}$$
$$y = r\sin\tfrac{vt}{r}$$

The components of the velocity can now be found with a differentiation.

$$v_x = \frac{dx}{dt} = \frac{d\left(r\cos\tfrac{vt}{r}\right)}{dt} = -r\frac{v}{r}\sin\tfrac{vt}{r} = -v\sin\tfrac{vt}{r}$$

$$v_y = \frac{dy}{dt} = \frac{d\left(r\sin\tfrac{vt}{r}\right)}{dt} = r\frac{v}{r}\cos\tfrac{vt}{r} = v\cos\tfrac{vt}{r}$$

Let's see if we're on the right track by calculating those components from different angles.

At $\theta = 0°$ ($vt/r = 0°$), the components are

$$v_x = -v\sin 0° = 0 \qquad\qquad v_y = v\cos 0° = v$$

At $\theta = 90°$ ($vt/r = 90°$), the components are

$$v_x = -v\sin 90° = -v \qquad\qquad v_y = v\cos 90° = 0$$

At $\theta = 180°$ ($vt/r = 180°$), the components are

$$v_x = -v\sin 180° = 0 \qquad\qquad v_y = v\cos 180° = -v$$

At $\theta = 270°$ ($vt/r = 270°$), the components are

$$v_x = -v\sin 270° = v \qquad\qquad v_y = v\cos 270° = 0$$
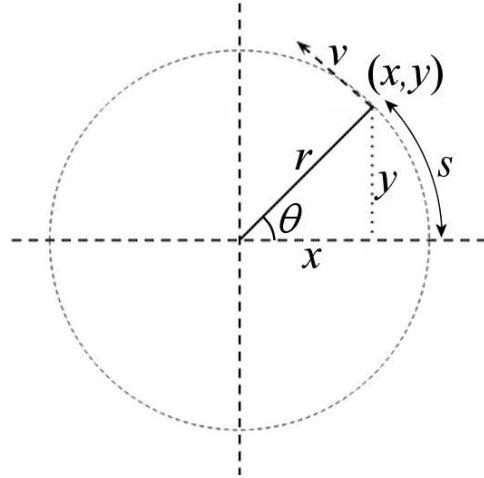
It is easy to see that the magnitude and the direction of the velocity at these four angles make sense.
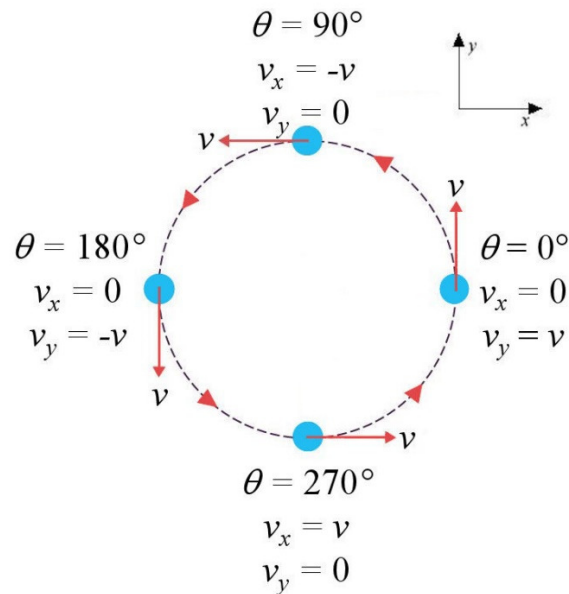
The components of the acceleration can now be found with a differentiation of the components of the velocity.

$$a_x = \frac{dv_x}{dt} = \frac{d\left(-v\sin\frac{vt}{r}\right)}{dt} = -v\frac{v}{r}\cos\frac{vt}{r} = -\frac{v^2}{r}\cos\frac{vt}{r}$$

$$a_y = \frac{dv_y}{dt} = \frac{d\left(v\cos\frac{vt}{r}\right)}{dt} = -v\frac{v}{r}\sin\frac{vt}{r} = -\frac{v^2}{r}\sin\frac{vt}{r}$$

Let's calculate the value of those components from different angles.

At $\theta = 0°$, the components are

$$a_x = -\frac{v^2}{r}\cos 0° = -\frac{v^2}{r} \qquad\qquad a_y = -\frac{v^2}{r}\sin 0° = 0$$

At $\theta = 90°$, the components are

$$a_x = -\frac{v^2}{r}\cos 90° = 0 \qquad\qquad a_y = -\frac{v^2}{r}\sin 90° = -\frac{v^2}{r}$$

At $\theta = 180°$, the components are

$$a_x = -\frac{v^2}{r}\cos 180° = \frac{v^2}{r} \qquad\qquad a_y = -\frac{v^2}{r}\sin 180° = 0$$

At $\theta = 270°$, the components are

$$a_x = -\frac{v^2}{r}\cos 270° = 0 \qquad\qquad a_y = -\frac{v^2}{r}\sin 270° = \frac{v^2}{r}$$

The following diagram shows the magnitude and direction of the acceleration for these different angles.

$$\theta = 90°$$
$$a_x = 0$$
$$a_y = -v^2/r$$

$$\theta = 180°$$
$$a_x = v^2/r$$
$$a_y = 0$$

$$\theta = 0°$$
$$a_x = -v^2/r$$
$$a_y = 0$$

$$\theta = 270°$$
$$a_x = 0$$
$$a_y = v^2/r$$
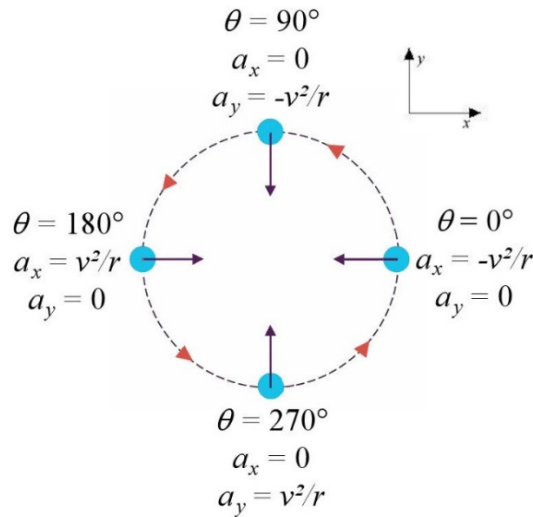
It is obvious that the acceleration is always v^2/r and is always directed towards the centre of the circle. This acceleration is called the *centripetal acceleration*, where *centripetal* means "towards the centre". This acceleration is denoted a_c.

Centripetal Acceleration

$$a_c = \frac{v^2}{r}$$

Towards the centre
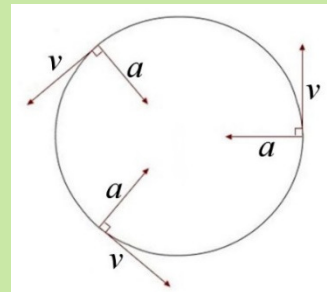
The following video shows the direction of the velocity and the acceleration vectors during a circular motion.
https://www.youtube.com/watch?v=wJutTmcHE4s

Example 2.4.1

An aircraft makes a motion with a constant speed following a circular trajectory with a radius of 2 km. What is the magnitude of the centripetal acceleration if the speed of the aircraft is 360 km/h?

www.embibe.com/study/uniform-circular-motion-in-a-horizontal-plane-concept

The magnitude of the acceleration is

$$a_c = \frac{v^2}{r}$$

$$= \frac{\left(100\,\frac{m}{s}\right)^2}{2000m}$$

$$= 5\,\frac{m}{s^2}$$

A formula of the centripetal acceleration using the period T of the circular motion can also be obtained. The period corresponds to the time it takes the object to go around the circular path once. This time is

$$T = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{2\pi r}{v}$$

(This equation is valid only if the speed is constant.) So v can be eliminated in the centripetal acceleration formula

$$a_c = \frac{v^2}{r}$$

$$= \frac{(2\pi r / T)^2}{r}$$

to obtain

Centripetal Acceleration (Valid only if the Speed is Constant)

$$a_c = \frac{4\pi^2 r}{T^2}$$

Towards the centre

Example 2.4.2

An aircraft makes a motion with a constant speed following a circular trajectory with a radius of 2 km. What is the magnitude of the centripetal acceleration if the period is 2 minutes?

The magnitude of the acceleration is

$$a_c = \frac{4\pi^2 r}{T^2}$$

$$= \frac{4\pi^2 \cdot 2000m}{(120s)^2}$$

$$= 5.483 \tfrac{m}{s^2}$$
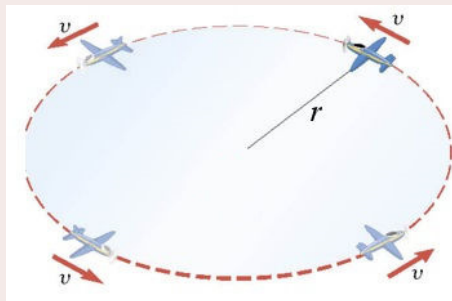
⚠️ Common Mistake: Using the Equations of Motion for Constant Acceleration for a Circular Motion

It is important not to use the centripetal acceleration in the equations for a motion with constant acceleration because this acceleration is not constant. Indeed, the previous diagram shows that the direction of the acceleration is continuously changing, which means that it is not constant.

2.5 NON-UNIFORM CIRCULAR MOTION

In a non-uniform circular motion, the object still follows a circular path, but with a changing speed. To analyze this kind of motion, the acceleration is resolved into two components.
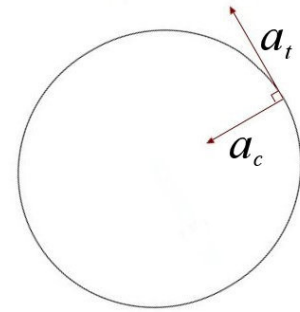
The first component, a_c, is the centripetal acceleration. It causes the path to curve. Its magnitude is always given by v^2/r, where v is the instantaneous speed of the object.

Centripetal acceleration: cause the path to curve

$$a_c = \frac{v^2}{r}$$ Towards the centre

Common Mistake: Using $a_c = 4\pi^2 r/T^2$ to Calculate the Centripetal Acceleration in a Non-Uniform Circular Motion.

This formula was obtained by assuming that the speed is constant. Do not use it in a non-uniform circular motion, where the speed changes.

The second component, a_t, changes the speed of the object. It is tangent to the circle and this is why it is called the *tangential acceleration*. If it is in the same direction as the velocity, the speed of the object increases and if it is in the opposite direction as the velocity, the speed of the object decreases. This acceleration corresponds to the rate of change of the speed and is, therefore,

Tangential Acceleration: Changes the Speed

$$a_t = \frac{dv}{dt}$$ Tangent to the circle

The total acceleration (a) of the object can be found from these two components with

Total Acceleration in a Circular Motion

$$a = \sqrt{a_c^2 + a_t^2}$$

Example 2.5.1

The diagram to the right shows the magnitude and direction of the acceleration of a car in a curve. The radius of the curve is 8 m.

a) What is the magnitude of the centripetal acceleration of the car?

   The centripetal acceleration of the car corresponds to the component of the acceleration towards the centre.
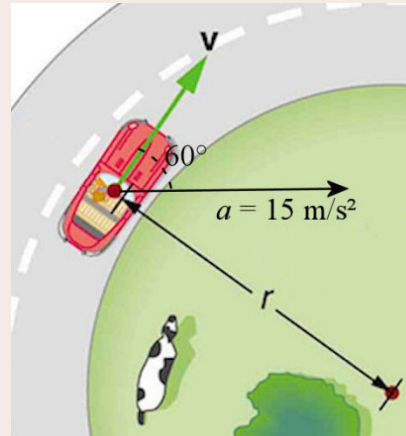
   According to the diagram to the right, this component is

$$\frac{a_c}{a} = \cos 30°$$

$$\frac{a_c}{15 \frac{m}{s^2}} = \cos 30°$$

$$a_c = 12.99 \frac{m}{s^2}$$
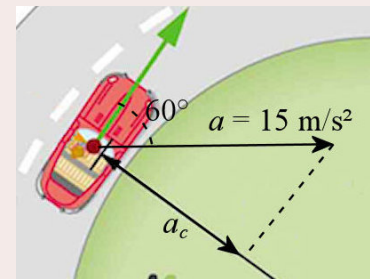
b) What is the speed of the car?

   The speed is found with the formula for the centripetal acceleration.

$$a_c = \frac{v^2}{r}$$

$$12.99 \frac{m}{s^2} = \frac{v^2}{8m}$$

$$v = 10.19 \frac{m}{s}$$

c) What is the magnitude of the tangential acceleration of the car?

   The tangential acceleration is the component of the acceleration perpendicular to the radius. According to the diagram to the right, this component is

$$\frac{a_t}{a} = \cos 60°$$

$$\frac{a_t}{15 \frac{m}{s^2}} = \cos 60°$$

$$a_t = 7.5 \frac{m}{s^2}$$
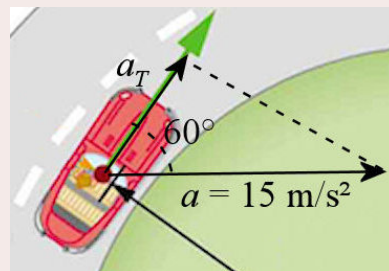
d) Is the speed of this car increasing or decreasing?

As the tangential acceleration is in the same direction as the velocity, the speed of the car is increasing.

In a circular motion, the position of the object along the circular path is denoted s (thus, Δs is the distance travelled along the circle). The speed of the object corresponds to the rate at which this position changes which means that

$$v = \frac{ds}{dt}$$

As this formula and

$$a_t = \frac{dv}{dt}$$

are exactly the same formulas as in chapter 1, except that the position is denoted s instead of x or y, the solutions are the same. If the tangential acceleration is constant, the solutions are
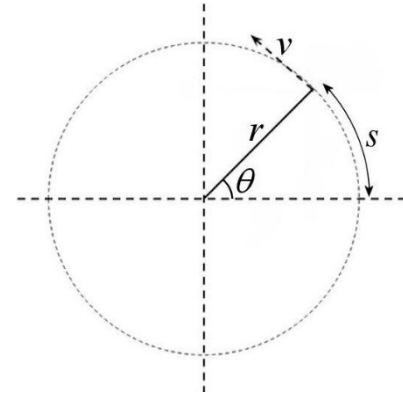
Circular Motion Kinematics Equations if a_t Is Constant

$$v = v_0 + a_t t$$
$$s = s_0 + v_0 t + \frac{1}{2} a_t t^2$$
$$2a_t (s - s_0) = v^2 - v_0^2$$
$$s = s_0 + \frac{1}{2}(v_0 + v)t$$

Example 2.5.2

A plane is following a circular trajectory. The speed increases at a constant rate of 8 m/s², and the initial velocity is 100 m/s. The radius of curvature is 5000 m. After a while, the plane has travelled a distance equal to 1/12 of the circumference of the circle.
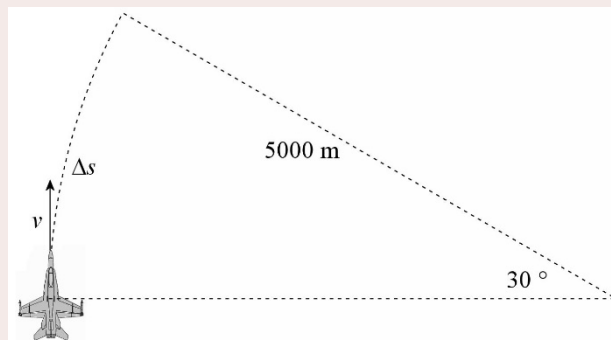
a) What is the distance travelled by the plane at this instant (Δs on the diagram)?

The distance corresponds to 1/12 of a circle. So, the displacement is

$$\Delta s = \frac{1}{12} 2\pi r$$

$$= \frac{1}{12} \cdot 2\pi \cdot 5000m$$

$$= 2618m$$

b) What is the speed of the plane at that instant?

With a constant tangential acceleration, the speed is found with the following equation (if the initial position is set at $s = 0$ m).

$$2a_t (s - s_0) = v^2 - v_0^2$$

$$2 \cdot 8 \tfrac{m}{s^2} \cdot (2618m - 0m) = v^2 - \left(100 \tfrac{m}{s}\right)^2$$

$$v = 227.8 \tfrac{m}{s}$$

c) What is the magnitude of the acceleration of the plane at that instant?

The two components of the acceleration must be found. The tangential acceleration is already known (8 m/s²). It remains to find the centripetal acceleration. Its magnitude is

$$a_c = \frac{v^2}{r}$$

$$= \frac{\left(227.8 \tfrac{m}{s}\right)^2}{5000m}$$

$$= 10.38 \tfrac{m}{s^2}$$

The acceleration is, therefore,

$$a = \sqrt{a_c^2 + a_t^2}$$

$$= \sqrt{\left(8 \tfrac{m}{s^2}\right)^2 + \left(10.38 \tfrac{m}{s^2}\right)^2}$$

$$= 13.10 \tfrac{m}{s^2}$$
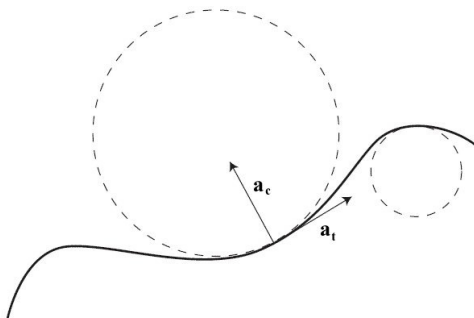
Note that any motion can be analyzed using these formulas since any small portion of a curved path can be considered to be part of a circle.

Everywhere, the centripetal acceleration, perpendicular to the motion, deflects the object from a straight path whereas the tangential acceleration, parallel to the motion, changes the speed of the object.

Of course, this use of centripetal and tangential components is not mandatory. For example, this procedure would greatly complicate the analysis of a projectile motion.

SUMMARY OF EQUATIONS

Position of an Object

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Displacement

$$\overrightarrow{\Delta r} = \vec{r}_2 - \vec{r}_1$$

Average Speed

$$\bar{\vec{v}} = \frac{\Delta \vec{r}}{\Delta t}$$

In components: $\bar{v}_x = \dfrac{\Delta x}{\Delta t}$ $\bar{v}_y = \dfrac{\Delta y}{\Delta t}$ $\bar{v}_z = \dfrac{\Delta z}{\Delta t}$

Instantaneous Speed

$$\vec{v} = \frac{d\vec{r}}{dt}$$

In components : $v_x = \dfrac{dx}{dt}$ $v_y = \dfrac{dy}{dt}$ $v_z = \dfrac{dz}{dt}$

Variation of Velocity

$$\Delta \vec{v} = \vec{v}_2 - \vec{v}_1$$

Average Acceleration

$$\bar{\vec{a}} = \frac{\Delta \vec{v}}{\Delta t}$$

In components: $\bar{a}_x = \dfrac{\Delta v_x}{\Delta t}$ $\bar{a}_y = \dfrac{\Delta v_y}{\Delta t}$ $\bar{a}_z = \dfrac{\Delta v_z}{\Delta t}$

Instantaneous Acceleration

$$\vec{a} = \frac{d\vec{v}}{dt}$$

In components: $a_x = \dfrac{dv_x}{dt}$ $a_y = \dfrac{dv_y}{dt}$ $a_z = \dfrac{dv_z}{dt}$

Three-Dimensional Motion with Constant Acceleration

$$v_x = v_{0x} + a_x t \qquad\qquad v_y = v_{0y} + a_y t \qquad\qquad v_z = v_{0z} + a_z t$$

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \qquad y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \qquad z = z_0 + v_{0z}t + \frac{1}{2}a_z t^2$$

$$2a_x(x - x_0) = v_x^2 - v_{x0}^2 \qquad 2a_y(y - y_0) = v_y^2 - v_{y0}^2 \qquad 2a_z(z - z_0) = v_z^2 - v_{z0}^2$$

$$x = x_0 + \frac{1}{2}(v_{x0} + v_x)t \qquad y = y_0 + \frac{1}{2}(v_{y0} + v_y)t \qquad z = z_0 + \frac{1}{2}(v_{z0} + v_z)t$$

Acceleration of a Projectile

$$a_x = 0$$
$$a_y = -9.8 m/s^2 = -g$$

To use these equations, you must have a perfectly horizontal *x*-axis and a perfectly vertical *y*-axis pointing upwards since we use a negative acceleration.

Basic Equations of the Projectile

$$x = x_0 + v_{0x}t$$
$$v_y = v_{0y} - gt$$
$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$
$$-2g(y - y_0) = v_y^2 - v_{y0}^2$$

Range of a Projectile Returning to the Ground at the Same Height it Was Launched

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

Link between *x* and *y* Positions for a Projectile

$$y = (\tan\theta)x - \frac{g}{2v_0^2 \cos^2\theta}x^2 \qquad \text{The starting point must be (0,0)}$$

Centripetal Acceleration

$$a_c = \frac{v^2}{r} \text{ or } a_c = \frac{4\pi^2 r}{T^2} \qquad \text{Towards the centre}$$

(The second formula is valid only if the speed is constant.)

Tangential Acceleration: Changes the Speed

$$a_t = \frac{dv}{dt} \qquad \text{Tangent to the circle}$$

Total Acceleration in a Circular Motion

$$a = \sqrt{a_c^2 + a_t^2}$$

Circular Motion Kinematics Equations if a_t is Constant

$$v = v_0 + a_t t \qquad\qquad 2a_t\left(s - s_0\right) = v^2 - v_0^2$$

$$s = s_0 + v_0 t + \frac{1}{2}a_t t^2 \qquad\qquad s = s_0 + \frac{1}{2}\left(v_0 + v\right)t$$

EXERCISES

2.1 The Laws of Kinematics in Two or Three Dimensions

1. At $t = 0$ s, an object is at $x = 0$ m and $y = 0$ m.
   Its speed is then 15 m/s and points at an angle
   of 30° above the positive x-axis. At $t = 2$ s,
   the object is now at $x = 8$ m and $y = 6$ m. Its
   speed is now 20 m/s, and it points at an angle
   of 60° above the negative x-axis.

   a) What is the average velocity between
      $t = 0$ s and $t = 2$ s?
   b) What is the average acceleration
      between $t = 0$ s and $t = 2$ s?
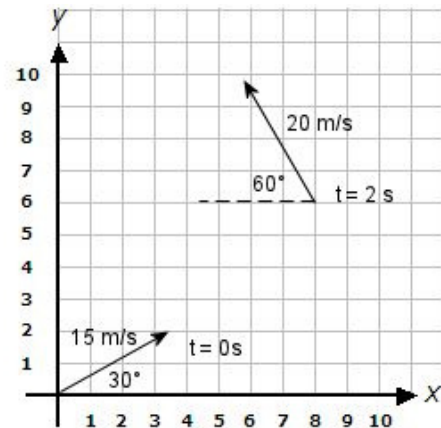
www.helpingwithmath.com/printables/worksheets/geometry/geo0701coords11.htm

2. The diagram shows the position of the Earth on its motion around the Sun at two
   moments (1 and 2). Remember that the Earth rotates around the Sun once every
   year.

   a) What is the displacement between
      positions 1 and 2?
   b) What is the distance travelled
      between positions 1 and 2?
   c) What is the average velocity
      between positions 1 and 2?
   d) What is the average acceleration
      between positions 1 and 2?
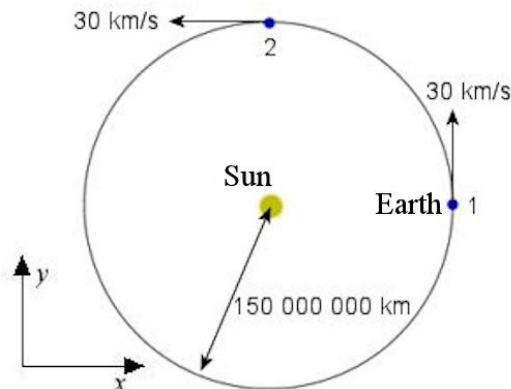
3. The position of an object is given by the formulas

$$x = -3\tfrac{m}{s^2}t^2 + 2\tfrac{m}{s}t - 4m$$
$$y = -2\tfrac{m}{s^3}t^3 + 6\tfrac{m}{s^2}t^2 + 1m$$

   a) What are the speed and the direction of the velocity at $t = 1$ s?
   b) What is the acceleration (magnitude and direction) at $t = 1$ s?

4. Here are the position, the velocity and the acceleration of an object at $t = 0$ s.

$$\vec{a} = \left(\vec{i} - 2\vec{j}\right)\tfrac{m}{s^2}$$
$$\vec{v}_0 = \left(2\vec{i} + 4\vec{j} - \vec{k}\right)\tfrac{m}{s}$$
$$\vec{r}_0 = \left(2\vec{i} - 2\vec{j} + 2\vec{k}\right)m$$

What will the position of the object at $t = 5$ s be if the acceleration is constant?

5. Here are the position and the velocity of an object at two moments.

$$t = 0s \qquad \vec{r} = \left(\vec{i} - \vec{j} - 2\vec{k}\right)m \qquad \vec{v} = \left(-2\vec{i} + 1\vec{k}\right)\tfrac{m}{s}$$
$$later \qquad \vec{r} = \left(2\vec{i} + \vec{j} - 4\vec{k}\right)m \qquad \vec{v} = \left(5\vec{i} + 6\vec{j} - 7\vec{k}\right)\tfrac{m}{s}$$

What is the acceleration, knowing that it is constant?

2.2 Projectile Motion

6. Anatole is making a car jump to impress his friends. So, he reaches the top of a hill at 180 km/h and then flies off the top of the hill at an angle of 32 °.

180 km/h

32°
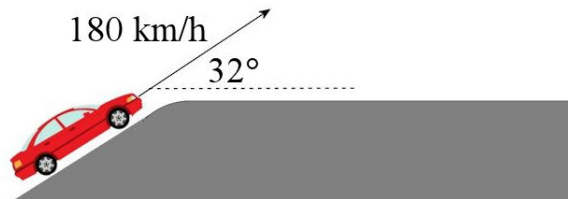
fr.depositphotos.com/2577683/stock-illustration-Car.html

The jump is going to look a little like this.
https://www.youtube.com/watch?v=5kMTHwjy6Z8
(Do not try that, it's really dangerous. You cannot control the car when it is in the air and this can be catastrophic if the car turns a little while flying.)

   a) What is the flight time of the car?
   b) What is the maximum height reached by the car?
   c) How far from the hill does the car hit the ground?

7. Justin Bieber is thrown from the top of a 265 m high cliff with a speed of 45 m/s and at an angle of 60 °.

   a) What is the maximum height reached by Justin? (From the top of the cliff)
   b) What is Justin's flight time?
   c) What is Justin's range?
   d) What are the speed and the direction of the velocity of Justin just before he hits the ground?
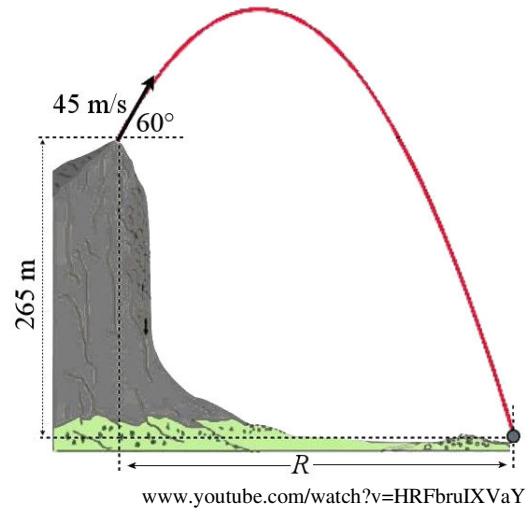
8. How fast has Alissia thrown the ball to make a basket in the situation shown in the diagram?
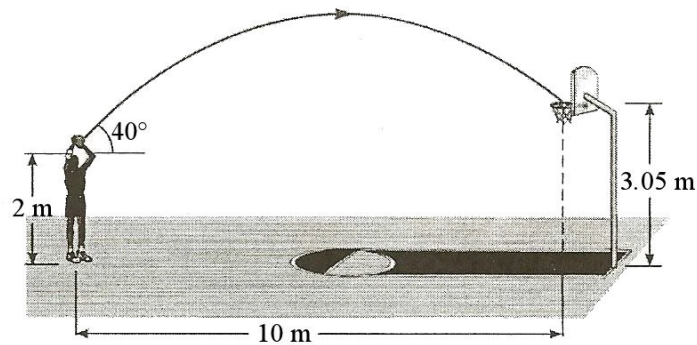
9. What are the initial speed and launch angle θ of this projectile if the flight time is 0.5 s?
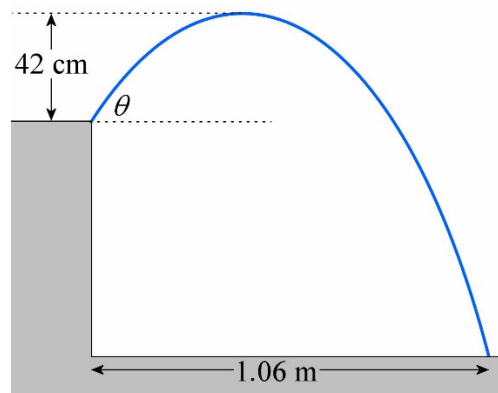
10. Ruprecht's mission is to jump over the Corinth Canal on his motorcycle. Ruprecht's motorcycle goes at 126 km/h, and the canal has a width of 70 m. It is assumed that the starting point is at the same height as the end point.

   a) What must be the launch angle?
   b) For how long will Ruprecht be in the air?

11. During an explosion of the Arenal volcano in Costa Rica, a stone is projected as shown in this diagram.
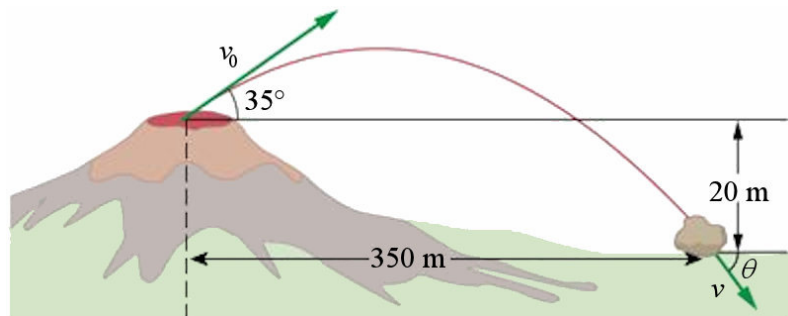
   a) What was the initial speed of the stone?
   b) What was the flight time of the stone?
   c) What was the maximum height reached by the stone, measured from its starting position?
   d) What were the speed and the direction of the velocity of the stone just before it hits the ground?

12. An object is launched from the ground. The following diagram gives the speed and the direction of the velocity of the ball 3 seconds later.

   a) What is the range?
   b) What is the initial speed of the ball?
   c) What is the launch angle of the ball?

$t = 3$ s
$v = 20$ m/s

$t = 0$
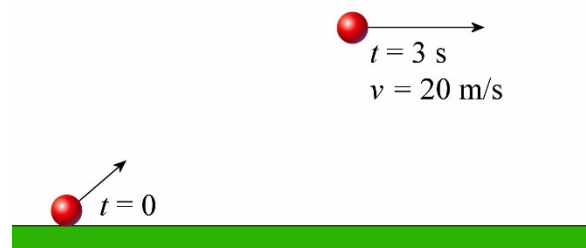
13. An object is launched from the ground with a speed of 30 m/s and a launch angle of 60° (position 1). Later (position 2), the object is still going upwards, but with a speed of only 20 m/s.

   a) How long has it taken for the stone to move from position 1 to position 2?
   b) What is the distance between position 1 and position 2?
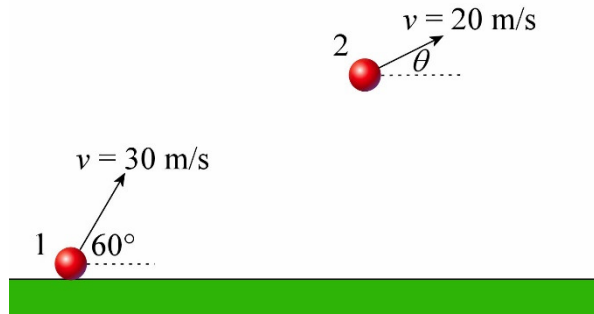
14. In the situation shown in the diagram, what should be the speed of the projectile so that it hits the target? (The cannon fires the ball horizontally.)
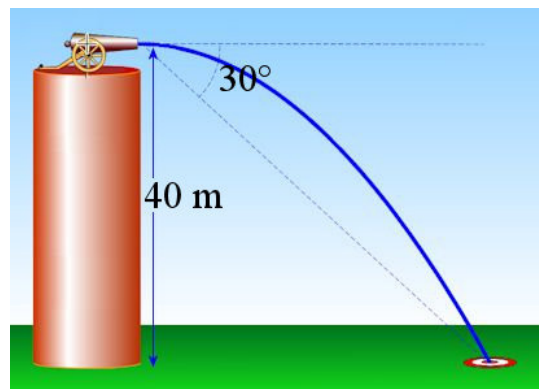
15. When a gun fires a projectile horizontally at a speed of 30 m/s, the projectile passes to 2 m below the target.
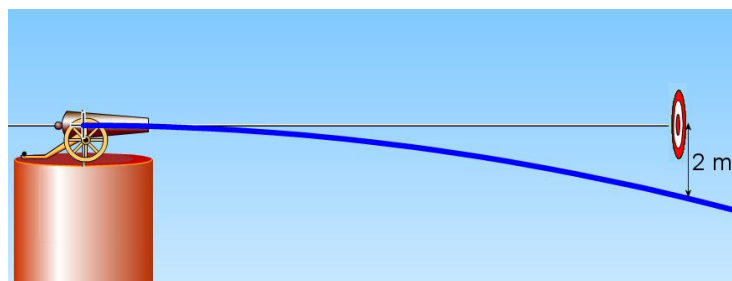
At what angle should the cannon be fired to hit the centre of the target?
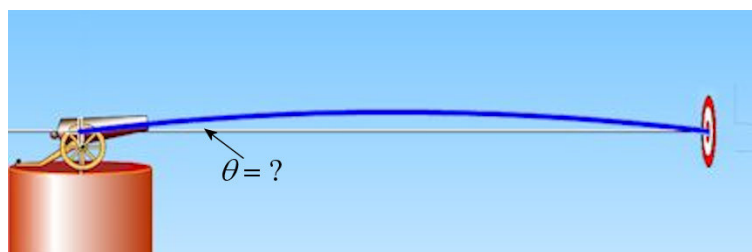
16. In the situation shown in the diagram...
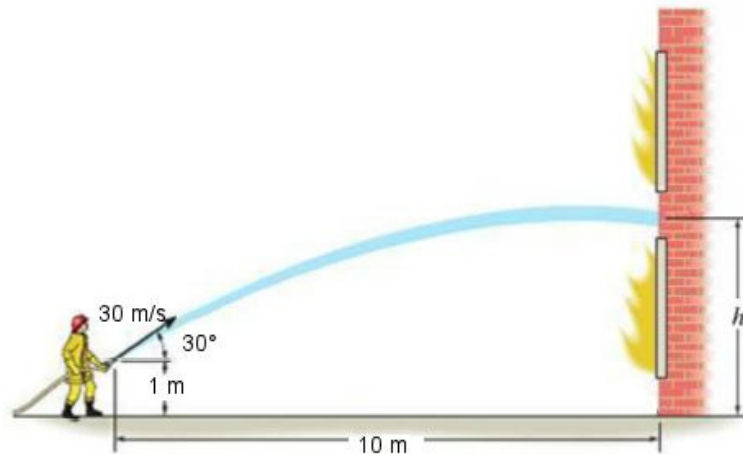
a) At what height does the water hit the wall?
b) What are the speed and the direction of the velocity of the water when it hits the wall?

17. A snowball cannon launches snowballs with a speed of 20 m/s and a launch angle of 35°. Will the balls fall in front of the wall, hit the wall, or go over the wall?
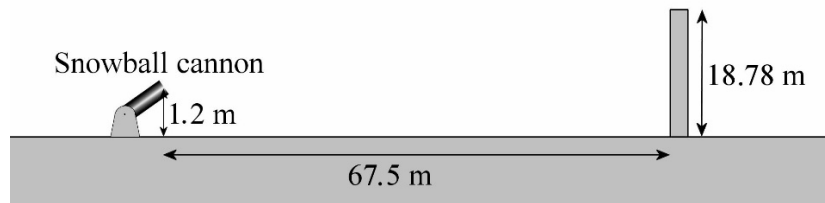
18. Jules kicks a ball with a speed of 24 m/s and an angle of 40 °. The ball then hits the edge of the roof of an 8 m high building. What is the distance between the building and Jules?
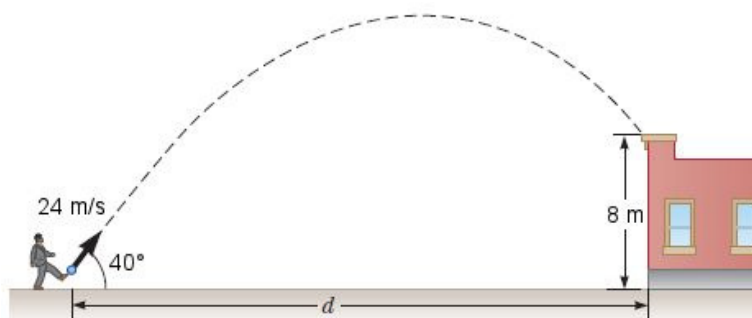
(Note that the path shown in the diagram is not necessarily the true path. The ball could be going upwards when it hits the roof.)

19. A stone is thrown from the top of a cliff. What is the height of the cliff if the speeds at the top and at the bottom of the cliff are those given in this diagram?
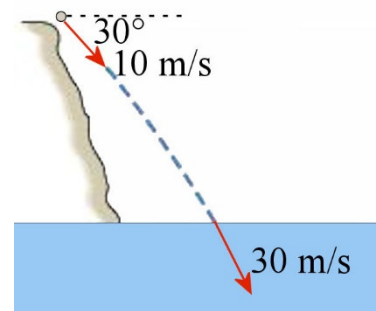
20. Two projectiles were launched from the same place with the same speed (50 m/s). One of the projectiles was launched with an angle of 60° and the other projectile was launched with an angle of 45°. However, the two projectiles hit the same place at the top of a cliff. What is the height of the cliff?
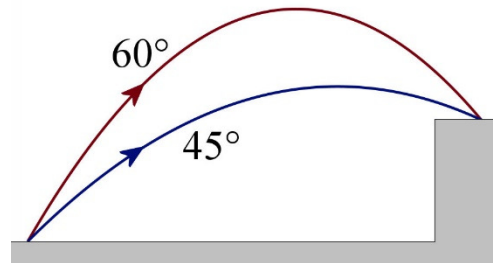
21. At what speed should this projectile be launched in order to hit the target?
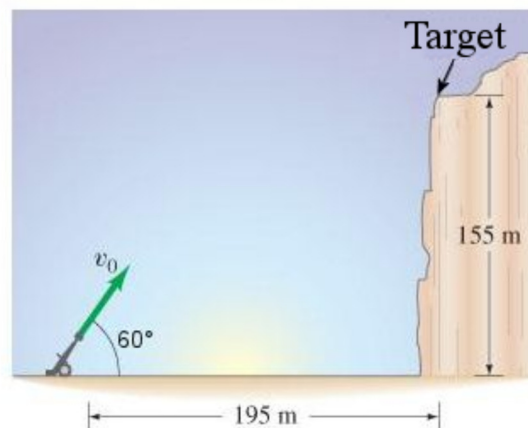
22. At what angle should this projectile be launched to hit the balloon?
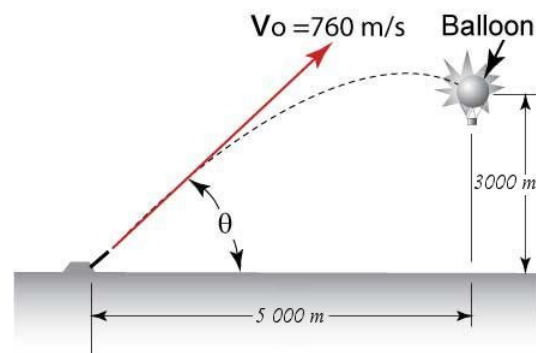
23. When a projectile is thrown at an angle of 28 degrees it falls back 40 m from its starting point, at the same height. How far from the starting point would a projectile fall if it was thrown at the same speed but with a 32-degree angle (it still falls at the same height)?
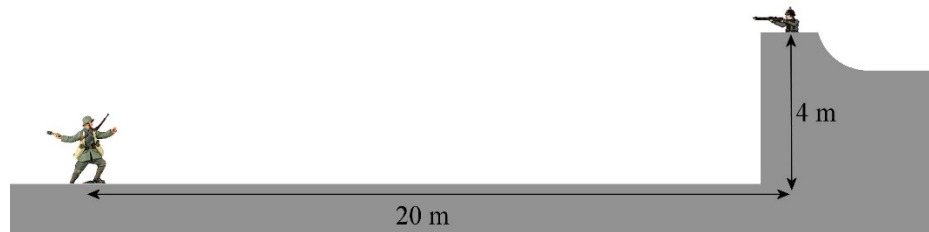
24. A projectile hits the ground at the same height it was launched. Knowing that the range is 40 m and that the maximum height reached by the projectile is 15 m, find the starting angle of the projectile.

Hint to solve this problem:
$$\sin 2\theta = 2\sin\theta\cos\theta$$
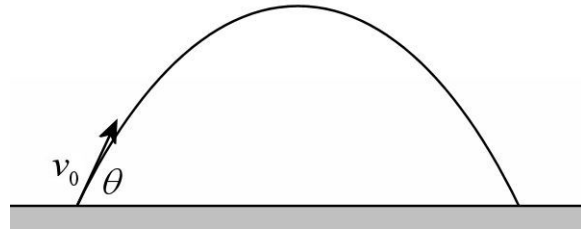
25. Klaus must throw a grenade on an enemy position located at the position shown in the diagram. The grenade explodes 2 seconds after it was thrown. How fast and at what angle should Klaus throw the grenade so that it explodes immediately upon arriving at the enemy position?

26. A projectile hits the ground at the same height it was launched. The height of the projectile was h at $t = 1$ s and $t = 5$ s. What is the value of h?
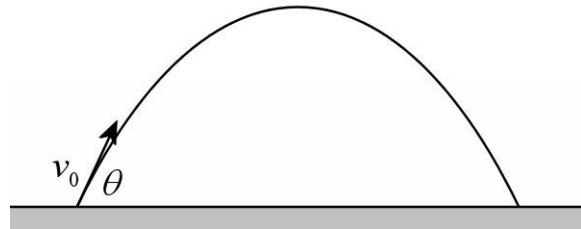
27. A rocket takes off from rest at an angle of 50 °. As long as the engine is running, the rocket makes a rectilinear motion with an acceleration of 20 m/s². After 20 seconds, the engine stops, and the rocket is now free-falling.
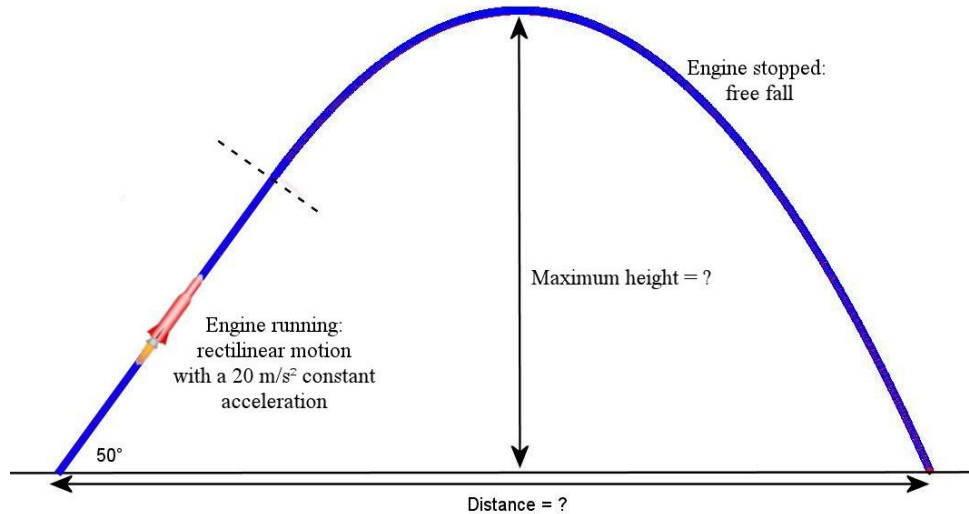
a) What is the maximum height reached by the rocket?
b) What is the total time flight of the rocket?
c) How far from its starting point does the rocket fall?

2.3 Uniform Circular Motion

28. The Moon orbits the Earth at a speed of 1024 m/s. The radius of the orbit of the Moon is 384 400 km. What is the magnitude of the centripetal acceleration of the Moon?

29. The children in this ride do one revolution every 5 s. If the circumference of their path is 50 m, what is the magnitude of the centripetal acceleration of the children?
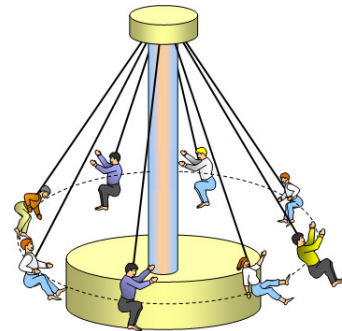
30. An electric train moves at a constant speed on a circular track. The magnitude of the centripetal acceleration of the train is 20 m/s². How long will the train take to do 50 laps if the track has a diameter of 1 m?
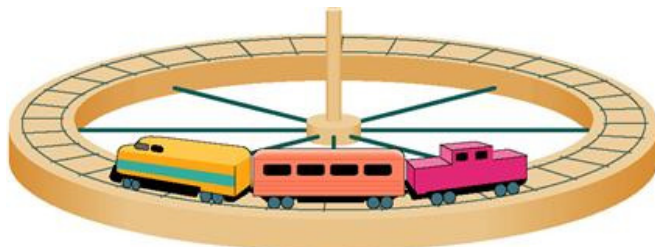
31. This car moves with a constant speed of 36 m/s on a circular track. What is the magnitude of the centripetal acceleration if the car goes around the track in 24 s?
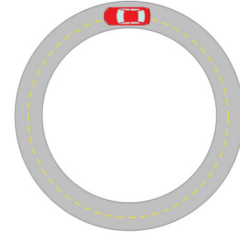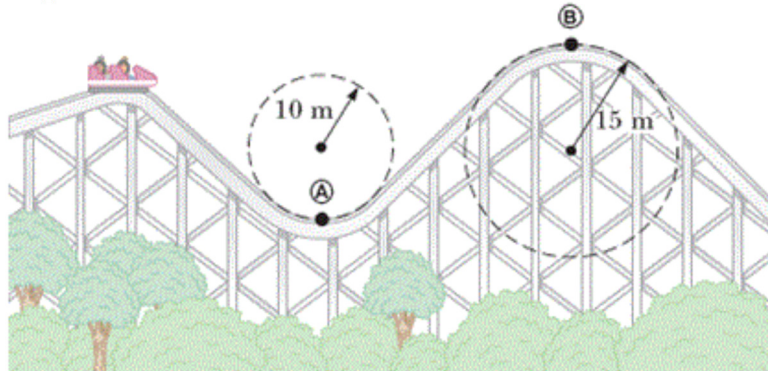
32. Show that the centripetal acceleration in a uniform circular motion can also be calculated with the following formula.

$$a_c = \frac{2\pi v}{T}$$

2.4 Non-Uniform Circular Motion

33. The roller coaster carriage of this diagram moves at a speed of 20 m/s at point A and 12 m/s at point B.

What are the magnitude and direction of the centripetal acceleration of the carriage at point A and B?

34. The car shown on this 100-metre diameter track has an initial speed of 90 km/h. The car brakes so that it stops in two laps with a constant tangential acceleration.

   a) What is the magnitude of the tangential acceleration of the car?
   b) What are the magnitude and the direction of the centripetal acceleration after the car has done one lap?
   c) What are the magnitude and the direction of the acceleration when the car has done one lap? (For the direction, give the angle between the velocity and the acceleration.)
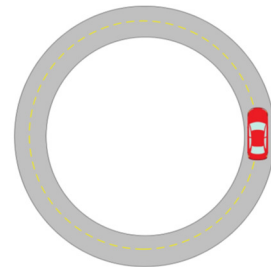
35. A car is heading north at 30 m/s. The acceleration of the car is shown in the diagram.
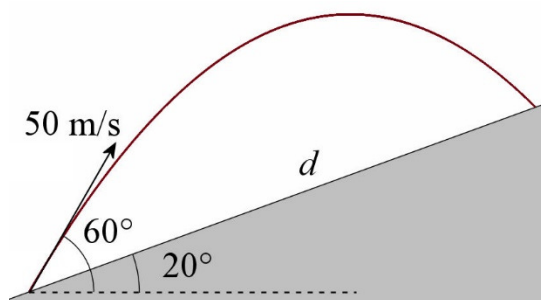
   a) Is the speed of the car increasing or decreasing?
   b) What is the tangential acceleration?
   c) Is this car turning eastwards or westwards?
   d) What is the radius of curvature of the road?
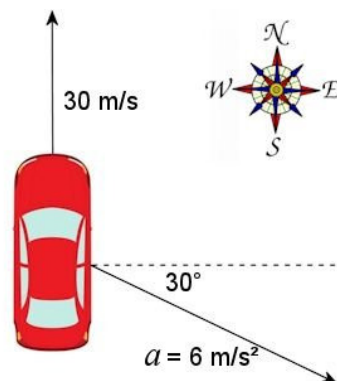
Challenges
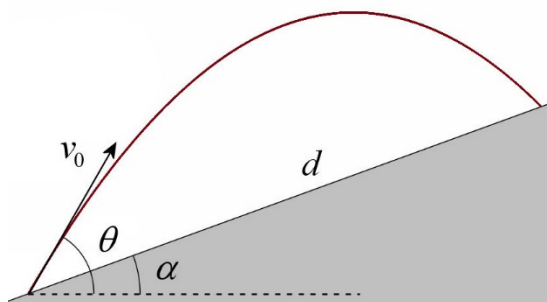(Questions more difficult than the exam questions.)

36. What is the value of d in this situation?

37. An object is launched on a slope, as shown on the diagram. Show that to reach the greatest distance on the slope, one must have

$$\theta = \frac{1}{2}\left(90° + \alpha\right)$$

and that the maximum distance reached is

$$d_{max} = \frac{1 - \sin\alpha}{\cos^2\alpha}\frac{v_0^2}{g}$$
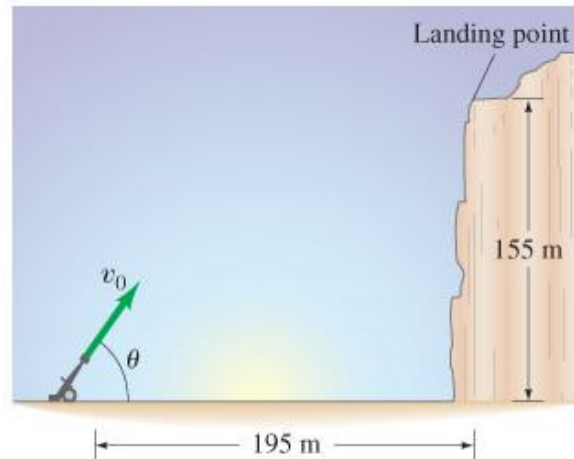
N.B.: These identities may be useful.

$$2\sin\theta\cos\theta = \sin 2\theta$$
$$\cos(-\alpha) = \cos\alpha \qquad \sin(-\alpha) = -\sin\alpha$$
$$\cos(90° - \alpha) = \sin\alpha \qquad \sin(90° - \alpha) = \cos\alpha$$
$$\cos(90° + \alpha) = -\sin\alpha \qquad \sin(90° + \alpha) = \cos\alpha$$
$$\cos^2\left(\frac{x}{2}\right) = \frac{1 + \cos x}{2}$$

38. A projectile must reach the landing place shown in the diagram.

   a) What is the minimum speed that allows the projectile to reach this place?
   b) What is the launch angle that the projectile must have at this minimum speed?

Hit to solve this problem:

$$2\sin\theta\cos\theta = \sin 2\theta$$

ANSWERS

2.1 The Laws of Kinematics in Two or Three Dimensions

1. a) $\vec{v} = \left(4\vec{i} + 3\vec{j}\right)\frac{m}{s}$ b) $\vec{a} = \left(-11.495\vec{i} + 4.91\vec{j}\right)\frac{m}{s^2}$
2. a) $\overrightarrow{\Delta s} = \left(-1.5\times10^{11}\vec{i} + 1.5\times10^{11}\vec{j}\right)m$ b) 2.3562 x 10^{11} m
    c) $\vec{v} = \left(-19\,013\vec{i} + 19\,013\vec{j}\right)\frac{m}{s}$ d) $\vec{a} = \left(-0.0038\vec{i} - 0.0038\vec{j}\right)\frac{m}{s^2}$
3. a) 7.21 m/s at 123.7°   b) 6 m/s² at 180°
4. $\vec{r} = \left(24.5\vec{i} - 7\vec{j} - 3\vec{k}\right)m$
5. $\vec{a} = \left(10.5\vec{i} + 9\vec{j} - 12\vec{k}\right)\frac{m}{s^2}$

2.2 Projectile Motion

6. a) 5.407 s   b) 35.82 m   c) 229.3 m
7. a) 77.49 m above its starting point   b) 12.34 s   c) 277.6 m
    d) 85.0 m/s at -74.6°
8. 10.67 m/s
9. 3.57 m/s at 53.5°
10. a) 17.03°   b) 2.09 s
11. a) 58.09 m/s   b) 7.355 s   c) 56.65 m   d) 61.37 m/s at -39.16°
12. a) 120 m   b) 35.56 m/s   c) 55.8°
13. a) 1.301 s   b) 32.11 m
14. 24.25 m/s
15. 6.02° or 83.98°
16. a) 6.048 m   b) 28.3 m/s at 23.4°

17. It falls in front of the wall.
18. 12.04 m or 45.85 m
19. 40.82 m
20. 50.04 m
21. 63.86 m/s
22. 33.46° or 87.50°
23. 43.37 m
24. 56.3°
25. 15.47 m/s at 49.72°
26. 24.5 m
27. a) 7855 m   b) 91.3 s   c) 20 905 m

2.3 Uniform Circular Motion

28. 0.002728 m/s²
29. 12.57 m/s²
30. 49.67 s
31. 9.425 m/s²
32. See the proof in the solution document

2.4 Non-Uniform Circular Motion

33. Point A: 40 m/s² upwards   Point B: 9.6 m/s² downwards
34. a) 0.4974 m/s² in a direction opposite to the velocity of
    the car   b) 6.25 m/s² towards the center of the circle
    c) 6.27 m/s² at 94.4° from the velocity (see diagram)

35. a) It slows down   b) -3 m/s²   c) eastwards
    d) 173.2 m
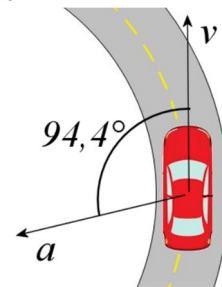
Challenges

36. 185.7 m
37. a) 62.93 m/s   b) 64.24°