8 WORK

Erwan, whose mass is 65 kg, goes Bungee jumping. He has been in free-fall for 20 m when the bungee rope begins to stretch. What will the maximum stretching of the rope be if the rope acts like a spring with a 100 N/m constant?

www.lifeinsuranceinsights.com/life-insurance-2/what-will-your-hobby-cost-you.html

Discover the answer to this question in this chapter.

8.1 DEFINITION OF WORK

The Work Done by a Constant Force on an Object Moving in a Straight Line

If a constant force acts on an object moving along a straight line, then some work is done. This work is the scalar product of the force and the displacement.

The Work Done by a Constant Force Acting on an Object Moving in a Straight Line

$$W = \vec{F} \cdot \overrightarrow{\Delta s}$$

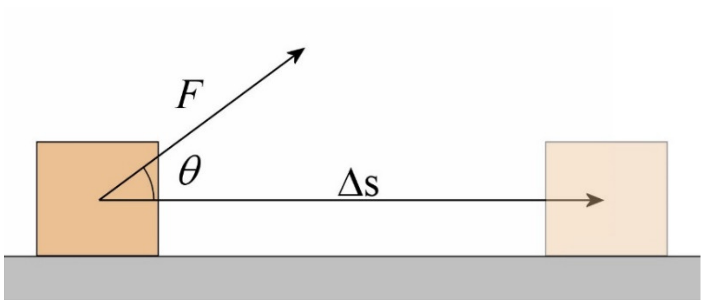
This scalar product can be calculated with these 2 formulas.

$$W = F\Delta s\cos\theta$$

or

$$W = F_x\Delta x + F_y\Delta y + F_z\Delta z$$

Δs is the displacement of the object and θ is the angle between the displacement and the force.



The unit used to measure work is the Nm. Another name was given to this unit: the joule (J).

The Unit of Work: Joule

$$1\ J = 1\ Nm = 1\tfrac{kgm^2}{s^2}$$

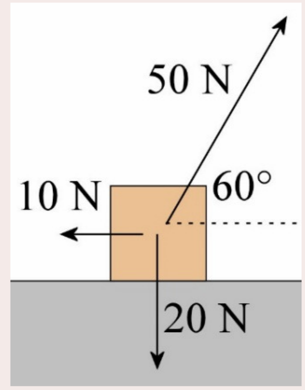
If there are several forces acting on an object, the sum of the work done by each of these forces is the net work.

The Net Work Done on an Object

$$W_{net} = \sum W = W_1 + W_2 + W_3 + \ldots$$

To read a short history of the concept of work, you can consult the following document.
https://physique.merici.ca/mechanics/HistoryWork.pdf

Example 8.1.1



What is the net work done on this box if it moves 3 metres towards the right?

Let's calculate the work done by each of the forces.

The work done by the 20 N force is

$$\begin{aligned} W_1 &= 20N \cdot 3m \cdot \cos 90° \\ &= 0J \end{aligned}$$

The angle is 90° because the force is directed downwards and the displacement is towards the right.

The work done by the 10 N force is

$$\begin{aligned} W_2 &= 10N \cdot 3m \cdot \cos 180° \\ &= -30J \end{aligned}$$

The angle is 180° because the force is to the left and the displacement is towards the right.

The work done by the 50 N force is

$$\begin{aligned} W_3 &= 50N \cdot 3m \cdot \cos 60° \\ &= 75J \end{aligned}$$

The angle is 60° as shown in the diagram.

Therefore, the net work is

$$\begin{aligned} W_{net} &= W_1 + W_2 + W_3 \\ &= 0J + -30J + 75J \\ &= 45J \end{aligned}$$

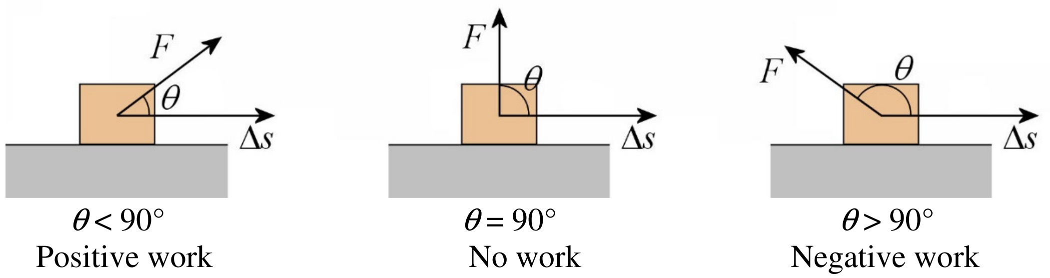
Here are some commentaries.

1) F and Δs are never negative. These are always the magnitudes of the force and the displacement.

2) There is no need to resolve the forces into components when calculating the work with $F\Delta s \cos\theta$. The magnitude of the force and the displacement must be put in the formula, not the components.

3) The work can be positive, negative, or zero. As F and Δs are always positive, the value of the angle determines the sign of the work. As the cosine is positive for an

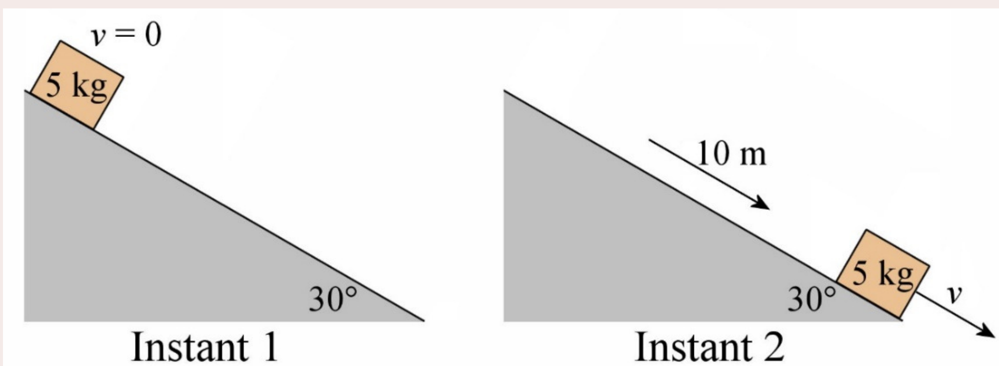
angle smaller than 90° and negative for an angle between 90° and 180°, the sign of the work is as follows.



4) The angle is always positive. Always use the smallest angle between the force and the displacement, and there's no sign for this angle. The angle will therefore always be between 0° and 180°.

Example 8.1.2

A 5 kg box slides 10 m downhill on a 30° slope. The coefficient of friction between the slope and the box is 0.2. What is the net work done on the box between these two instants?

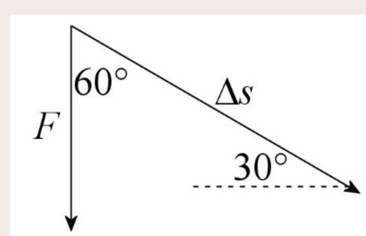


To find the work done by the forces, the forces acting on the box must be found first. There are 3 forces acting on the 5 kg box.

1) The weight.
2) A normal force.
3) A friction force.

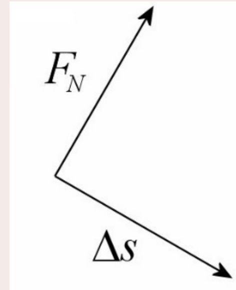
The work done by the weight is

$$
\begin{aligned}
W_g &= mg\Delta s\cos\theta \\
&= 5kg \cdot 9.8\tfrac{N}{kg} \cdot 10m \cdot \cos 60° \\
&= 245J
\end{aligned}
$$



The angle between the force and the displacement is 60° according to the diagram.

The work done by the normal force is



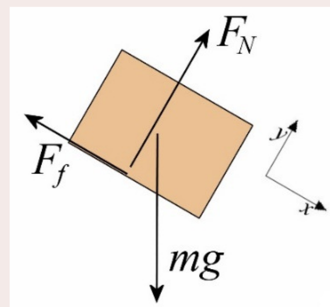
$$\begin{aligned} W_N &= F_N \cdot 10m \cdot \cos 90° \\ &= 0J \end{aligned}$$

The angle between the force and the displacement is 90° according to the diagram.

The work done by the friction force is

$$W_f = \mu_k F_N \cdot 10m \cdot \cos 180°$$

The angle is 180° because the friction force is directed uphill and the displacement is directed downhill. To find this work, the magnitude of the normal force must be found. Without going too much into details (because many examples of calculations of the normal force on a slope were done in the previous chapters), the equation of the *y*-component of the forces is



$$\begin{aligned} \sum F_y &= ma_y \\ F_N + mg\sin(-60°) &= 0 \end{aligned}$$

The normal force is then

$$\begin{aligned} F_N &= mg \sin 60° \\ &= 5kg \cdot 9.8\tfrac{N}{kg} \cdot \sin 60° \\ &= 42.435N \end{aligned}$$

and the work done by the friction force is, therefore,

$$\begin{aligned} W_f &= \mu_c F_N \cdot 10m \cdot \cos 180° \\ &= 0.2 \cdot 42.435N \cdot 10m \cdot (-1) \\ &= -84.87J \end{aligned}$$

Thus, the net work is

$$\begin{aligned} W_{net} &= W_g + W_N + W_f \\ &= 245J + 0J + -84.87J \\ &= 160.12J \end{aligned}$$

The scalar product can also be calculated from the components of the vectors.

Example 8.1.3

The force $\vec{F} = \left(3\vec{i} + 4\vec{j} + 5\vec{k}\right)N$ acts on an object moving from the point (1 m, 2 m, 3 m) to the point (5 m, 6 m, -3 m). What is the work done on the object?

Going from (1 m, 2 m, 3 m) to (5 m, 6 m, -3 m), the components of the displacement are

$$\Delta x = 4m \qquad \Delta y = 4m \qquad \Delta z = -6m$$

Therefore, the work is

$$\begin{aligned} W &= F_x \Delta x + F_y \Delta y + F_z \Delta z \\ &= 3N \cdot 4m + 4N \cdot 4m + 5N \cdot (-6m) \\ &= 12J + 16J - 30J \\ &= -2J \end{aligned}$$

The Work Done if F or θ Are not Constants

If the magnitude of the force or the angle between the force and the displacement changes, the displacement must be divided into parts where the force and the angle are constants. The work done is obtained by summing the work done in each of these parts.

The Work Done if F or θ change

$$W = \sum_{\text{F and } \theta \text{ constant}} \vec{F} \cdot \overrightarrow{\Delta s}$$

This scalar product can be calculated with these 2 formulas.

$$W = \sum_{F \text{ and } \theta \text{ constant}} F \Delta s \cos\theta$$

or

$$W = \sum_{F_x, F_y \text{ and } F_z \text{ constant}} F_x \Delta x + F_y \Delta y + F_z \Delta z$$

Example 8.1.4

A force acts on an object whose displacement is 6 m towards the right. The force is 5 N towards the right over a distance of 5 m and then 3 N towards the left over a distance of 1 m. What is the work done on the object between these two instants?

As the force changes, the displacement must be divided into two parts. The work done during the first part is

$$\begin{aligned} W_1 &= F\Delta s\cos\theta \\ &= 5N \cdot 5m \cdot \cos 0° \\ &= 25J \end{aligned}$$

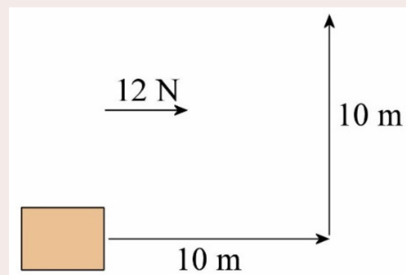
The work done during the second part is

$$\begin{aligned} W_2 &= F\Delta s\cos\theta \\ &= 3N \cdot 1m \cdot \cos 180° \\ &= -3J \end{aligned}$$

Therefore, the total work is

$$\begin{aligned} W &= W_1 + W_2 \\ &= 25J + -3J \\ &= 22J \end{aligned}$$

Example 8.1.5



A 12 N force acting towards the right acts on an object that move following the path shown in the diagram. What is the work done on the object?

As the angle between the force and the displacement changes, the displacement must be divided into parts. The work done during the first part (motion towards the right) is

$$\begin{aligned} W_1 &= F\Delta s\cos\theta \\ &= 12N \cdot 10m \cdot \cos 0° \\ &= 120J \end{aligned}$$

The work done during the second part (upwards motion) is

$$\begin{aligned} W_2 &= F\Delta s\cos\theta \\ &= 12N \cdot 10m \cdot \cos 90° \\ &= 0J \end{aligned}$$

Therefore, the total work is

$$\begin{aligned} W &= W_1 + W_2 \\ &= 120J + 0J \\ &= 120J \end{aligned}$$

But what should be done if the force or the angle are continuously changing? It seems that the displacement could not be divided into segments where the force is constant then.

Actually, it can be done. The displacement is divided into very short distances, so small that they become infinitesimally small. The work done on such a distance is

$$dW = \vec{F} \cdot \overrightarrow{ds}$$

If the works done during these small displacements are then added, the total work is obtained.

Work Done on an Object (Most General Formula)

$$W = \int \vec{F} \cdot d\vec{s}$$

There's an integral in this formula because this is what you get when you sum infinitesimally small quantities. Properly speaking, this is a line integral as it is the result of a calculation along a line (since the object passes from one place to another following a path, which is a line).

It is possible to calculate this kind of integral for three-dimensional trajectories, but this is beyond college curriculum. Only displacements along an axis (which will be denoted x) will be considered here. In such a case, the infinitesimal work is

$$\vec{F} \cdot d\vec{s} = F_x dx$$

and the work is

Work Done by a Variable Force on an Object Moving Along the *x*-axis (the Object Goes from *x* to *x'*)

$$W = \int_x^{x'} F_x dx$$

Example 8.1.6

A variable force F = (3 N/m x + 2 N) acts on an object going from x = 1 m to x = 3 m. What is the work done on the object between these two instants?

The work is

$$W = \int_{1m}^{3m} \left(3\tfrac{N}{m} x + 2N\right) dx$$

$$= \left(\frac{3\frac{N}{m} x^2}{2} + 2N \cdot x \right) \Bigg|_{1m}^{3m}$$

$$\begin{aligned}&=\left(\frac{3\frac{N}{m}\cdot(3m)^2}{2}+2N\cdot(3m)\right)-\left(\frac{3\frac{N}{m}\cdot(1m)^2}{2}+2N\cdot(1m)\right)\\&=16J\end{aligned}$$

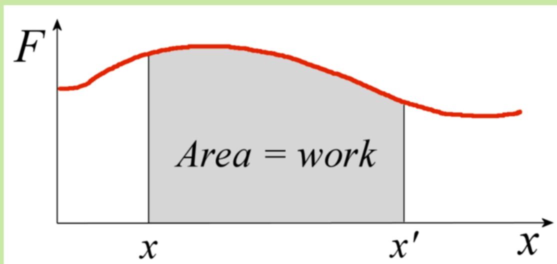
Graphical Representation of Work

Since the work is

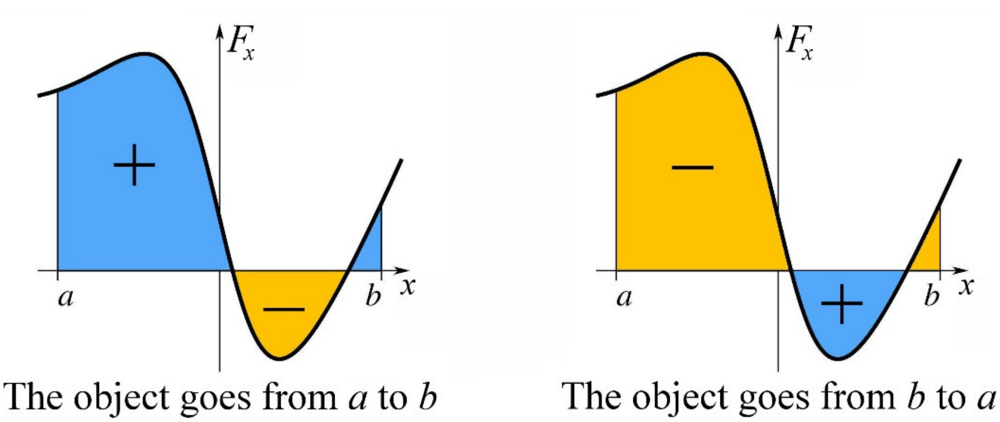
$$W=\int_x^{x'}F_x dx$$

and since an integral represents the area under the curve, we conclude that

The work done on an object is the area under the curve of the force acting on the object as a function of position.



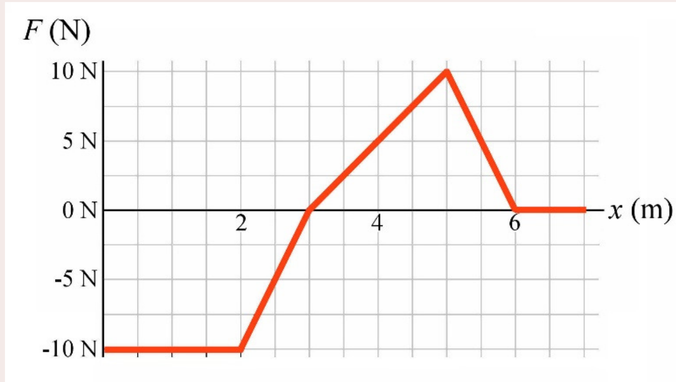
As the work can sometimes be negative, this area can also be negative. As in calculus, the area is positive if it is above the x-axis and negative if it is below the x-axis. However, these signs are inverted if the object moves from a higher value of x to a smaller value of x. This comes from the fact that the sign of an integral changes if the limits of an integral are inverted.

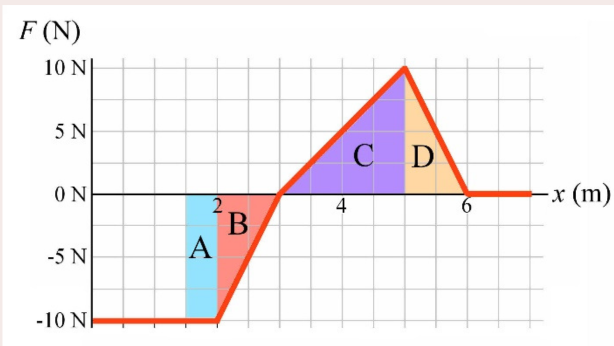


en.wikipedia.org/wiki/Integral

Example 8.1.7

Here is the graph of the force acting on an object as a function of position. What work is done on this object if it moves from $x = 1.5$ m to $x = 6$ m?





To find the work, the area under the curve between $x = 1.5$ m and $x = 6$ m must be calculated. To calculate this area, the area will be split into 4 regions (identified by the letters A, B, C and D on the graph).

The area of region A is 10 N · 0.5 m = 5 J. The work on this distance is therefore -5 J.

The area of region B is ½·(10 N · 1 m) = 5 J. The work on this distance is therefore -5 J.

The area of region C is ½· (10 N · 2 m) = 10 J. The work on this distance is therefore 10 J.

The area of region D is 1/2· (10 N · 1 m) = 5 J. The work on this distance is therefore 5 J.

Therefore, the total work is

$$\begin{aligned} W &= -5J + -5J + 10J + 5J \\ &= 5J \end{aligned}$$

The Work Done by a Spring

With the definition of work obtained previously, the work done by a spring on an object can be calculated.

As the force made by a spring is $F = -kx$, the work done by a spring on an object moving from x to x' is

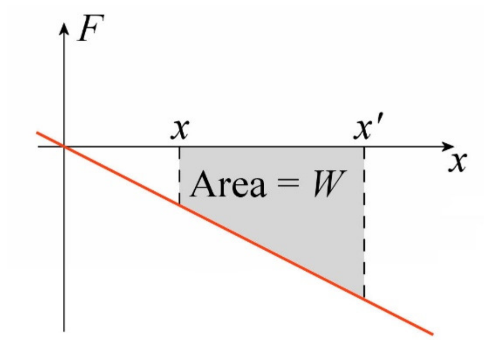
$$W_{sp} = \int_x^{x'} (-kx)\, dx = \left(\frac{-kx^2}{2} \right) \Bigg|_x^{x'}$$
$$= \frac{-kx'^2}{2} - \frac{-kx^2}{2}$$

This can be simplified to obtain

Work Done by a Spring

$$W_{sp} = -\frac{k}{2}\left(x'^2 - x^2 \right)$$

This calculation could also have been done by calculating the area under the curve of the force. Since the graph of -*kx* is a straight line with slope - *k* passing through the origin, the area under the curve between *x* and *x*' corresponds to the grey area in the grap to the right



This area is

Area = (Area of the triangle ranging from the origin to *x′*) – (Area of the triangle ranging from the origin to *x*)

$$Area = \left(\frac{x' \cdot F'}{2} \right) - \left(\frac{x \cdot F}{2} \right)$$

Since the magnitude of the force is *kx*, the area becomes

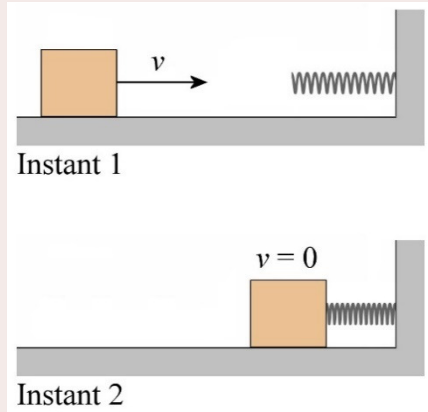
$$Area = \left(\frac{x' \cdot kx'}{2} \right) - \left(\frac{x \cdot kx}{2} \right)$$
$$= \frac{kx'^2}{2} - \frac{kx^2}{2}$$

As this area is under the *x*-axis, the work is negative. Therefore, the work is

$$W_{sp} = -\left(\frac{kx'^2}{2} - \frac{kx^2}{2} \right)$$
$$= -\frac{k}{2}\left(x'^2 - x^2 \right)$$

This is the same result as the one obtained with the integral.

Example 8.1.8



A box sliding on a horizontal surface hits a spring, which compresses the spring to 20 cm. The spring constant is 1000 N/m. What is the work done by the spring on the box between these two instants?

The work done by the spring is

$$W_{sp} = -\frac{k}{2}\left(x'^2 - x^2\right)$$

At instant 1, the spring is not compressed. Therefore, $x = 0$ m.

At instant 2, the spring is compressed 20 cm. Therefore, $x' = 20$ cm.

Thus, the work done is

$$\begin{aligned} W_{sp} &= -\frac{1000\frac{N}{m}}{2}\cdot\left((0.2m)^2 - (0m)^2\right) \\ &= -20J \end{aligned}$$

It is normal for the work to be negative in this case, because the force exerted by the spring on the box is directed towards the left while the displacement of the box is towards the right.

8.2 WORK-ENERGY THEOREM

Proof of the Work-Energy Theorem

Let's now examine why it can be useful to calculate the work done on an object. The proof starts with the definition of net work.

$$W_{net} = \int_s^{s'} \vec{F}_{net} \cdot d\vec{s}$$

where s and s' are the positions at two different moments. Since $F_{net} = ma$ according to Newton's second law and the acceleration is the derivative of the velocity, this equation becomes

$$\begin{aligned} W_{net} &= \int_s^{s'} m\vec{a} \cdot d\vec{s} \\ &= \int_s^{s'} m\frac{d\vec{v}}{dt} \cdot d\vec{s} \\ &= \int_v^{v'} m\frac{d\vec{s}}{dt} \cdot d\vec{v} \end{aligned}$$

As the derivative of the position is the velocity, this becomes

$$W_{net} = \int_{v}^{v'} m\vec{v} \cdot d\vec{v}$$

But since

$$d\left(v^2\right) = d\left(\vec{v} \cdot \vec{v}\right) = \vec{v} \cdot d\vec{v} + d\vec{v} \cdot \vec{v} = 2\vec{v} \cdot d\vec{v}$$

the net work can be written as

$$W_{net} = \int_{v}^{v'} m\vec{v} \cdot d\vec{v} = \int_{v}^{v'} m\frac{1}{2}\, d\left(v^2\right)$$

If this integral is done, the work is

$$\begin{aligned} W_{net} &= \frac{1}{2} m \int_{v}^{v'} d\left(v^2\right) \\ &= \frac{1}{2} m \left[v^2\right]_{v}^{v'} \\ &= \frac{1}{2} m \left(v'^2 - v^2\right) \\ &= \frac{1}{2} m v'^2 - \frac{1}{2} m v^2 \end{aligned}$$

The terms found in the last line were given the name *kinetic energy*.

Kinetic Energy

$$E_k = \frac{1}{2} m v^2$$

Therefore, the work is

$$\begin{aligned} W_{net} &= \frac{1}{2} m v'^2 - \frac{1}{2} m v^2 \\ &= E_k' - E_k \end{aligned}$$

Finally, the work-energy theorem is obtained.

Work-Energy Theorem

$$W_{net} = \Delta E_k$$

Therefore, the net work done on an object gives the change in kinetic energy. This theorem allows us to easily find the change of speed of an object.

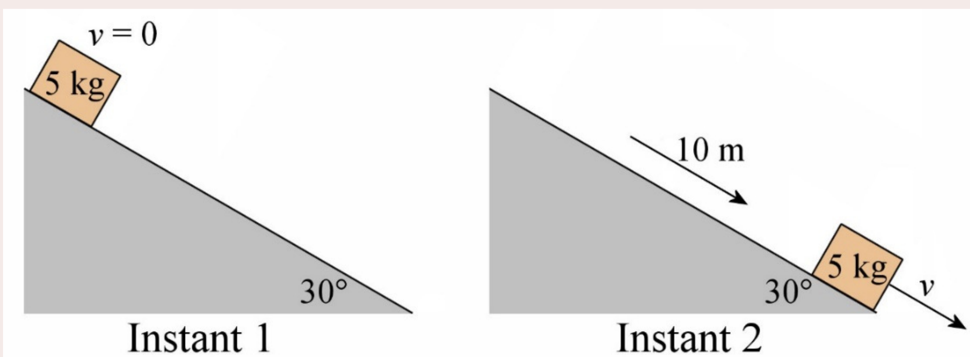
The sign of net work thus has a new meaning.

1- If the net work is positive, then the kinetic energy increases, and the speed of the object increases.
2- If the net work is negative, then the kinetic energy decreases, and the speed of the object decreases.
3- If the net work is zero, then the kinetic energy is constant, and the speed of the object is constant.

The kinetic energy theorem appeared quite late in the history of mechanics. While Newton's laws date from 1687, the kinetic energy theorem was only formulated explicitly in the 1820s by French Gaspard-Gustave Coriolis, Claude-Louis-Henri Navier and Jean-Victor Poncelet.

Example 8.2.1

A 5 kg box slides 10 m down on a 30° slope. The coefficient of friction between the slope and the box is 0.2. What is the speed of the box after the 10 m slide if the box was initially at rest?



The work-energy theorem will be applied here for the motion between these two instants:

Instant 1: The box is at the top of the slope.
Instant 2: The box located is 10 m down the slope.

W_{net} Calculation

The net work done on the box between these two instants was already calculated in an example of the previous section (example 8.1.2). The net work was

$$W_{net} = 160.12 J$$

ΔE_k Calculation

With an initial velocity of zero, the change in kinetic energy is

$$\begin{aligned}\Delta E_k &= \frac{1}{2}mv'^2 - \frac{1}{2}mv^2\\ &= \frac{1}{2}5kg \cdot v'^2 - \frac{1}{2}5kg \cdot \left(0\tfrac{m}{s}\right)^2\\ &= 2.5kg \cdot v'^2\end{aligned}$$

Application of the Work-Energy Theorem

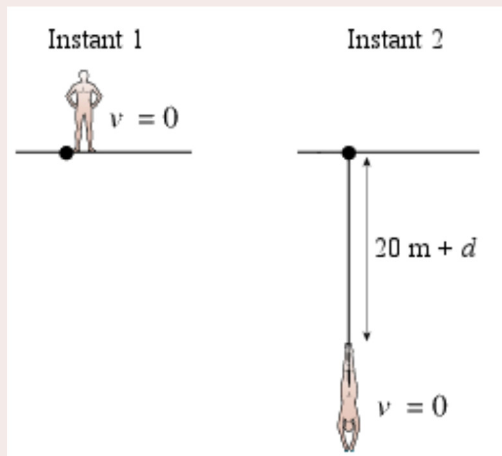
$$\begin{aligned}W_{net} &= \Delta E_k\\ 160.12J &= 2.5kg \cdot v'^2\\ v' &= 8.003\tfrac{m}{s}\end{aligned}$$

Example 8.2.2

Erwan, whose mass is 65 kg, goes Bungee jumping. He has been in free-fall for 20 m when the bungee rope begins to stretch. What will the maximum stretching of the rope be if the rope acts like a spring with a 100 N/m constant?

The work-energy theorem will be applied here for the motion between the two instants shown in the diagram.

In this diagram, d is the stretching of the rope.



www.physicsforums.com/showthread.php?t=199894

ΔE_k Calculation

As Erwan has no speed at both instants, the change in kinetic energy is

$$\begin{aligned}\Delta E_k &= \frac{1}{2}mv'^2 - \frac{1}{2}mv^2\\ &= \frac{1}{2}65kg \cdot \left(0\tfrac{m}{s}\right)^2 - \frac{1}{2}65kg \cdot \left(0\tfrac{m}{s}\right)^2\\ &= 0J\end{aligned}$$

W_{net} Calculation

To calculate the net work done between the two instants, the forces acting on the object must be found. The forces are:

1) A 637 N weight directed downwards.

2) The force made by the rope (like a spring) directed upwards. This force will act only after an initial downwards displacement of 20 m.

The work done by the weight is

$$\begin{aligned} W_g &= F_g \cdot \Delta s \cdot \cos\theta \\ &= 637N \cdot (20m + d) \cdot \cos 0° \\ &= 637N \cdot (20m + d) \end{aligned}$$

The work done by the spring is

$$W_{sp} = -\frac{k}{2}\left(x'^2 - x^2\right)$$

At the beginning, the stretching is zero ($x = 0$), whereas it is d at the end ($x' = d$). Therefore, the work done by the spring is

$$\begin{aligned} W_{sp} &= -\frac{k}{2}\left(d^2 - 0^2\right) \\ &= -50\tfrac{N}{m} \cdot d^2 \end{aligned}$$

Thus, the net work is

$$\begin{aligned} W_{net} &= W_g + W_{sp} \\ &= 637N\left(20m + d\right) - 50\tfrac{N}{m} d^2 \end{aligned}$$

Application of the Work-Energy Theorem

$$W_{net} = \Delta E_k$$

$$637N\left(20m + d\right) - 50\tfrac{N}{m} \cdot d^2 = 0J$$

$$50\tfrac{N}{m} \cdot d^2 - 637N \cdot d - 12,740J = 0$$

The solution of this quadratic equation is d = 23.56 m. (The other solution d = -10.82 m corresponds to a compression of the rope, which is impossible here.)

The rope thus stretches 23.56 m, for a total fall distance of 43.56 m.

Note that the solution of this problem would have been much longer if it had been made using only Newton's second law because it would have been necessary to do an integral to find the change of speed while the rope stretches. As the force is not constant, the acceleration is not constant and all the constant acceleration motion formulas cannot be used here. This is why an integral would have been required to find the displacement.

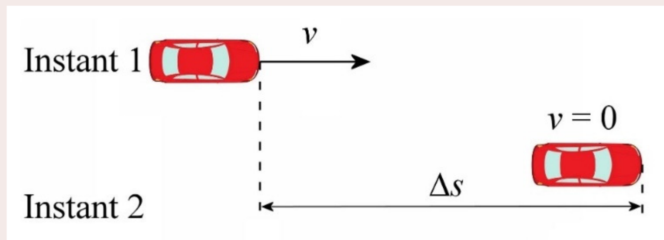
The following example is a road safety example. In it, it is shown that the braking distance increases with the square of the speed. So, don't go too fast, because if you do, you'll need a lot of distance to stop.

Example 8.2.3

What is the minimum braking distance of a car going at 90 km/h if the coefficient of static friction between the tires and the road is 0.9?

The work-energy theorem will be applied here for the motion between these two instants:

Instant 1: the car is moving at 90 km/h.
Instant 2: the car is stopped, Δs farther down the road.



fr.depositphotos.com/2577683/stock-illustration-Car.html

ΔE_k Calculation

As the speed changes from 90 km/h to 0 km/h, the kinetic energy variation is

$$\begin{aligned}\Delta E_k &= \cancel{\frac{1}{2}mv'^2} - \frac{1}{2}mv^2 \\ &= -\frac{1}{2}mv^2\end{aligned}$$

W_{net} Calculation

To calculate the net work between these two instants, the forces acting on the car must be found. There are 3 forces acting on the car.

1) The weight, directed downwards.
2) A normal force directed upwards.
3) A friction force opposed to the displacement.

The work done by the weight and the normal force are both zero because these forces are perpendicular to the velocity and, therefore, to the displacement. Only the friction force does work on the car here. Thus

$$\begin{aligned}W_{net} &= W_f \\ &= F_f \Delta s \cos\theta\end{aligned}$$

As the force is in the opposite direction to the displacement, this becomes

$$W_{net} = F_f \Delta s \cos 180°$$
$$= -F_f \Delta s$$

Application of the Work-Energy Theorem

$$W_{net} = \Delta E_k$$
$$-F_f \Delta s = -\frac{1}{2}mv^2$$
$$F_f \Delta s = \frac{1}{2}mv^2$$

As the static friction force must be smaller than $\mu_s F_N$

$$F_f \leq \mu_s F_N$$

The following equation must also be true.

$$F_f \Delta s \leq \mu_s F_N \Delta s$$

Since $F_f \Delta s = \frac{1}{2}mv^2$, the equation is

$$\frac{1}{2}mv^2 \leq \mu_s F_N \Delta s$$

As the normal force is equal to the weight in this case, the equation becomes

$$\frac{1}{2}mv^2 \leq \mu_s mg\Delta s$$
$$\frac{1}{2}\cancel{m}v^2 \leq \mu_s \cancel{m}g\Delta s$$
$$\Delta s \geq \frac{v^2}{2\mu_s g}$$

Therefore, the minimum breaking distance is

$$\Delta s_{\min} = \frac{v^2}{2\mu_s g}$$
$$= \frac{\left(25\frac{m}{s}\right)^2}{2 \cdot 0.9 \cdot 9.8\frac{N}{kg}}$$
$$= 35.43m$$

Thus, the minimum braking distance of a car is proportional to the square of the speed.

$$\Delta s_{min} = \frac{v^2}{2\mu_s g}$$

This increase of the breaking distance with the speed can be seen in these clips.
http://www.youtube.com/watch?v=lm9GG8OxmQo
http://www.youtube.com/watch?v=9kV24bhdzLI

The formula also indicates that the coefficient of friction also influences the braking distance. A larger coefficient of friction means a smaller the stopping distance. The coefficient of friction between the road and formula 1 tires being much greater than for an ordinary car tires, the braking distance of a Formula 1 is much smaller.
http://www.youtube.com/watch?v=R1jCiU-4K5Y

It is also important not to lock the wheels while braking. To minimize the breaking distance, you must brake as hard as possible, without locking the wheels. If the wheels are locked, then they slide on the road. This means that you then have kinetic friction instead of static friction. As the coefficient of kinetic friction is smaller than the coefficient of static friction, the braking distance increases if the wheels are locked. This is why anti-lock braking systems (ABS) are used: they prevent the locking of the wheels. The other major utility (in fact, the main) of this system is to enable the driver to maintain control of his vehicle while breaking. Indeed, it is impossible to steer a car when the wheels are locked. Even if you turn the wheel, the car does not turn then. By preventing the wheels from locking, the system allows the driver to keep control of his car even during an intense braking.

8.3 POWER

Definition

In a general way, the power is defined by

Power

$$P = \frac{\textit{Energy} \text{ or } \textit{work}}{\textit{time}}$$

As the energy or work is in joule and time is in second, this power is in J/s. The name watt (W) was given to this unit.

Unit of Power: Watt

$$1W = 1\tfrac{J}{s} = 1\tfrac{kgm^2}{s^3}$$

For example, a 60 W lightbulb consumes 60 joules of electrical energy per second. A BBQ that has a power of 10,000 W generates 10,000 joules of thermal energy per second.

Another unit is sometimes used for power: the horsepower (hp), which is worth 746 W. Actually, a horse can provide much more power than one horsepower. This 746 W is the result of an estimate made in the 19th century of the **average** power of a horse when it works without too much effort. This must not be confused with the cheval-vapeur, which is worth only 736 W!

Unit of Power: Horsepower

$$1hp = 746W$$

Power of a Force

In this chapter, we will focus on the power of a force. In that case, the average power is the work done by this force divided by the time required to do this work.

Average Power of a Force

$$\overline{P} = \frac{W}{\Delta t}$$

The instantaneous power is calculated exactly as average power, but by considering the work done in a very short time (an infinitesimal time). The instantaneous work is thus

Power of a Force

$$P = \frac{dW}{dt}$$

Power corresponds, therefore, to the rate at which work is done.

Using what is known about work, the power can be written as

$$\overline{P} = \frac{W}{\Delta t} = \frac{F\Delta s\cos\theta}{\Delta t} \text{ (If the force is constant)}$$

$$P = \frac{dW}{dt} = \frac{Fds\cos\theta}{dt}$$

This becomes

Average and Instantaneous Power

$$\overline{P} = F\overline{v}\cos\theta = \vec{F}\cdot\vec{\overline{v}} \text{ (If the force is constant)}$$
$$P = Fv\cos\theta = \vec{F}\cdot\vec{v}$$

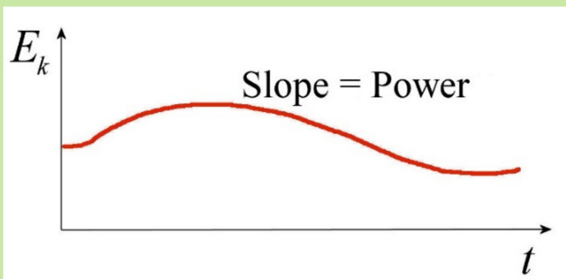
Graphical Interpretation

Since the net work is equal to the variation of kinetic energy, we have

$$P_{net} = \frac{dW_{net}}{dt}$$
$$P_{net} = \frac{dE_k}{dt}$$

The latter equation is the equation of the slope on a graph of kinetic energy as a function of time. This means that

On a graph of the kinetic energy of an object as a function of time, the slope is the power of the net force acting on the object.



Work from Power

If the power is constant, the following equation

$$P = \frac{W}{\Delta t}$$

means that the work can be obtained from the power with

Work from Power if the Power Is Constant

$$W = P\Delta t$$

If the power is not constant, the following formula must be used

$$P = \frac{dW}{dt}$$

to obtain the work done for infinitesimal time.

$$dW = Pdt$$

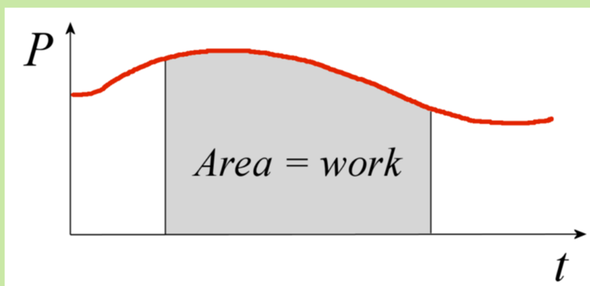
Then all these works are added to get the following equation.

Work from Power if the Power Is not Constant

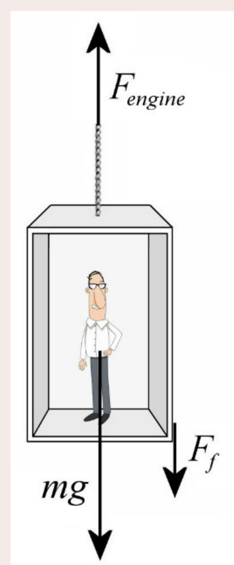
$$W = \int Pdt$$

On a graph, we thus have the following interpretation.

The work done is equal to the area under the curve of the power as a function of time.



Example 8.3.1



A 1000 kg (including the passenger) elevator moves upwards with a steady speed of 3 m/s. If the frictional force opposing the motion of the elevator is 4000 N, what is power (in hp) of the engine pulling the elevator?

The force made by the engine is found with the sum of the vertical forces.

$$\sum F_y = ma_y$$

$$-mg + F_{engine} - F_f = 0$$

$$F_{engine} = mg + F_f$$

$$F_{engine} = 9800N + 4000N$$

$$F_{engine} = 13,800N$$

Therefore, the power is

$$\begin{aligned}P_{engine} &= F_{engine} v \cos\theta \\ &= 13,800N \cdot 3\tfrac{m}{s} \cdot \cos 0° \\ &= 41,400W \\ &= 55.5hp\end{aligned}$$

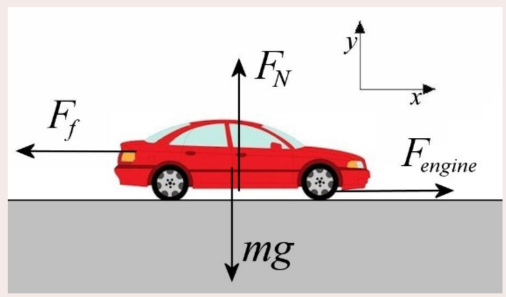
The angle is 0° since the force and the velocity are both directed upwards.

Example 8.3.2

The power of the engine of a 1000 kg car is 12 hp when the car moves on a horizontal surface with a constant speed of 80 km/h. What is the power of the engine if the same car moves uphill on a 10° slope with the same constant speed of 80 km/h?

Let's start by looking at what happens on the horizontal surface. This will allow us to know the frictional force acting on the car when it is travelling at 80 km/h.

The forces on the car are shown in the diagram.



fr.depositphotos.com/2577683/stock-illustration-Car.html

(The normal force is actually acting on all four wheels. The point of application of the force of the engine is at the contact of the wheels and the ground since it is the friction force between the wheels and the road which makes the car accelerate.)

If the speed is constant, then

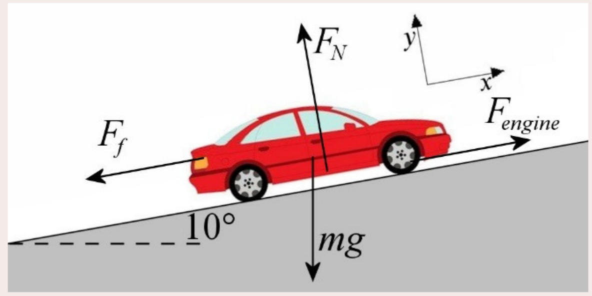
$$\sum F_x = ma_x$$
$$F_{engine} - F_f = 0$$
$$F_f = F_{engine}$$

The force of the engine can be known since the power of the engine is known. The force is

$$P_{engine} = F_{engine} v \cos\theta$$
$$12 \cdot 746W = F_{engine} \cdot 22.22\tfrac{m}{s} \cdot \cos 0°$$
$$F_{engine} = 402.8N$$

The angle is 0° as the force and the velocity are both directed towards the right. We now know that the magnitude of the friction force acting on the car is 402.8 N when it is moving at 80 km/h.

If the car now moves uphill on a 10° slope, the forces shown in the diagram acts on the car.



To find the power of the engine, the force made by the engine must be found. It is found with the sum of the x-component of the forces.

$$\sum F_x = ma_x$$
$$F_{engine} - F_f + mg\cos 100° = 0$$
$$F_{engine} = F_f - mg\cos 100°$$

Since the car is moving at the same speed as it was on the horizontal surface, the frictional force is the same (402.8 N). The equation now becomes

$$\begin{aligned} F_{engine} &= F_f - mg\cos 100° \\ &= 402.8N - 1000kg \cdot 9.8\tfrac{N}{kg} \cdot \cos 100° \\ &= 402.8N + 1701.8N \\ &= 2104.6N \end{aligned}$$

Therefore, the power is

$$\begin{aligned} P_{engine} &= F_{engine} v\cos\theta \\ &= 2104.6N \cdot 22.22\tfrac{m}{s} \cdot \cos 0° \\ &= 46,768W \\ &= 62.7hp \end{aligned}$$

It can be seen that the required engine power increases much when the car moves up the slope. Some cars could not even achieve this feat due to lack of power. The famous Chevettes of the ’80s, having only a 53 hp power (base model), would not have been able to move up this slope at 80 km/h.

en.wikipedia.org/wiki/Chevrolet_Chevette

Example 8.3.3

A 4 kg object is subjected to a force whose power is given by

$$P = 6\tfrac{W}{s^2}t^2 + 4\tfrac{W}{s}t$$

a) Knowing that the object was at rest at t = 0 s, determine the speed of the object at t = 3 s.

We will use the work-energy theorem $W = \Delta E_k$ to find the speed. The work done on the object is

$$
\begin{aligned}
W &= \int_0^{3s} P\,dt \\
&= \int_0^{3s} \left(6\tfrac{W}{s^2}t^2 + 4\tfrac{W}{s}t\right) dt \\
&= \left[2\tfrac{W}{s^2}t^3 + 2\tfrac{W}{s}t^2\right]_{0s}^{3s} \\
&= 2\tfrac{W}{s^2}\cdot(3s)^3 + 2\tfrac{W}{s}\cdot(3s)^2 \\
&= 72J
\end{aligned}
$$

Thus, the speed is

$$
\begin{aligned}
W &= \Delta E_k \\
72J &= \frac{1}{2}mv'^2 - \frac{1}{2}mv^2 \\
72J &= \frac{1}{2}4kg\cdot v'^2 - 0 \\
v' &= 6\tfrac{m}{s}
\end{aligned}
$$

b) Knowing that this is a motion along the x-axis, determine the force acting on the object at $t = 3$ s.

The force will be found with $P = Fv\cos\theta$. For a motion in one dimension, the angle can only be 0° or 180°. The power at $t = 3$ s is

$$
\begin{aligned}
P &= 6\tfrac{W}{s^2}t^2 + 4\tfrac{W}{s}t \\
&= 6\tfrac{W}{s^2}\cdot(3s)^2 + 4\tfrac{W}{s}\cdot 3s \\
&= 66W
\end{aligned}
$$

Thus, the formula for the instantaneous power gives

$$
\begin{aligned}
P &= Fv\cos\theta \\
66W &= F\cdot 6\tfrac{m}{s}\cdot\cos\theta
\end{aligned}
$$

Since the power is positive, the angle must be 0°. Therefore, the magnitude of the force is

$$
\begin{aligned}
66W &= F\cdot 6\tfrac{m}{s}\cdot\cos 0° \\
F &= 11N
\end{aligned}
$$

SUMMARY OF EQUATIONS

The Work Done by a Constant Force Acting on an Object Moving in a Straight Line

$$W = \vec{F} \cdot \overrightarrow{\Delta s}$$

This scalar product can be calculated with these 2 formulas.

$$W = F \Delta s \cos\theta$$

or

$$W = F_x \Delta x + F_y \Delta y + F_z \Delta z$$

The Net Work Done on an Object

$$W_{net} = \sum W = W_1 + W_2 + W_3 + \dots$$

The Work Done if *F* or *θ* change

$$W = \sum_{\text{F et } \theta \text{ constants}} \vec{F} \cdot \overrightarrow{\Delta s}$$

This scalar product can be calculated with these 2 formulas.

$$W = \sum_{F \text{ and } \theta \text{ constants}} F \Delta s \cos\theta$$

or

$$W = \sum_{F_x,\, F_y \text{ and } F_z \text{ constants}} F_x \Delta x + F_y \Delta y + F_z \Delta z$$

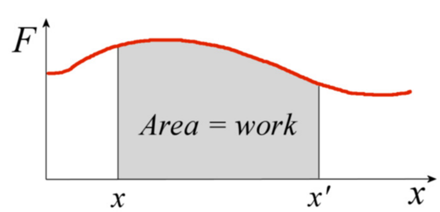
Work Done on an Object (Most General Formula)

$$W = \int \vec{F} \cdot d\vec{s}$$

Work Done by a Variable Force on an Object Moving along the x-axis (the Object Goes from *x* to *x'*)

$$W = \int_x^{x'} F_x dx$$

The work done on an object is the area under the curve of the force acting on the object as a function of position.



Work Done by a Spring

$$W_{sp} = -\frac{k}{2}\left(x'^2 - x^2\right)$$

Kinetic Energy

$$E_k = \frac{1}{2}mv^2$$

Work-Energy Theorem

$$W_{net} = \Delta E_k$$

Power

$$P = \frac{\textit{Energy} \text{ or } \textit{work}}{\textit{time}}$$

Average Power of a Force

$$\overline{P} = \frac{W}{\Delta t}$$

Power of a Force

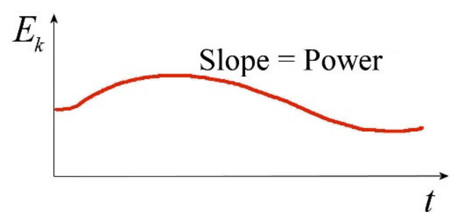
$$P = \frac{dW}{dt}$$

Average and Instantaneous Power

$$\overline{P} = F\overline{v}\cos\theta = \vec{F}\cdot\vec{\overline{v}} \text{ (If the force is constant)}$$

$$P = Fv\cos\theta = \vec{F}\cdot\vec{v}$$

On a graph of the kinetic energy of an object as a function of time, the slope is the power of the net force acting on the object.



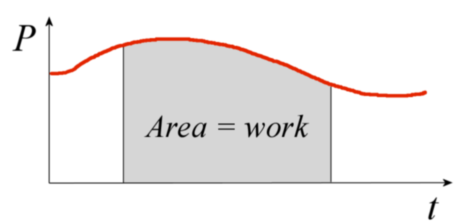
Work from Power if the Power Is Constant

$$W = P\Delta t$$

Work from Power if the Power Is not Constant

$$W = \int P dt$$

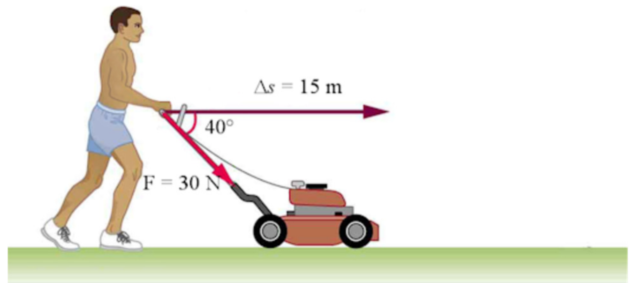
The work is the area under the curve of the power on a graph of the power as a function of time



EXERCISES

8.1 Definition of Work

1. What is the work done by Mustafa in this situation?



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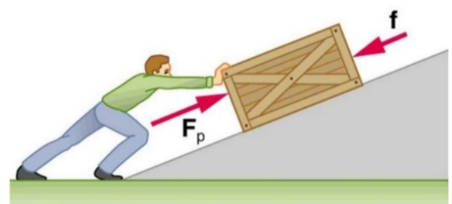
2. Honoré pushes a 30 kg crate up an 8° slope over a distance of 25 m. The crate has an acceleration of 1 m/s² towards the top of the slope, and there is a frictional force of f = 70 N opposed to the motion of the crate.

   a) What is the work done by the weight?
   b) What is the work done by the friction force?
   c) What is the work done by Honoré?
   d) What is the net work?



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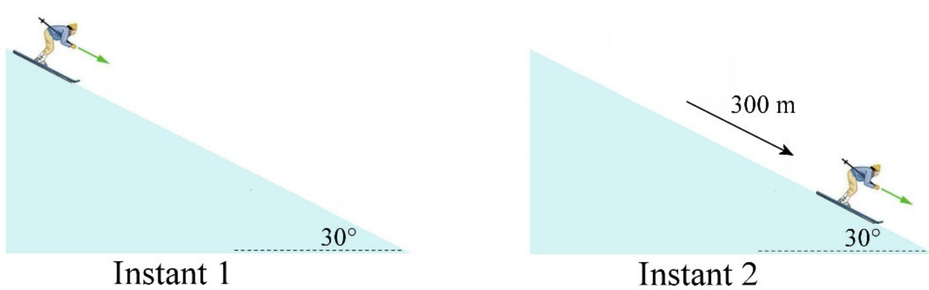
3. The two following forces act on an object

$$\vec{F}_1 = \left(2\vec{i} + \vec{j} - 4\vec{k}\right)N \qquad \vec{F}_2 = \left(-4\vec{i} + 5\vec{j} + 2\vec{k}\right)N$$

   What is the net work done on the object if it is moving in a straight line from the position (0 m, 1 m, 2 m) to the position (5 m, -2 m, - 3 m) then in a straight line once again from the position (5 m, -2 m, - 3 m) to the position (8 m, 2 m, -5 m)?

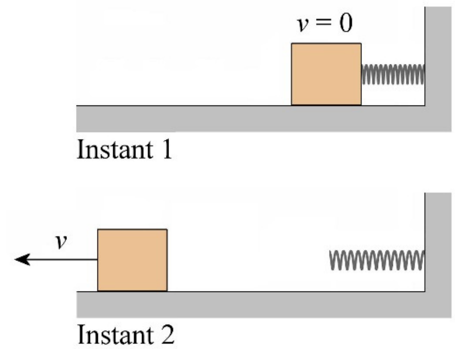
4. Rita, whose mass is 80 kg, skis downhill on a 30° slope. The coefficient of friction between the skis and the slope is 0.1. The distance travelled by Rita is 300 m.



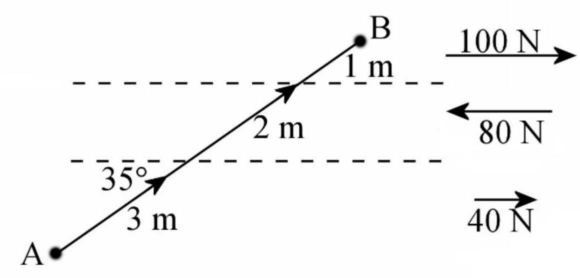
vhcc2.vhcc.edu/ph1fall9/frames_pages/openstax_problems.htm

   a) What is the work done by the weight between these two instants?
   b) What is the work done by the friction force between these two instants?
   c) What is the net work done on Rita between these two instants?

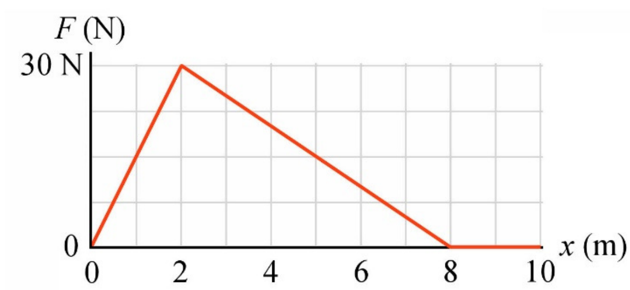
5. A spring initially compressed 50 cm pushes a 10 kg block. What is the work done by the spring if the spring constant is 2000 N/m?



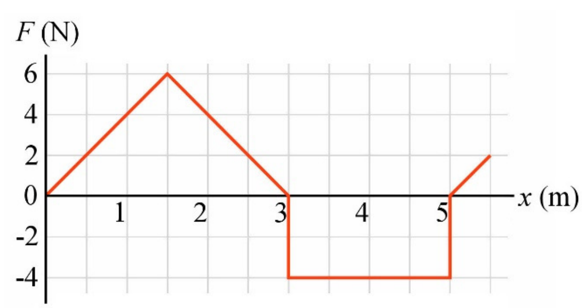
6. An object is moving 6 m following the path shown in the diagram. For the first 3 m, a 40 N force towards the right acts on the object. For the following 2 m, an 80 N force towards the left acts on the object. Finally, a 100 N force towards the right acts on the object for the last 1 m. What is the work done on the object?



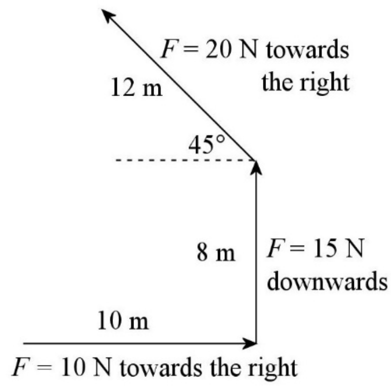
7. Here's the graph of the force acting on an object as a function of position. What is the work done on the object by this force when the object moves from $x = 0$ m to $x = 6$ m?



8. Here's the graph of the force acting on an object as a function of position. What is the work done on the object by this force when the object moves from $x = 4$ m to $x = 0$ m?



9. An object moves along the path shown in the diagram. This diagram indicates the force exerted on the object on each straight part of the trajectory. What is the work done on the object?

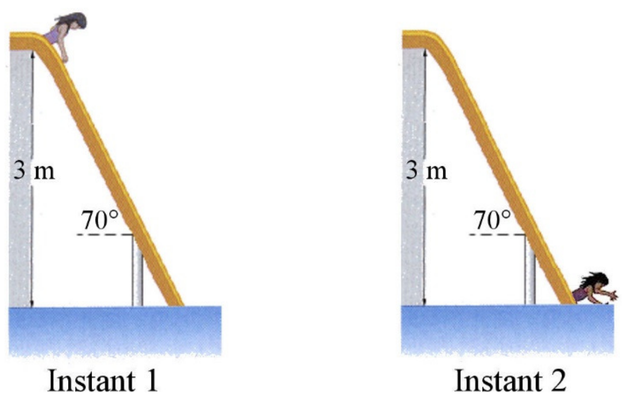


10. The force exerted on an object is given by the formula $F_x = 18\frac{N}{m^2} \cdot x^2$. What is the work done on the object by this force when the object moves from $x = -1$ m to $x = 3$ m?

8.2 Work-Energy Theorem

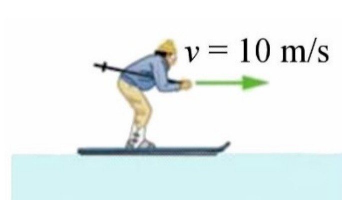
11. A 430 g soccer ball is launched from the ground with an upwards speed of 30 m/s. Use the work-energy theorem to find the speed of the ball when it is 20 m above its starting point if the drag is neglected.

12. Mara, whose mass is 25 kg, slides down the water slide shown in the diagram. The coefficient of kinetic friction between Mara and slide is 0.1. Use the work-energy theorem to determine the speed of Mara at the bottom of the slide.



www.physicsforums.com/showthread.php?t=362115

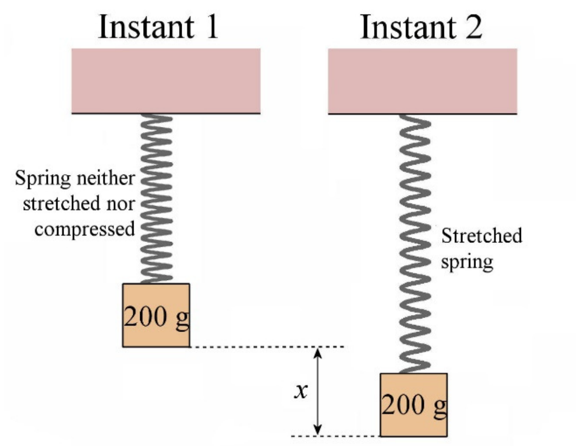
13. The frictional force does –3000 J of work to completely stop a skier who slides on a horizontal surface with an initial speed of 10 m/s. What would be the speed of the skier if the friction force had only done –1500 J of work?



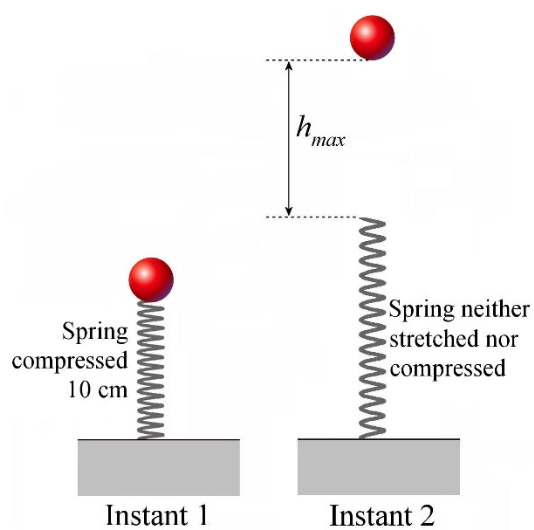
vhcc2.vhcc.edu/ph1fall9/frames_pages/openstax_problems.htm

14. René, whose mass is 55 kg, starts a free-fall motion from a stationary air balloon. 10 seconds after his departure, René has travelled 300 m, and his speed is 39.4 m/s. What is the work done by the drag force during these 10 seconds?

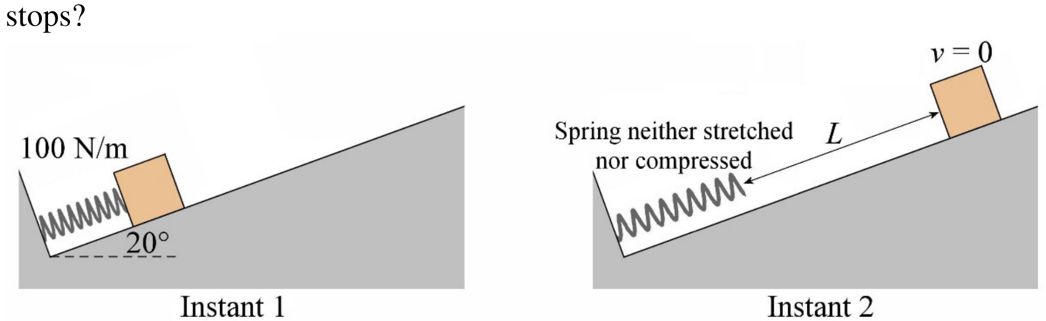
15. A 200 g mass is attached to a spring whose constant is 50 N/m. Originally, the spring is neither stretched nor compressed. Then the mass is released. What will the maximum stretching of the spring be?



16. A 500 g ball is placed on a vertical spring whose constant is 500 N/m. If the spring is compressed 10 cm from its equilibrium position and then released, the ball reaches a maximum height h_{max}. What is this maximum height, measured from the equilibrium position of the spring? (Neglect the drag force.)

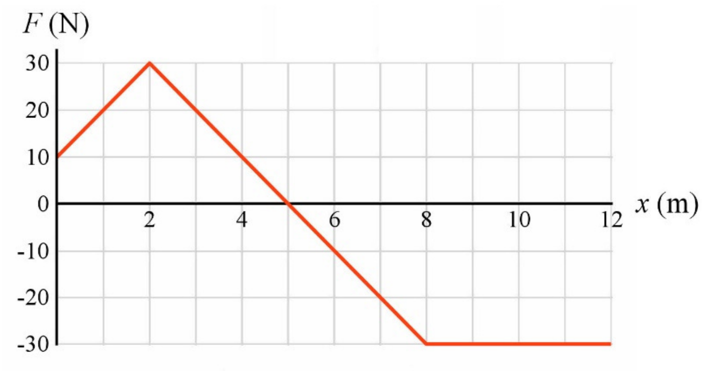


17. A 500 g block is placed in the spring gun shown in the diagram. Initially, the spring is compressed 50 cm and then the block is released. There is no friction between the block and the slope. What will the distance travelled by the block be before it stops?



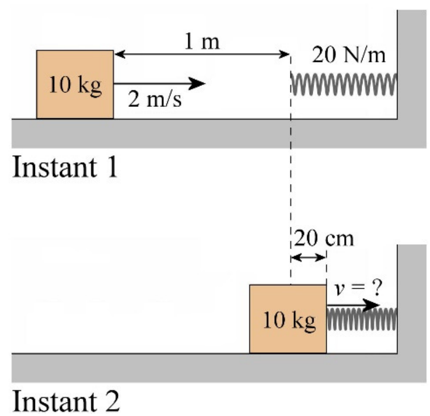
18. Here's the graph of the force acting on a 5 kg object as a function of its position. The object has a speed of 2 m/s towards the positive x-axis when it is at $x = 0$ m.

a) What is the speed of the object at $x = 5$ m?
b) What is the speed of the object at $x = 12$ m?

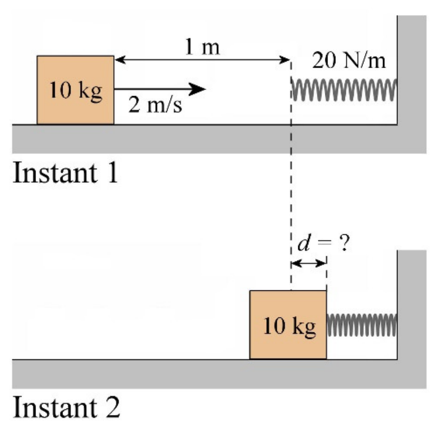


19. A 10 kg block slides on a horizontal surface. The friction coefficient between the block and the surface is 0.1. When the block is 1 m from a spring, whose constant is 20 N/m, it has a speed of 2 m/s.

a) What will the speed of the block be when the spring is compressed 20 cm?



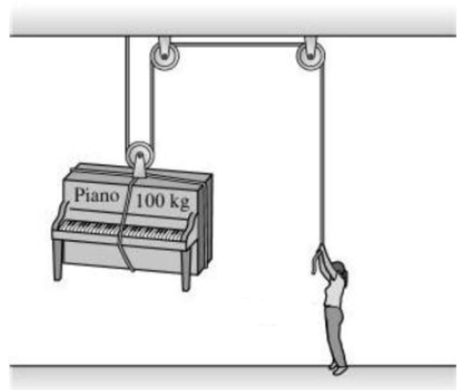
b) What will the maximum compression of the spring be when it is hit by the object?



8.3 Power

20. A BMW 335i 2007 is going at 120 km/h on a horizontal road. What is the power of the engine of the car (in hp), knowing that the value of C_xA is 0.632 m² for this car? (Assuming that only the drag force is opposed to the motion of the car and that the air density is 1.3 kg/m³.)

21. Laura lifts a piano 2 m above the ground in 20 seconds using the pulley system shown in the diagram. What is the average power of Laura (in hp)? (The piano has no speed at the beginning and no speed at the end of this motion.)



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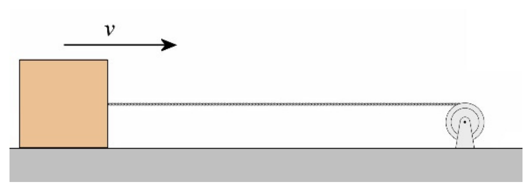
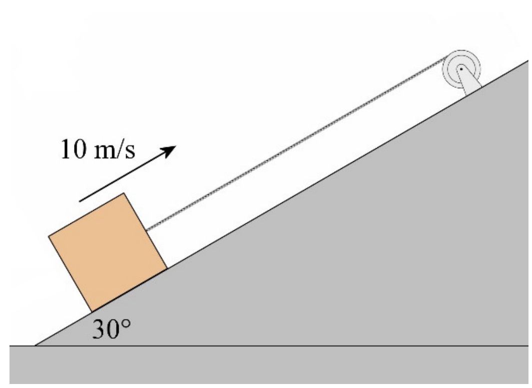
22. A 5 kg crate is initially at rest. Then a winch pulls it and gives it a speed of 20 m/s in 10 seconds. There is no friction between the crate and the surface. How much time will it take for a 100 kg crate initially at rest to reach a speed of 10 m/s if the same winch (with the same average power) is used to pull the crate?



23. A winch pulls a 50 kg box along a 30° slope with a constant speed of 10 m/s.

   a) What is the power of the winch if there is no friction between the box and the surface?
   b) What is the power of the winch if the coefficient of kinetic friction between the box and the surface is 0.3?



24. A 30 kg rocket toy lifts off vertically from the ground. The rocket then goes up with a constant acceleration. After 1 minute, its engine stops. The rocket has then a speed of 300 m/s.

a) What is the work done by the rocket engine (if air friction is neglected)?
b) What is the average power of the rocket engine?

(Actually, the mass of the rocket decreases as fuel is ejected, but we will proceed as if the mass remains constant.)

25. A 20 kg object is subjected to a force whose power is given by

$$P = 12\tfrac{W}{s^2}t^2$$

Initially, the object is at rest. This is a one-dimensional motion along the *x*-axis.

a) What is the work done on the object between $t = 0$ s and $t = 10$ s?
b) What is the speed of the object at $t = 10$ s?
c) What is the acceleration of the object at $t = 10$ s?
d) What is the force acting on the object at $t = 10$ s?
e) What is the displacement of the object between $t = 0$ s and $t = 10$ s?

Challenges

(Questions more difficult than the exam questions.)

26. The force acting on a 1 kg object initially at rest is given by the equation

$$F = 9N - 1\tfrac{N}{m^2}x^2$$

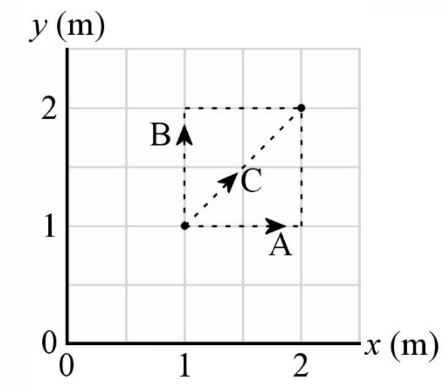
Where does the object reach its maximum speed and what is this maximum speed?

27. The force exerted on an object that can move in 2 dimensions is given by the formula

$$\vec{F} = \left(3\tfrac{N}{m^2}x^2 + 1\tfrac{N}{m}y\right)\vec{i} + \left(2\tfrac{N}{m}y + 1\tfrac{N}{m}x\right)\vec{j}$$

This object is moving from the point (1 m, 1 m) to the point (2 m, 2 m).

a) What is the work done on the object if it follows path A?
b) What is the work done on the object if it follows path B?
c) What is the work done on the object if it follows path C?



ANSWERS

8.1 Definition of Work

1. 344.7 J
2. a) -1022.9 J b) -1750 J c) 3522.9 J d) 750 J
3. 4 J
4. a) 117,600 J b) -20,369 J c) 97,231 J
5. 250 J
6. 49.15 J
7. 110 J
8. -5 J
9. -189.71 J
10. 168 J

8.2 Work-Energy Theorem

11. 22.54 m/s
12. 7.527 m/s
13. 7.071 m/s
14. -119,010 J
15. 7.84 cm
16. 41.02 cm
17. 6.959 m
18. a) 6.164 m/s b) The object cannot be at x = 12 m
19. a) 1.252 m/s b) 63.25 cm

8.3 Power

20. 20.4 hp
21. 0.1314 hp
22. 50 s
23. a) 2450 W b) 3723 W
24. a) 3,996,000 J b) 66,600 W
25. a) 4000 J b) 20 m/s c) 3 m/s² d) 60 N e) 80 m

Challenges

26. 6 m/s at x = 3 m
27. a) 13 J b) 13 J c) 13 J