6 CIRCULAR MOTION

In a rotor ride such as the one shown in this image, what is the maximum period of rotation that the rotor ride can have so that people do not slip down the wall if the coefficient of friction between the wall and people is 0.7 and if the radius of the rotor ride is 2.5 m?

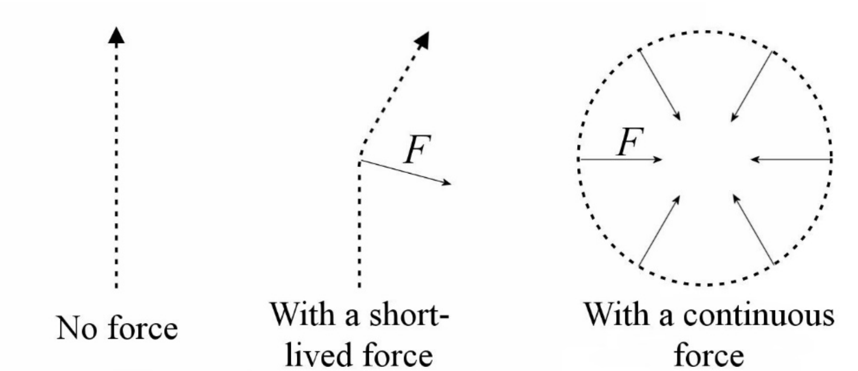
www.retronaut.com/2013/01/rotor-rides/

Discover the answer to this question in this chapter.

6.1 CENTRIPETAL FORCE

Centripetal Force Formula

As seen previously, there is an acceleration towards the centre in a uniform circular motion. This conclusion is reinforced if this motion is examined with Newton's laws. If there is no force, the object continues in a straight line (diagram to the left).



If the trajectory changes, it is because a force acted on the object. In the centre diagram, a force acted towards the right for a short time in order to deflect the object towards the right. In a circular motion (diagram to the right), it is necessary to deflect the object continuously. This means that there is always a force directed towards the centre of the circle. This confirms that there is an acceleration towards the centre, as seen in chapter 2.

If Newton's second law

$$\sum \vec{F} = m\vec{a}$$

is combined with the fact that

$$a = \frac{v^2}{r}$$

towards the centre, the following conclusion is reached.

Uniform Circular Motion

In a uniform circular motion, there must be a net force directed towards the centre of the circle whose magnitude is $m\dfrac{v^2}{r}$.

The net force directed towards the centre of the circle is called the *centripetal force* (Newton coined the word *centripetal* in 1684). This is not a new type of force; it is just a name given to forces already known, like gravity, normal force, tension force or friction force, when their action result in a circular motion. In this example, the tension force is the centripetal force.

Christiaan Huygens discovered the centripetal force formula in 1659, but he did not publish it until 1673. In fact, Huygens could not give the formula in the form mv^2/r since he did not know $F = ma$ and the concept of mass was not really used before Newton. Huygens was examining the tension of a rope that holds an object in a circular motion. He tells us that the tension increases with v^2, that it increases with $1/r$ and that it is equal to the weight of the object if the speed of the rotating object is equal to the speed that the object would have if it fell from rest on a distance equal to half the radius of the circular trajectory. It is not simple, but it is the same thing as the law given for the centripetal force. Around 1680, Edmond Halley, Christopher Wren, and Robert Hooke discovered that the force is directed toward the center.

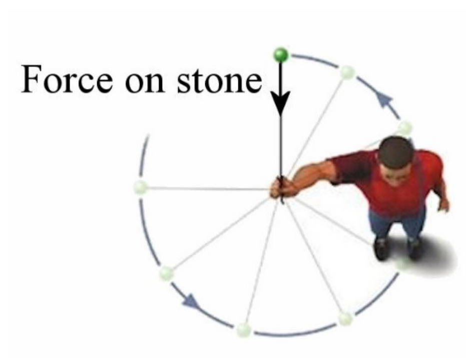
Force Making the Centripetal Force

Let's say it again, the centripetal force is not a new type of force; it is just a name given to forces already known, like gravity, normal force, tension force or friction force, when their action result in a circular motion. Let's look at some situations with to see what forces make the centripetal force in these cases.

Object Rotating at the End of a String

An object fixed at the end of a rope rotates at a constant speed. The force directed towards the centre of the circle is exerted by the tension force of the string acting on the stone.

The tension force is the centripetal force here.

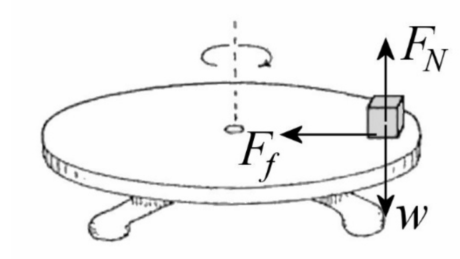


astronomy.nmsu.edu/tharriso/ast105/Ast105week04.html

Object on a Turntable

Now imagine an object placed on a turntable.

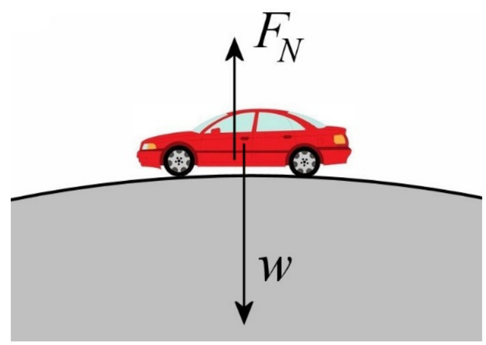
In this case, the normal force and the weight cannot make the centripetal force as they are not directed towards the centre of the circular motion. The only force that can be directed towards the centre of the circle is the friction force. The friction force is thus the centripetal force here.



dev.physicslab.org/Document.aspx?doctype=5&filename=Compilations_CPworkbook_CentripetalForce.xml&asid=5679&siteid=%7B9CF1626F-C264-436A-B5DE-DF386F0F2F11%7D

Car on a Bump

Now suppose that a car passes over a bump. The forces acting on the car are shown in this diagram.



fr.depositphotos.com/2577683/stock-illustration-Car.html

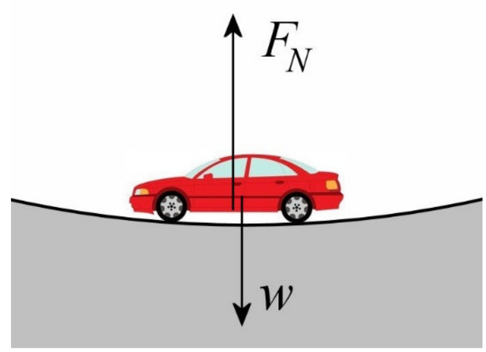
Since the car makes a circular motion, the net force must be directed towards the centre of the circle. This force must then be directed downwards when the car is in the position shown in the diagram. This means that the weight must be greater than normal force in order to have a resulting force directed downwards.

$$w > F_N$$

In this case, the normal force cancels part of the weight and the remaining part of the weight makes the downwards force. Here, the centripetal force is thus made by a part of the gravitational force.

Car in a Dip

Now suppose that a car passes in a dip. The forces acting on the car are shown in this diagram.



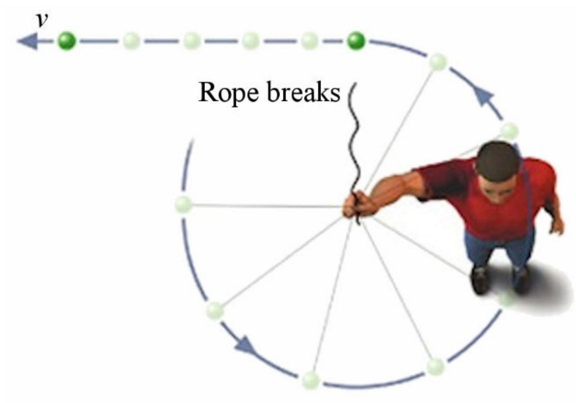
Since the car is a circular motion, the net force must be pointing towards the centre of the circle. Here the net force must be directed upwards. This means that the normal force must be greater than the weight so that there is a resultant upwards force.

$$w < F_N$$

In this case, a part of the normal force cancels the weight, and the remaining part of the normal force makes the upwards force. Here, the centripetal force is thus made by a part of the normal force.

What Happens If the Centripetal Force Disappears?

If the centripetal force suddenly disappears, then the circular motion stops, and the object continues its motion in a straight path because there is no longer any force acting on it. In the case illustrated in the diagram, the force made by the rope disappears when the rope breaks. The object can no longer be in a circular motion as there is no longer any force towards the centre. The object then goes in a straight line in the direction of the velocity that the object had when the rope broke. Since the velocity is always tangent to the circle in a circular motion, the straight-line motion is tangent to the circle as shown in the diagram.



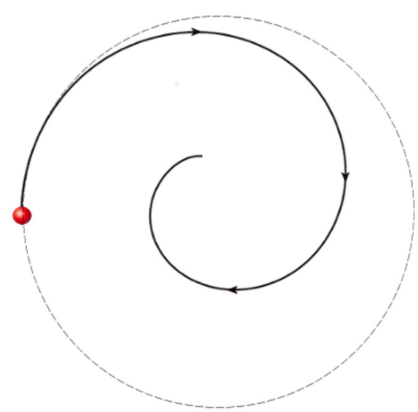
astronomy.nmsu.edu/tharriso/ast105/Ast105week04.html

This is what happens in this video. The person in the ride can make a circular motion because the normal force exerted by the metal railing on the back of the person is directed towards the centre. When the person loses contact with the metal railing, the force disappears, and the person can no longer be in a circular motion.
http://www.youtube.com/watch?v=bi-YKulwvUs
The person then continued its motion in a straight line and is thrown to the ground.

Too Much or Not Enough Inwards Force

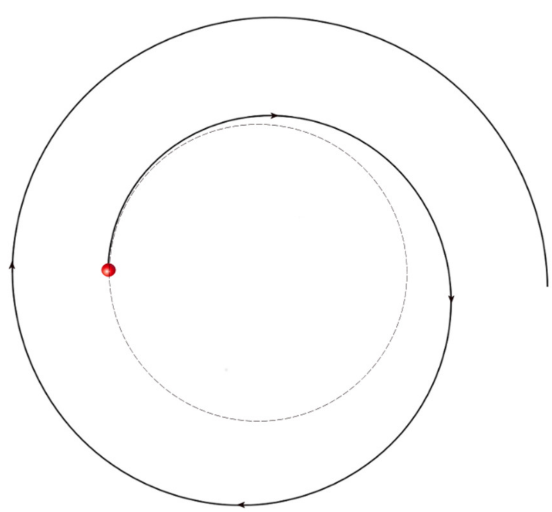
Too Much Inwards Force

If the sum of forces towards the centre circle is bigger than mv^2/r, then the net force is greater than the force required to make a circular motion with a constant radius. This excess of force gives too much acceleration towards the centre of the circle, and the object then gets closer to the centre of the circle. Here's the trajectory obtained if the sum of the forces is always greater than mv^2/r.



Not Enough Inwards Force

If the sum of forces towards the centre circle is less than mv^2/r, then the net force is smaller than the force required to make a circular motion with a constant radius. This lack of force does not give enough acceleration towards the centre of the circle, and the object then moves away from the centre of the circle. Here's the trajectory obtained if the sum of the forces is always smaller than mv^2/r.



A Movie Mistake

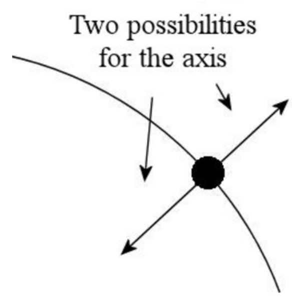
The curved bullet trajectories in the film *wanted*, recreated here
http://www.youtube.com/watch?v=bCCIvXqcaMQ
is, therefore, impossible. Even if the gun is rotated, there is no centripetal force acting on the bullet once it has left the gun. Thus, the bullet is supposed to move in a straight line (while falling to the ground because of gravity). It is, therefore, impossible for the path to be curved as seen in the clip. Many people think that there is some sort of conservation of rotational motion, but there is none.

6.2 EQUATIONS FOR CIRCULAR MOTION

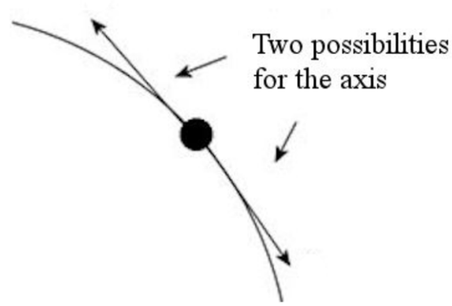
Axes to Use to Describe Circular Motion

Problem solving with circular movements (uniform or not) is much easier with the following choice of axes.

1) Use an axis in the radial direction (2 possibilities: towards the centre of the circular motion or away from the centre of the circular motion).



2) Use another axis in the tangential direction. (2 possibilities: in the direction of the velocity of the object or in the opposite direction to the velocity of the object.)



3) If a third axis is needed, it must be perpendicular to the other two axes.

This choice of axes is perfect to solve any problems of circular motion, uniform or non-uniform.

Actually, this is the same rule as the rule used in chapters 4 and 5: an axis must be placed in the direction of the velocity of the object or in the opposite direction to the velocity of the object. This axis is the axis tangent to the circle. As the other axis must be perpendicular to the axis in the direction of the velocity, it is automatically in the radial direction.

Equations for Circular Motion

With axes directed as specified, here's what is obtained if Newton's laws are applied to a circular motion.

Equations of Circular Motion

Radial direction (towards the centre or away from the centre)

$$\sum F = ma_c$$

Tangential direction (in the direction of velocity or opposite to the direction of velocity)

$$\sum F = ma_T$$

Third axis

In these notes, the acceleration is always zero in the direction of this axis.

$$\sum F = 0$$

There are two important cases for the circular motion: uniform circular motion (in which the speed is constant) and non-uniform circular motion.

Uniform Circular Motion

In a uniform circular motion, two formulas can be used for the centripetal acceleration. The sign of the acceleration changes according to the direction of the axis. Thus,

$$a_c = \frac{v^2}{r} \qquad or \qquad a_c = \frac{4\pi^2 r}{T^2} \qquad \text{If the axis points towards the centre.}$$

$$a_c = -\frac{v^2}{r} \qquad or \qquad a_c = -\frac{4\pi^2 r}{T^2} \qquad \text{If the axis points away from the centre.}$$

Since the speed is constant, the tangential acceleration is

$$a_T = 0$$

Non-Uniform Circular Motion

In a non-uniform circular motion only one formula can be used for centripetal acceleration.

$$a_c = \frac{v^2}{r} \qquad \text{If the axis points towards the centre.}$$

$$a_c = -\frac{v^2}{r} \qquad \text{If the axis points away from the centre.}$$

As the speed changes, the tangential acceleration is not zero.

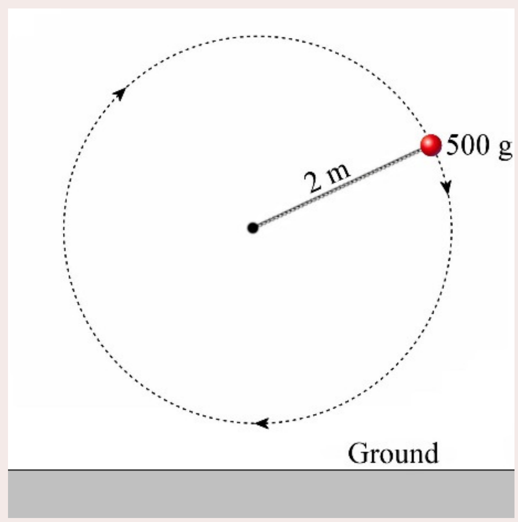
$$a_T \neq 0$$

6.3 UNIFORM CIRCULAR MOTION

Now, these ideas will be applied to the study of uniform circular motion.

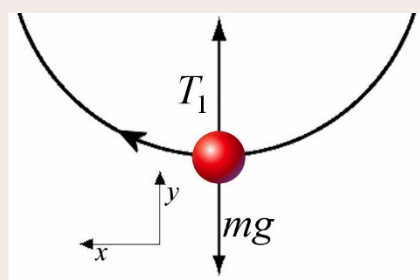
Example 6.3.1

A 500 g object attached at the end of a 2 m rope rotates in a vertical plane at a steady speed of 10 m/s.



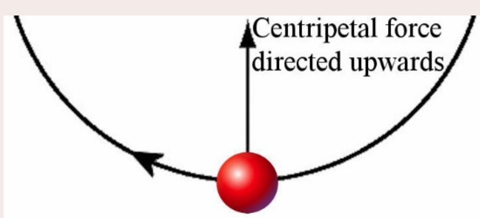
a) What is the tension of the rope when the object is at the lowest point of the trajectory?

Forces Acting on the Object



The forces acting on the stone at the lowest point are:

1) The weight (mg) directed downwards.
2) The tension force (T_1) directed upwards.



Here, only the tension force is in the right direction to make the centripetal force. The tension force should, therefore, cancel the force of gravity and also make the centripetal force.

Sum of the Forces

We will use a y-axis pointing towards the centre an x-axis pointing in the direction of the velocity of the object.

With those axes, the sums of the forces are

$$\sum F_x = 0$$
$$\sum F_y = -mg + T_1$$

Newton's Second Law

The centripetal acceleration is directed upwards since the centre of the circle is above the object. Thus, Newton's second law gives

$$\sum F_x = ma_T \qquad \to \qquad 0 = 0$$
$$\sum F_y = ma_c \qquad \to \qquad -mg + T_1 = m\frac{v^2}{r}$$

Solving the Equations

With the second equation, the tension can be found.

$$\begin{aligned} T_1 &= m\frac{v^2}{r} + mg \\ &= 0.5kg \cdot \frac{\left(10\frac{m}{s}\right)^2}{2m} + 0.5kg \cdot 9.8\frac{N}{kg} \\ &= 25N + 4.9N \\ &= 29.9N \end{aligned}$$

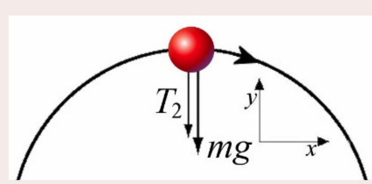
Of this 29.9 N tension force, 4.9 N are required to cancel gravity, and 25 N are required to make the centripetal force.

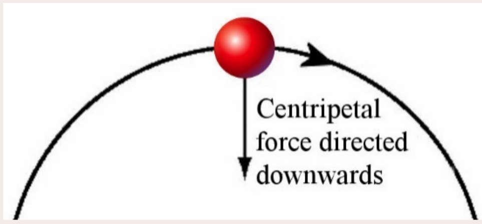
b) What is the tension of the rope when the object is at the highest point of the trajectory?

Forces Acting on the Object

The forces acting on the stone at the lowest point are:

1) The weight (mg) directed downwards.
2) The tension force (T_2) directed upwards.





In this case, the tension force and the weight are in the right direction to make the centripetal force. The tension should, therefore, be simply equal to what is missing from the gravitation force to make the centripetal force.

Sum of the Forces

We will use a *y*-axis pointing away from the centre an *x*-axis pointing in the direction of the velocity of the object.

With those axes, the sums of the forces are

$$\sum F_x = 0$$
$$\sum F_y = -mg - T_2$$

Newton's Second Law

The centripetal acceleration is directed downwards since the centre of the circle is below the object. Thus, Newton's second law gives

$$\sum F_x = ma_T \qquad \rightarrow \qquad 0 = 0$$
$$\sum F_y = ma_c \qquad \rightarrow \qquad -mg - T_2 = -m\frac{v^2}{r}$$

There is a minus sign in front of mv^2/r since the centripetal force is towards the centre of the circle, which is downwards when the object is at the highest point. (An axis pointing upwards is used.)

Solving the Equations

With the second equation, the tension can be found.

$$\begin{aligned} T_2 &= m\frac{v^2}{r} - mg \\ &= 0.5kg \cdot \frac{\left(10\frac{m}{s}\right)^2}{2m} - 0.5kg \cdot 9.8\tfrac{N}{kg} \\ &= 25N - 4.9N \\ &= 20.1N \end{aligned}$$

There are already 4.9 N in the right direction coming from the weight. The tension must then be (20.1 N) to have the required 25 N centripetal force for this motion.

Here are two comments concerning this previous example:

1) Should the rope break in such a motion, it would break at the lowest point because the rope must compensate for the force of gravity and make the centripetal force at this point. It is, therefore, at this location that the tension is largest.

2) If the speed of the object were to be lower, say 2 m/s, then there would be a problem at the highest point since a calculation similar to the one done in the previous example would give us a tension of -3.9 N. As the tension cannot be negative (that would mean that the rope pushes), this situation is impossible. At this speed, the centripetal force required is only 1 N, while the weight is 4.9 N. As there is nothing to cancel the weight, there is an excess of centripetal force. The object would then move closer to the centre of the circle, and the rope would slacken.

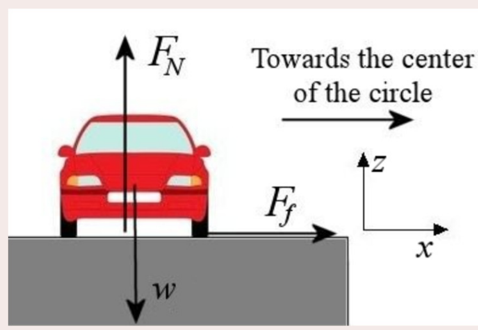
Example 6.3.2

At what maximum speed can a 1000 kg car take a curve having a 10 m radius if the coefficient of static friction between the tires and the road is 0.8?

Forces Acting on the Object

Three forces are acting on the car:

1) The weight (w) directed downwards.
2) The normal force (F_N) directed upwards.
3) The static friction force (F_f) parallel to the ground.



fr.depositphotos.com/2577683/stock-illustration-Car.html

(The frictional force is static because the tires do not slip on the asphalt.) Only the friction force is in the right direction to make the centripetal force. Therefore, the centripetal force must be made by the static frictional force.

www.physicsclassroom.com/class/circles/u6l1c.cfm

Sum of the Forces

The following axes will be used: an axis (x) towards the centre of the circular motion, an axis (y) in the direction of the motion of the car (out of the page in the diagram) and a third axis (z) upwards.

Therefore, the sums of the forces are

$$\sum F_x = F_f$$
$$\sum F_y = 0$$
$$\sum F_z = -mg + F_N$$

Newton's Second Law

The acceleration is towards the right since the centre of circle is towards the right. Thus, Newton's second law is

$$\sum F_x = ma_c \qquad \to \qquad F_f = m\frac{v^2}{r}$$
$$\sum F_y = ma_T \qquad \to \qquad 0 = 0$$
$$\sum F_z = 0 \qquad \to \qquad -mg + F_N = 0$$

Solving the Equations

With static friction, F_f and F_N must be found and then those values must be used in the equation $F_f \leq \mu_s F_N$.

The friction force can be found with the first equation.

$$F_f = m\frac{v^2}{r}$$

The normal force can be found with the second equation.

$$F_N = mg$$

Using those values in the equation $F_f \leq \mu_s F_N$, the result is

$$F_f \leq \mu_s F_N$$
$$m\frac{v^2}{r} \leq \mu_s mg$$
$$v \leq \sqrt{\mu_s rg}$$

Thus, the maximum speed is

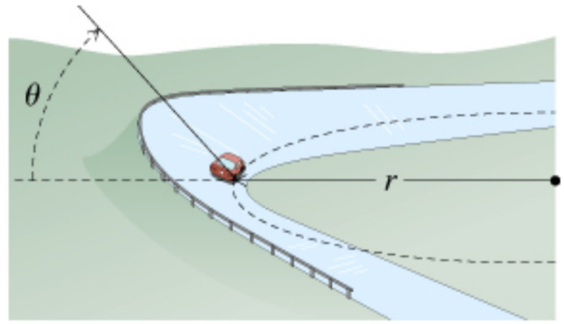
$$v_{\max} = \sqrt{\mu_s rg}$$

Note that this speed is independent of the mass of the car.

With the values given in this problem, the maximum speed is

$$\begin{aligned} v_{\max} &= \sqrt{0.8 \cdot 10m \cdot 9.8 \tfrac{N}{kg}} \\ &= 8.854 \tfrac{m}{s} \\ &= 31.9 \tfrac{km}{h} \end{aligned}$$

This speed increases if the road is banked, as shown here.

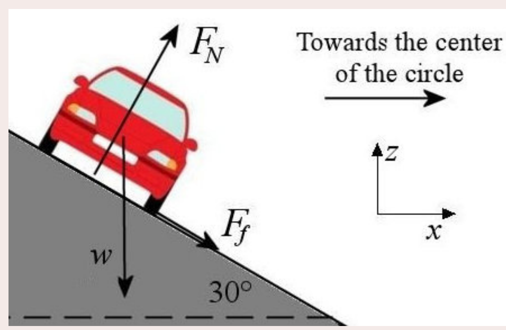


www.masteringphysicssolutions.net/mastering-physics-solutions-banked-frictionless-curve-and-flat-curve-with-friction/

Let's see why the speed increases with an example.

Example 6.3.3

At what speed can a 1000 kg car take a curve having a 10 m radius if the coefficient of static friction between the tires and the road is 0.8 and if the road is banked 30° from the horizontal?



Forces Acting on the Object

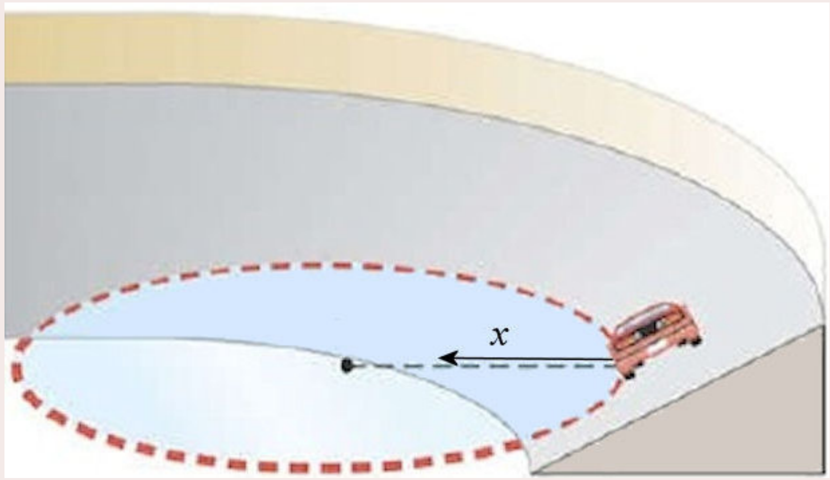
There are still 3 forces acting on the car.

1) The weight (w) directed downwards.
2) The normal force (F_N) perpendicular to the road.
3) The friction force (F_f) parallel to the road.

The weight is still not in the right direction to make the centripetal force, but the normal force has a horizontal component, which now contributes to the centripetal force. On the other hand, the frictional force is not in the horizontal direction anymore. Only the horizontal component of friction force can contribute to the centripetal force. The centripetal force is, therefore, made here by a component of the normal force and a component of the friction force.

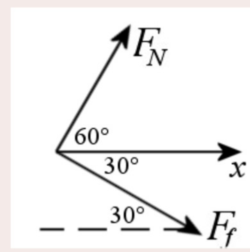
Sum of the Forces

The following axes will be used: an axis (x) towards the centre of the circular motion, an axis (y) in the direction of the motion of the car (out of the page in the diagram) and a third axis (z) upwards. The x-axis is still horizontal because the centre of the circular motion is in this direction.



www.slideshare.net/cscottthomas/ch-5-circular-motion-online

The diagram to the right shows the angle between the positive x-axis and the forces



Thus, the table of forces is

Forces	x	z
Weight	0	$-mg$
Normal force	$F_N \cos 60°$	$F_N \sin 60°$
Friction force	$F_f \cos(-30°)$	$F_f \sin(-30°)$

Therefore, the sums of the forces are

$$\sum F_x = F_N \cos 60° + F_f \cos(-30°)$$
$$\sum F_y = 0$$
$$\sum F_z = -mg + F_N \sin 60° + F_f \sin(-30°)$$

Newton's Second Law

The acceleration is towards the right since the centre of circle is towards the right. Thus, Newton's second law is

$$\sum F_x = ma_c \qquad \to \qquad F_N \cos 60° + F_f \cos(-30°) = m\frac{v^2}{r}$$
$$\sum F_y = ma_T \qquad \to \qquad 0 = 0$$
$$\sum F_z = 0 \qquad \to \qquad -mg + F_N \sin 60° + F_f \sin(-30°) = 0$$

Solving the Equations

Normally, one would find F_f and F_N and then use these values in $F_f \leq \mu_s F_N$. However, these two variables are not easy to obtain here, so another method will be used.

As the maximum speed is sought, the maximum force towards the centre will be used. This maximum force occurs when the friction is at its maximum $F_{fmax} = \mu_s F_N$. Thus

$$F_N \cos 60° + \mu_s F_N \cos(-30°) = m\frac{v_{\max}^2}{r}$$
$$-mg + F_N \sin 60° + \mu_s F_N \sin(-30°) = 0$$

The normal force can be found with the second equation.

$$-mg + F_N\left(\sin 60° + \mu_s \sin(-30°)\right) = 0$$
$$F_N = \frac{mg}{\sin 60° + \mu_s \sin(-30°)}$$
$$F_N = 2.1458 \cdot mg$$
$$F_N = 21{,}028N$$

Substituting this normal force into the sum of the *x*-component of forces, the maximum speed is obtained.

$$2.1458\cancel{m}g \cos 60° + \mu_s 2.1458\cancel{m}g \cos(-30°) = \cancel{m}\frac{v_{\max}^2}{r}$$
$$v_{\max}^2 = 2.1458 \cdot rg\left(\cos 60° + \mu_s \cos(-30°)\right)$$
$$v_{\max}^2 = 2.1458 \cdot 10m \cdot 9.8\tfrac{N}{kg} \cdot \left(\cos 60° + 0{,}8 \cdot \cos(-30°)\right)$$
$$v_{\max} = 15.84\tfrac{m}{s} = 57\tfrac{km}{h}$$

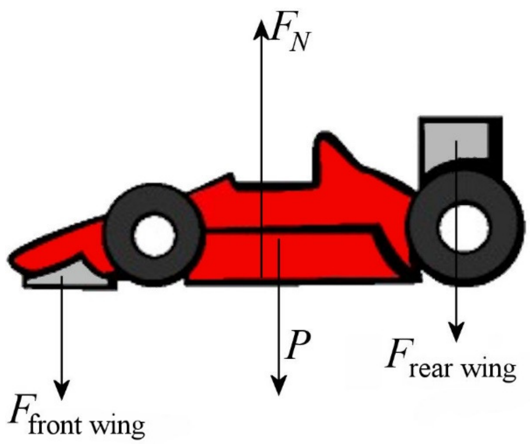
The mass of the car does not matter since it no longer appears in the equation of the maximum speed. It can also be noted that the maximum speed has increased compared to the previous example. A part of normal force now acts as a centripetal force, and this allows the car to move faster in the curve without slipping.

It is possible to take a banked curve even if there is no friction. In this case, only the horizontal component of the normal force acts as the centripetal force. However, as the component of the normal force has a very precise value, the car must move at exactly the right speed to take such a curve without slipping if there is no friction.

The road can be banked even more in very tight curves. There is no limit to the banking angle given to the road, as shown in this video.
http://www.youtube.com/watch?v=ury7J6bK7bc

As the wall is almost vertical, the centripetal force is made almost exclusively by the normal force. Here, the friction force prevents the car from sliding down the wall.

Banking the road is not the only option to increase the maximum speed of a car in a curve. Formula 1 race cars may take non-banked curves at much higher speed than the speed found in example 6.2.2 ($v_{max} = \sqrt{\mu_s rg}$). To achieve this, wings are used to increase the normal force between the car and the ground. When the air strikes the wings, it exerts a downwards force. The equation of force is then



www.123rf.com/photo_11141455_silhouette-drawing-of-f1-racing-car-cockpit-view.html

$$\sum F_y = F_N - w - F_{wings} = 0$$
$$F_N = w + F_{wings}$$

This shows that normal force is greater with wings. In fact, at high speeds, the force made by the wing is much greater than the weight and the normal force is way larger than it would be without wings. This greater normal force greatly increases the maximum static friction force ($\mu_s F_N$). This allows the drivers to take curves at much higher speeds because the maximum centripetal force is much greater (since the static friction force is the only force acting as the centripetal force in a non-banked curve). If one of the wings breaks, the normal force drops, and the driver can no longer take a curve at high speed.

http://www.youtube.com/watch?v=jhwPSAI-dOo

When the rear wing breaks, the normal force on the rear wheels suddenly drops, which causes a sudden drop of the frictional force and thus of the centripetal force. There is then a lack of centripetal force on the rear wheels so that the back of the car cannot take this curve as quickly. Therefore, the rear of the car drifts towards the outside of the curve, causing the accident.

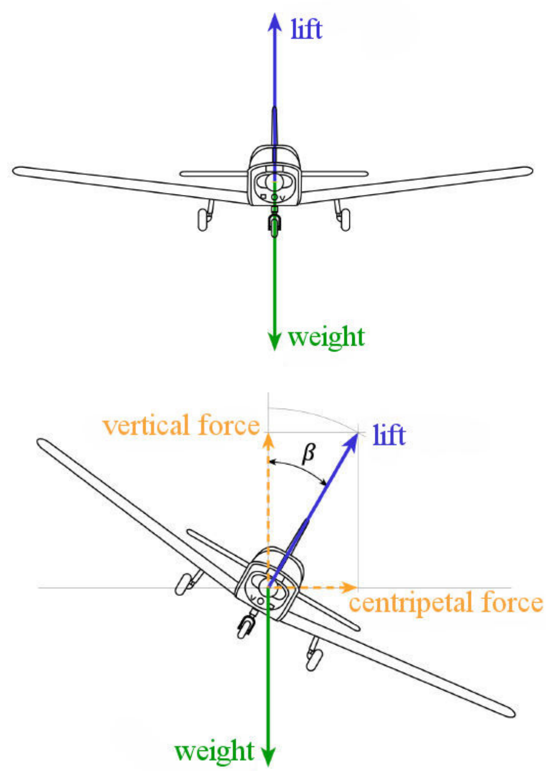
The drag, however, exerts a force on the wings which opposes the motion of the formula 1. Formula 1 drag coefficients are very high mainly because of this drag force on the wings. This higher drag force is, of course, a handicap for a race car but ultimately is it better to have wings. The cars go a little slower on the straights, but they can go so much faster in curves that, in the end, they can go around the track much more quickly with wings.

The forces exerted by the wings are so large at high speeds that it would be possible for a Formula 1 race car to drive upside down on a ceiling without falling (at speed larger than about 160 km/h) since the force made by the wings, which are pushing the car towards the ceiling, would be greater than the weight.

Now let's look at how a plane turn. In order for an aircraft to turn, there must be a force towards the center of the circular motion, so towards the left or towards the right. The problem is that there is no force acting in that direction on an airplane. The lift (force made by air on the wings) is upwards, the weight is downwards, the drag is towards the rear of the aircraft and the thrust of the engines is towards the front of the airplane. There is no force towards the right or towards the left! For a car, you could have friction. For an aircraft, there can be no friction since the aircraft does not touch the ground.

There is only one force that can be used as a centripetal force, and that is lift. If the plane tilts, the lift will also tilt.

Lift now has a horizontal component. This component then acts as the centripetal force.



www.aerodrome-ecuvillens.ch/pilote%20guide/vol_virage.pdf

Example 6.3.4

A Cessna travelling at 144 km/h must make a complete horizontal circle in 2 minutes.

a) What is the radius of the turn?

The circumference of the circle is

$$\ell = 2\pi r$$

As the aircraft travels at 40 m/s for two minutes, the circumference of the circle is also

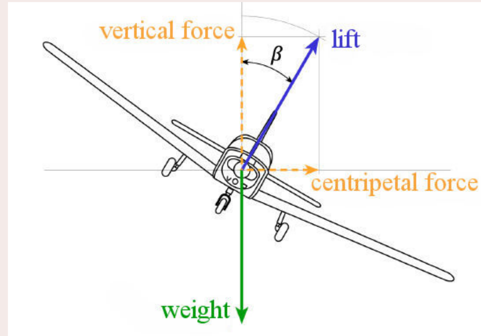
$$\begin{aligned} \ell &= vt \\ &= 40\tfrac{m}{s} \cdot 120s \\ &= 4800m \end{aligned}$$

Therefore

$$\begin{aligned} 4800m &= 2\pi r \\ r &= 763.9m \end{aligned}$$

a) How much does the plane have to tilt the plane to make this turn (β on the diagram)?

Forces Acting on the Object



There are 2 forces acting on the aircraft.

1) The weight (mg) directed downwards.
2) The lift (F_L) tilted by an angle β.

Sum of the Forces

The following axes will be used: an axis (x) towards the centre of the circular motion, an axis (y) in the direction of the motion of the aircraft (out of the page in the diagram) and a third axis (z) upwards.

Therefore, the sums of the forces are

$$\sum F_x = F_L \cos(90° - \beta)$$
$$\sum F_y = 0$$
$$\sum F_z = -mg + F_L \sin(90° - \beta)$$

Newton's Second Law

The acceleration is towards the right since the centre of circle is towards the right. Thus, Newton's second law is

$$\sum F_x = ma_c \quad \rightarrow \quad F_L \cos(90° - \beta) = m\frac{v^2}{r}$$
$$\sum F_y = ma_T \quad \rightarrow \quad 0 = 0$$
$$\sum F_z = 0 \quad \rightarrow \quad -mg + F_L \sin(90° - \beta) = 0$$

Solving the Equations

If the 3rd equation is solved for the lift

$$F_L = \frac{mg}{\sin(90° - \beta)}$$

And the result is use in the 1st equation, the result is

$$F_L \cos(90° - \beta) = \frac{mv^2}{r}$$

$$\frac{mg}{\sin(90°-\beta)}\cos(90°-\beta)=\frac{mv^2}{r}$$

$$\frac{g}{\tan(90°-\beta)}=\frac{v^2}{r}$$

Therefore,

$$\frac{9.8\frac{N}{kg}}{\tan(90°-\beta)}=\frac{\left(40\frac{m}{s}\right)^2}{763.9m}$$

$$\tan(90°-\beta)=4.679$$

$$90°-\beta=77.94°$$

$$\beta=12.06°$$

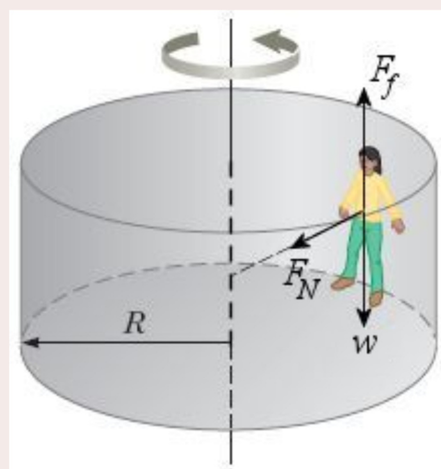
Example 6.3.5

In a rotor ride such as this one http://www.youtube.com/watch?v=2Sd9a0CSeiw (you'll see at 1:10 that the floor is removed and that the people remain stuck on the wall), what is the maximum period of rotation that the rotor ride can have so that people do not slip down the wall if the coefficient of friction between the wall and people is 0.7 and if the radius of the rotor ride is 2.5 m?

Forces Acting on the Object

There are three forces on the person.

1) The weight (w) directed downwards.
2) The normal force (F_N) directed towards the centre of the circular motion (inwards).
3) The friction force (F_f) directed upwards.



www.chegg.com/homework-help/questions-and-answers/advanced-physics-archive-2012-november-12

Sum of the Forces

The following axes will be used: an axis (x) towards the centre of the circular motion, an axis (y) in the direction of the motion of the person and a third axis (z) upwards.

Therefore, the sums of the forces are

$$\sum F_x=F_N \qquad \sum F_y=0 \qquad \sum F_z=-mg+F_f$$

Newton's Second Law

With a centripetal acceleration to the positive *x*, Newton's Second Law gives the following equations.

$$\sum F_x = ma_c \qquad \rightarrow \qquad F_N = m\frac{4\pi^2 r}{T^2}$$
$$\sum F_y = ma_T \qquad \rightarrow \qquad 0 = 0$$
$$\sum F_z = 0 \qquad \rightarrow \qquad -mg + F_f = 0$$

As the rotation period is sought, the centripetal force formula with the rotation period is used.

Solving the Equations

According to the third equation, the friction force is

$$F_f = mg$$

According to the first equation, the normal force is

$$F_N = m\frac{4\pi^2 r}{T^2}$$

Using those values in $F_f \leq \mu_s F_N$, we have

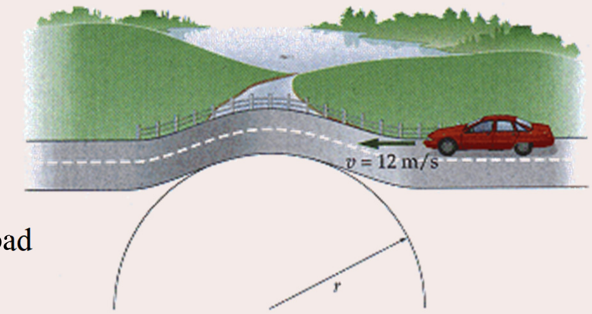
$$F_f \leq \mu_s F_N$$
$$mg \leq \mu_s m\frac{4\pi^2 r}{T^2}$$
$$g \leq \mu_s \frac{4\pi^2 r}{T^2}$$
$$T \leq \sqrt{\frac{4\pi^2 r\mu_s}{g}}$$

Therefore, the maximum rotation period is

$$T_{\max} = \sqrt{\frac{4\pi^2 \cdot 2.5m \cdot 0.7}{9.8\,\frac{N}{kg}}}$$
$$= 2.655s$$

Sometimes, an object loses contact with another during a circular motion. In order to know whether this happens or not, the magnitude of the normal force between the two objects in contact must be found, assuming they are in contact. A positive value for the normal force means that the objects do indeed touch each other whereas a negative value for the normal force means that this situation is impossible and that the objects do not touch each other.

Example 6.3.5



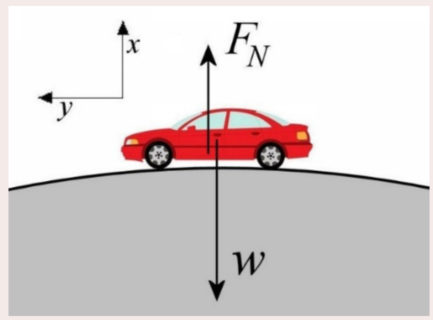
A car going at 12 m/s passes over a bump having a 5 m radius, as shown in the diagram.

a) Is the car still in contact with the road at the top of the bump?

www.chegg.com/homework-help/questions-and-answers/physics-archive-2011-september-22

Forces Acting on the Object

Let's assume that the car is still in contact with the road at the top of the bump. The forces on the car are:



1) The weight (w) directed downwards.
2) The normal force (F_N) directed upwards.

Sum of the Forces

The following axes will be used: an axis (x) towards the centre of the circular motion and an axis (y) in the direction of the motion of the car.

The sum of vertical forces is

$$\sum F_x = -mg + F_N$$

Newton's Second Law

With a centripetal acceleration directed downwards, Newton's Second Law gives the following equation.

$$\sum F_x = ma_c \qquad \to \qquad -mg + F_N = -m\frac{v^2}{r}$$

mv^2/r is negative because the centripetal force is directed downwards since the centre of the circle is under the car.

Solving the Equation

The two questions can now be answered. Let's first check whether the car stays in touch with the road. To find out, the normal force must be found.

$$F_N = mg - m\frac{v^2}{r}$$

$$= m\left(g - \frac{v^2}{r}\right)$$

$$= m\left(9.8\tfrac{N}{kg} - \frac{\left(12\tfrac{m}{s}\right)^2}{5m}\right)$$

$$= m\left(-19\tfrac{N}{kg}\right)$$

Regardless of the value of the mass, the normal force is negative, which is impossible. Therefore, the car is not in contact with the road.

In fact, only the force of gravity was in the right direction here to act as the centripetal force. However, this gravitational force is not sufficiently large to make the centripetal force, and there is, therefore, a lack of centripetal force. Thus, the car moves away from the centre of the circle, thereby losing contact with the road.

b) What maximum speed can a car passing on this bump have without losing contact with the road?

Now let's calculate the maximum speed to stay in touch with the road. The car is in contact with the road if normal is positive.

If normal is positive, then it means that

$$F_N = m\left(g - \frac{v^2}{r}\right) > 0$$

$$g > \frac{v^2}{r}$$

$$v < \sqrt{rg}$$

$$v < \sqrt{5m \cdot 9.8\tfrac{N}{kg}}$$

$$v < 7\tfrac{m}{s}$$

Therefore, the maximum speed is 7 m/s.

6.4 CENTRIFUGAL FORCES DO NOT EXIST

Common Mistake: Thinking That There Is a Centrifugal Force Directed Outwards in a Circular Motion.

You surely heard the term "centrifugal force" before. When a person is ejected from a merry-go-round, the centrifugal force is often invoked as the cause of this ejection. That is wrong. The lack of centripetal force is really the cause of the ejection.

A Lack of Centripetal Force

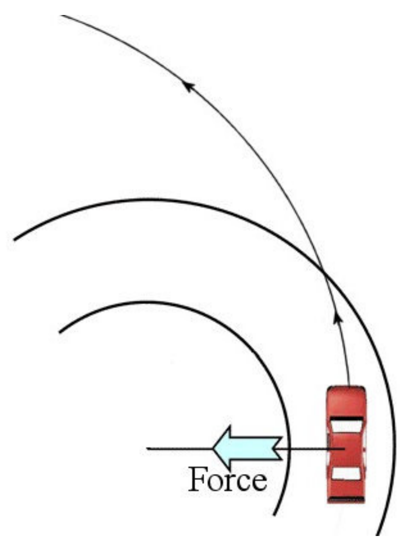
Consider the situation shown in this small animation to illustrate our point.
http://www.youtube.com/watch?v=1_UBPOiNHj8
There is an object on a rolling cart and there is no friction between the object and the cart. When the cart starts to turn, the small object cannot follow this circular motion since no force can act as the centripetal force. Only the weight and the normal force are acting on the small object, and none is in the right direction to deflect the trajectory to allow the object to turn with the cart. Without a centripetal force, the small object continues in a straight line as the cart turns. As a result, the small object falls from the cart. No centrifugal forces are pushing the object towards the outside of the curve. It is instead the absence of centripetal force which caused the object to fall from the cart since it has not been able to follow the circular motion of the cart.

If a car tries taking a curve with too much speed, it will leave the road, falling into the outside ditch of the curve. This is not because a centrifugal force pushes the car outwards, but because there is a lack of centripetal force. In a curve, the vehicle turns because there is a frictional force between the tires and the road acting as centripetal force. As there is a maximum to this frictional force, there is a maximum to the centripetal force. If the car is going too fast in the curve, it is possible that the friction force, even at its maximum value, is smaller than the centripetal force required to take this curve.

www.physicsclassroom.com/class/circles/u6l1c.cfm

The friction deflects the path of the car, but not enough to follow the curvature of the road. The car thus makes a circular motion, but with a greater radius of curvature than the road, causing the car to fall into the outside ditch. A centrifugal force does not push the car into the ditch; it fell there because there was a lack of centripetal force.

This is what happens to these cars.
http://www.youtube.com/watch?v=Cj8iaI_TNXw
http://www.youtube.com/watch?v=Dn7_bph2lhU

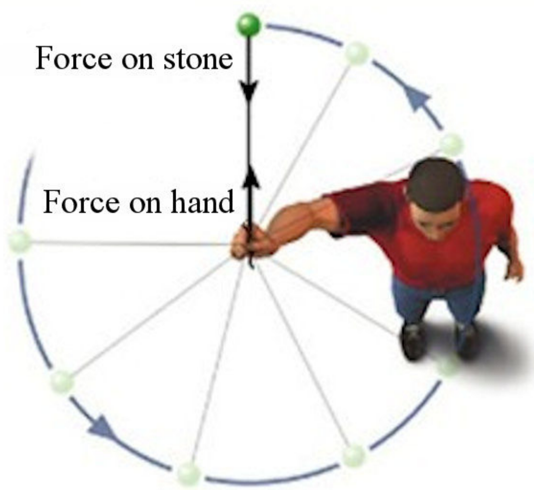


stockcarscience.com/scienceTopics/scsRacing_CentrifugalForce.php

Let's look at the people in this video, especially the person who wears red pants.
http://www.youtube.com/watch?v=hW57tLMcqt4
The normal made by the back of the bench makes the centripetal force. However, let's look at the forces acting on the person's upper body. Initially, the muscles of the person make the force towards the center on the upper body that allows it to make the circular motion. When the speed of rotation becomes too great, the muscles are no longer strong enough and there is then a lack of centripetal force. The person's torso can no longer follow the circular motion, and then moves away from the axis of rotation. When the torso becomes horizontal, the normal made by the backrest that was making the centripetal force up to this point changes direction. It is now upwards and therefore can no longer act as a centripetal force. Then, there is no centripetal force anymore, and the person can no longer follow the circular motion. So, he leaves following a straight-line motion.

There Are Forces Directed Outwards

When a stone rotates at the end of a rope and a person holds the rope, it is possible to think that there is a centrifugal force because the person feels that the rope pulls on its hand towards the outside of the circle. That is true, but the forces made by the rope at each end of the rope must be looked at very carefully. The rope pulls on the stone towards the inside of the circle. This is the centripetal force required for the stone to make its circular motion. The rope also pulls with equal force to the other end of the rope. So, the rope pulls on the hand towards the outside of the circle. This is indeed a force directed outwards, but this force does not act on the object making the circular motion. The net force on the object making the circular motion is always directed towards the centre of the circular path. The force can be directed outwards on other objects.



astronomy.nmsu.edu/tharriso/ast105/Ast105week04.html

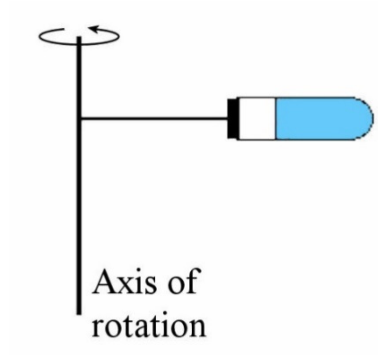
Even Huygens spoke of centrifugal force and not centripetal force. Actually, Huygens was looking for the force that the rotating object makes on the rope. As the rope pulls the object towards the centre, the object pulls the rope outwards. The force found by Huygens is thus really directed outwards.

In a car taking a curve, there seem to be forces directed outwards. Suppose you're sitting on the passenger side, without a seatbelt, on a very slippery seat. If the car turns left, you slide outwards until you come into contact with the door. You slide towards the door because there is no centripetal force acting on you. You then continue your motion in a straight line while the car is turning. If you continue in a straight line and the car is turning, you eventually come into contact with the side door. With this contact, there is now a normal force pushing you inwards that acts as the centripetal force to allow you to turn with the car. As a contact force is always a repulsive force, the force that the door exerts on you is indeed directed towards the centre of the circular motion. If the door pushes on you inwards, then, according to Newton's third law, you must exert a force directed outwards on the door. Does this mean that there is an outwards force on the car and that there is, therefore, a centrifugal force? No. There is indeed this outwards force on the car, but the net force on the car must be directed inwards. When you exert an outwards force on the door, the inwards friction force between the road and the tires must increase slightly to compensate for this outwards force.

The Centrifuge

Although the centrifugal force does not exist, centrifuges exist. However, the centrifugal force does not explain why substances settle at the bottom of test tubes.

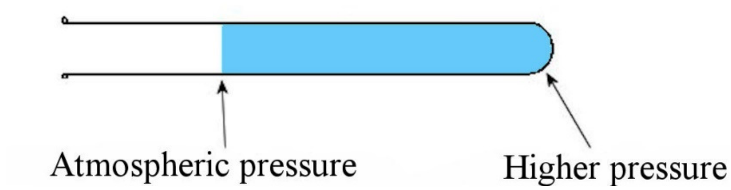
Here's a simplified centrifuge model, used to understand the mechanism. A test tube is simply rotating as shown in the diagram.



fr.openclassrooms.com/sciences/cours/la-physique-chimie-en-seconde/a-centrifugation

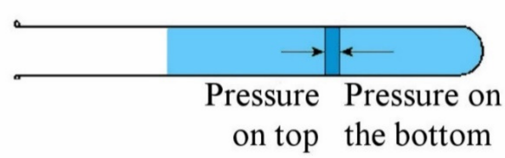
A Test Tube Filled Only With Water

Consider first a test tube only filled with water. As the water makes a circular motion, there must be a force acting on the water. This force is made by the water pressure. When the test tube is rotated, a pressure variation appears in the water. The pressure is largest at the bottom of the tube and decreases towards the surface of the liquid.



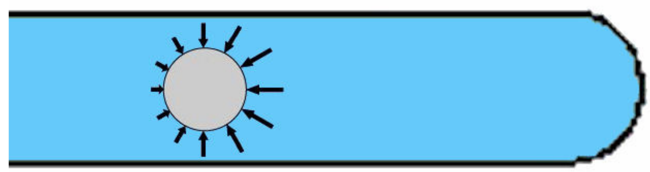
matoumatheux.ac-rennes.fr/num/fractions/6/eprouvette2.htm

Let's examine the force on a small slice of water in the test tube. The pressure force exerted below the layer is greater than the pressure force exerted above the layer. The excess pressure force towards the centre of the circular path is the force that makes the centripetal force of the water slice.



In fact, the pressure as a function of depth could be calculated based on the assumption that the pressure increase with depth should generate the right centripetal force on any slice of water.

A Test Tube Filled with Water Containing Particles



Now imagine that there is a particle in the water. The water pressure then acts on the particle. As the pressure gets larger towards the bottom of the test tube, the pressure force is greater on this side. The sum of these forces acting on the particle is a force towards the axis of rotation.

This force is, in fact, a buoyant force and is in the right direction to act as a centripetal force. The magnitude of this force can easily be found, using the formula of the buoyant force. However, the value of the gravitational acceleration must be replaced by the centripetal acceleration of the water in this formula. Therefore, the force is

$$F_B = \rho_{water}\left(\frac{v^2}{r}\right)V_f$$

The sum of the x-component of the force is then (using a positive x-axis pointing to the left, so towards the axis of rotation)

$$\sum F_x = ma$$

$$\rho_{water}\left(\frac{v^2}{r}\right)V_f = ma$$

From this equation, the acceleration can be found

$$\rho_{water}\left(\frac{v^2}{r}\right) = \frac{m}{V_f}a$$

$$\rho_{water}\left(\frac{v^2}{r}\right) = \rho_{particle}a$$

$$a = \frac{\rho_{water}}{\rho_{particle}}\left(\frac{v^2}{r}\right)$$

If the particle density is the same as the water density, the acceleration of the particle is exactly v^2/r. This means that the buoyant force has just the right magnitude to make the centripetal force. The particle then makes its circular motion and always remains at the same distance from the axis of rotation. The particle, therefore, remains at the same position in the tube. This occurs only if the particle has the same density as water.

If the particle has a higher density than water, then $a < v^2/r$. The buoyant force is not large enough, and there is a lack of centripetal force. Thus, the particle cannot follow the circular motion with a constant radius. With a lack of centripetal force, the particle makes a circular motion while moving away from the axis of rotation, until it reaches the bottom of the tube. The higher the density of the particle, the larger the lack of centripetal force is, and the faster the particle reaches the bottom of the cylinder. The denser particles will, therefore, reach the bottom first, followed by other less dense particles, and then by others less dense particles and so on. There is then a separation of the particle according to their density at the bottom of the test tube.

If the particle has a smaller density than water, then $a > v^2/r$. The buoyant force is too large, and there is an excess of centripetal force. In this case, the particle makes a circular motion while moving towards the axis of rotation, until it reaches the surface of the water in the test tube.

In the end, we get a test tube in which the particles are deposited in order of density at the bottom of the tube for particles denser than water and where the particles less dense than water float to the surface of the tube. The faster the rotation, the faster this separation is done. The centrifugal force does not make this separation.

6.5 NON-UNIFORM CIRCULAR MOTION

In a non-uniform circular motion, the centripetal acceleration is

$$a_c = \frac{v^2}{r} \qquad \text{If the axis points towards the centre.}$$

$$a_c = -\frac{v^2}{r} \qquad \text{If the axis points away from the centre.}$$

and the tangential acceleration is not zero.

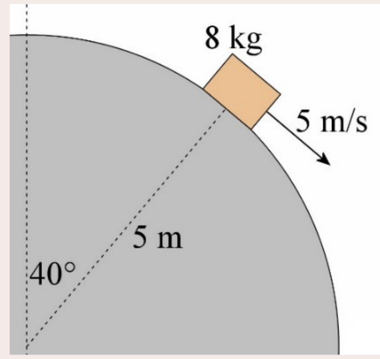
$$a_T \neq 0$$

Remember this common mistake.

Common Mistake: Using $a_c = \frac{4\pi^2 r}{T^2}$ to Calculate the Centripetal Acceleration in a Non-Uniform Circular Motion.

This formula was obtained by assuming that the speed is constant. Do not use it in a non-uniform circular motion, where the speed changes.

Example 6.5.1

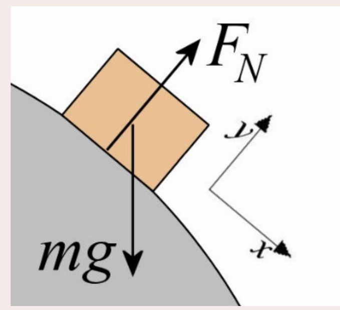


An object is sliding on a spherical surface. What are the tangential acceleration and the magnitude of normal force when the object is at the position shown in the diagram?

Forces Acting on the Object

The forces acting on the object are:

1) The weight (mg) directed downwards.
2) A normal force (F_N) perpendicular to the surface.



Sum of the Forces

Using the axes shown in the diagram, the sums of the forces are

$$\sum F_x = mg\cos(-50°)$$
$$\sum F_y = mg\sin(-50°) + F_N$$

Newton's Second Law

With a centripetal acceleration in the opposite direction to the y-axis and a tangential acceleration in the direction of the x-axis, Newton's second law gives

$$\sum F_x = ma_T \qquad \rightarrow \qquad mg\cos(-50°) = ma_T$$
$$\sum F_y = ma_c \qquad \rightarrow \qquad mg\sin(-50°) + F_N = -m\frac{v^2}{R}$$

Solving the Equations

With the first equation, the tangential acceleration can be found.

$$\begin{aligned} a_T &= g\cos(-50°) \\ &= 6.299\tfrac{m}{s^2} \end{aligned}$$

The normal force can be found with the second equation.

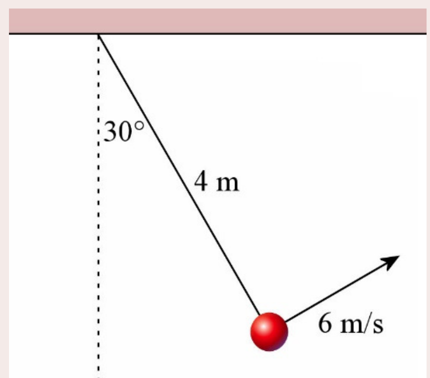
$$\begin{aligned} F_N &= -mg\sin(-50°) - m\frac{v^2}{R} \\ &= -8kg \cdot 9.8\tfrac{N}{kg} \cdot \sin(-50°) - 8kg \cdot \frac{\left(5\tfrac{m}{s}\right)^2}{5m} \\ &= 60.06N - 40N \\ &= 20.06N \end{aligned}$$

If the speed had been higher, say 7 m/s, the normal force would have been

$$\begin{aligned} F_N &= -mg\sin(-50°) - m\frac{v^2}{R} \\ &= -8kg \cdot 9.8\tfrac{N}{kg} \cdot \sin(-50°) - 8kg \cdot \frac{\left(7\tfrac{m}{s}\right)^2}{5m} \\ &= 60.06N - 78.4N \\ &= -18.34N \end{aligned}$$

This answer is impossible. It means that to have this circular motion at such a speed, the normal force must pull on the object, something a normal force cannot do. There is, therefore, a lack of centripetal force. Thus, the object would not be able to make a circular motion with such a small radius, and it would leave the surface of the sphere.

Example 6.5.2



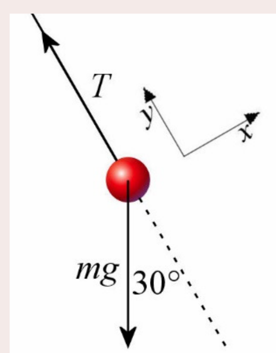
This pendulum has a speed of 6 m/s when the angle between the rope and the vertical is 30°.

a) What is the magnitude of the acceleration of the mass at the end of the rope?

To find the acceleration, the two components of the acceleration are needed. Those components are the centripetal and tangential accelerations.

The centripetal acceleration is easily found with

$$\begin{aligned} a_c &= \frac{v^2}{r} \\ &= \frac{\left(6\tfrac{m}{s}\right)^2}{4m} \\ &= 9\tfrac{m}{s^2} \end{aligned}$$



To find the tangential acceleration, the sum of the forces on the object must be calculated.

Forces Acting on the Object

There are 2 forces acting on the object.

1) The weight (mg) directed downwards.
2) The tension force (T).

Sum of the Forces

Using the axes shown in the diagram, the sums of the forces are

$$\sum F_x = mg\cos(-120°)$$
$$\sum F_y = mg\sin(-120°) + T$$

Newton's Second Law

With a centripetal acceleration in the direction of the y-axis and a tangential acceleration in the direction of the x-axis, the equations are

$$\sum F_x = ma_T \qquad \rightarrow \qquad mg\cos(-120°) = ma_T$$
$$\sum F_y = ma_c \qquad \rightarrow \qquad mg\sin(-120°) + T = m\frac{v^2}{R}$$

Solving the equation

The tangential acceleration can be found with the first equation.

$$\begin{aligned} a_T &= g\cos(-120°) \\ &= -4.9\tfrac{m}{s^2} \end{aligned}$$

With tangential acceleration, the magnitude of the total acceleration can now be found.

$$\begin{aligned} a &= \sqrt{a_c^2 + a_T^2} \\ &= \sqrt{\left(9\tfrac{m}{s^2}\right)^2 + \left(4.9\tfrac{m}{s^2}\right)^2} \\ &= 10.25\tfrac{m}{s^2} \end{aligned}$$

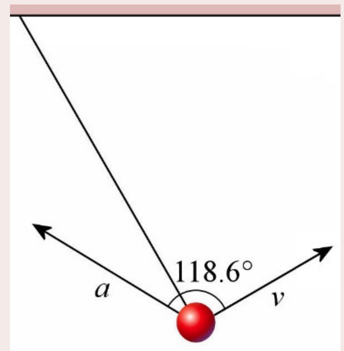
b) What is the angle between the velocity of the mass and the net force exerted on the mass?

As the force and the acceleration are in the same direction, the answer to this question can be found by searching for the angle between the acceleration and the velocity. As the velocity is in the direction of the x-axis, the answer to this question can be found by searching for the angle between the x-axis and the acceleration.

However, the angle between the acceleration vector and the x-axis can be found with

$$\tan\theta = \frac{a_y}{a_x}$$

With our choice of axes, the tangential acceleration is the x-component, and the centripetal acceleration is the y-component.



Therefore,

$$\begin{aligned}\theta &= \arctan\frac{a_y}{a_x}\\ &= \arctan\frac{9\,\frac{m}{s^2}}{-4.9\,\frac{m}{s^2}}\\ &= 118.6°\end{aligned}$$

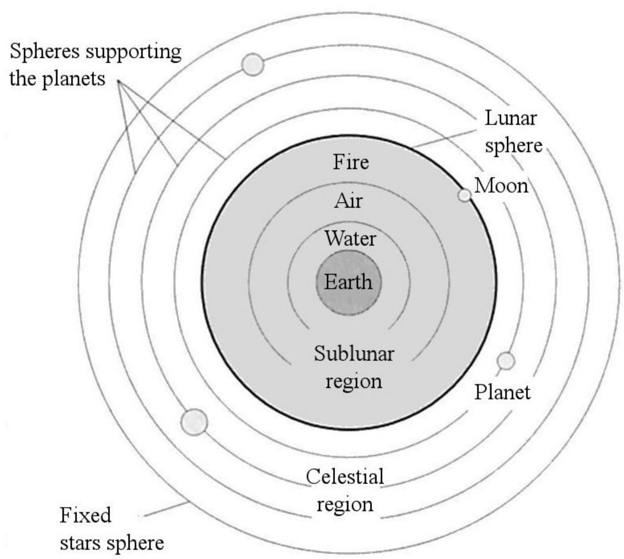
(180° were added to the answer given by the calculator since the x-component is negative.)

6.6 GRAVITATION AND CIRCULAR ORBITS

Now we will take a look at the circular motion of objects orbiting a planet or star. We will first see how the study of this movement led to the discovery of the law of gravitation. Then we will find formulas for the speed and the period of objects in orbit.

Old Theories

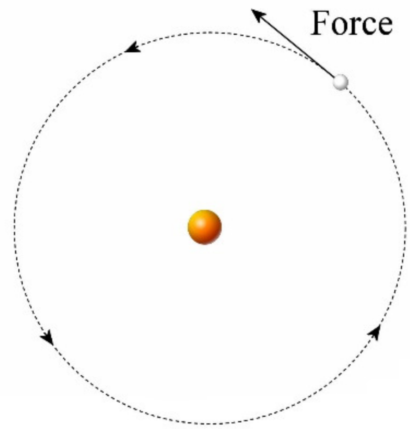
This is how Aristotle explained gravitation. The four elements of the Greeks each have a natural place in the universe. Starting from the center of the universe, there is earth, then water, then air and finally fire. According to this theory, a stone falls to the ground when it is lifted into air or water because the stone, mainly composed of the earth element, falls back to its natural place. In the same way, fire in air rises to return to its natural place, which is above the air.



Y. Gingras, P Keating, C. Limoges, Du scribe au savant, Éditions du boréal, 1998

Then there are the planets that revolve around the Earth. This means that there is a clear separation between 2 parts of this universe. In the sublunar world, the natural motion of objects is to go up or down according to their composition. In the heavens, the natural motion of the planets is a circular motion around the center. There are therefore 2 regions where the laws of physics are very different.

Since force was associated with speed, it was necessary to find out how the planets could move to revolve around the Earth. For a long time, they were looking for a force acting in the direction of the speed. When the force is associated with velocity, there had to be a force acting on the planets in the direction of their motion. It was even imagined at some point that angels exerted this force! As can be seen, planets are not attracted towards the center in these ancient models.

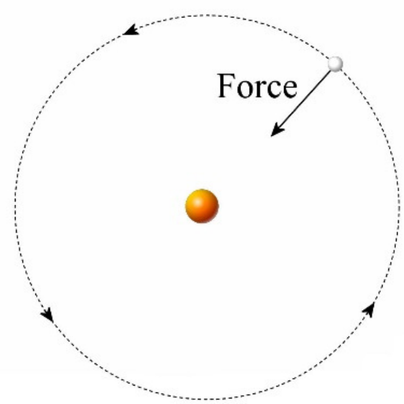


Attraction

With the discovery of the law of inertia (beginning of the 17th century), they began to understand that objects must move in **a straight line** if nothing alters their motion. If the planets are moving in a circular motion rather than in a straight line, then there must be a centripetal force that causes them to constantly change direction.

In fact, it took almost a century to arrive at a correct analysis of the motion of orbiting planets. It all starts with the discovery of the formula for centripetal force. Huygens discovered in 1659 that the tension of the rope that holds a rotating object at the end of a rope must be proportional to r/T^2. However, Huygens never thought of using his discoveries to study the motion of orbiting planets.

After the publication of Huygens's work in 1673, some, such as Robert Hooke, Edmund Halley, and Christopher Wren, used Huygens's discoveries to study the orbital motion of the planets. They understood, around 1680, that the force on the object in orbit must be directed towards the centre. They were beginning to understand that this means that the Sun attracts the planets into the Solar System.



Halley and Wren even concluded that the force made by the Sun must diminish with the square of the distance. To reach this conclusion, they started with Huygens's result

$$F \propto \frac{r}{T^2}$$

Next, a law that Kepler discovered for planets is used. In 1618, Kepler had noticed that

$$\frac{r^3}{T^2} = \text{constant}$$

for all planets revolving around the Sun (r is the radius of the planet's orbit and T is the rotation period around the Sun). As this relationship means that

$$r^3 \propto T^2$$

the force formula can be written in the following form.

$$F \propto \frac{r}{T^2}$$

$$F \propto \frac{r}{r^3} \quad \text{(Kepler's law is used)}$$

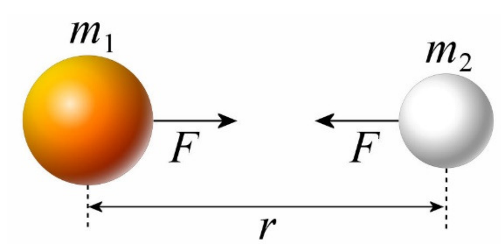
$$F \propto \frac{1}{r^2}$$

In 1684, Halley asked Newton what the orbits of the planets would be if the gravitational force varied with $1/r^2$. This request launched Newton, who had not been interested in mechanics since the late 1660s, in a very thorough study of the orbits of planets. This study leads him to a complete overhaul of mechanics that associates force with acceleration and culminate in 1687 with the publication of Newton's book, which revolutionized physics forever.

Newton can go even further. With his new laws of mechanics, he discovers that the force on the planet must also be proportional to the mass of the object on which the force acts (so that the masses cancel each other out in $F = ma$, giving all the objects the same acceleration in free fall).

$$F_g \propto \frac{m}{r^2}$$

But there's more. According to Newton's third law, if the planet having a mass m_1 exerts a force on the planet having a mass m_2, then the planet having a mass m_2 exerts the same force on the planet having a mass m_1. As the force is proportional to the mass of the planet it acts on, and as the force is acting on the two masses, then the force must be proportional to both masses.



$$F_g \propto \frac{m_1 m_2}{r^2}$$

This is Newton's law of gravitation. The result is clear. The Sun is not the only object that attracts the planets. All the massive objects in the universe attract all the other massive objects in the universe with a gravitational force. This is the universal gravitational attraction. It still took a lot of audacity to say that all masses attract each other because it is not at all obvious that all objects attract each other. A banana does not seem to be attracted by an apple placed nearby, but this is what the law of gravitation says. Of course, it was understood that the attraction was too weak to be perceived and it took nearly a century before someone (Henry Cavendish) managed to measure this small force of attraction between objects of a few kilograms.

In fact, the idea of gravitational attraction wasn't entirely new. Gilles Personne de Roberval (1644) and Robert Hooke (1665) had already formulated the idea that all objects in the universe attract each other (but without arriving at a formula for force).

Newton's law of gravitation leads to a great unification in physics. In Aristotle's system, there was gravitation near the Earth made by the natural positions of the 4 elements inside the orbit of the Moon (which was called the *sublunar region*) while there was no gravitation beyond the orbit of the Moon (which was called the *heavens*). There was a separation between these 2 regions. The physics was not the same in these 2 regions. With Newton, this separation between the sublunar world and the heavens is not needed anymore. The force of gravity acts on objects near the surface of the Earth and on the Moon following exactly the same laws.

The Law of Gravitation

In the equation

$$F_g \propto \frac{m_1 m_2}{r^2}$$

The *proportional to* sign can be replaced by a constant of proportionality (called G) to obtain the following law.

Law of Gravitation (General Formula)

1) Magnitude of the force

$$F_g = G\frac{m_1 m_2}{r^2}$$

where $G = 6.674 \times 10^{-11}$ Nm²/kg²

2) Direction of the force

Attraction of the two masses one towards the other.

3) Application point of the force

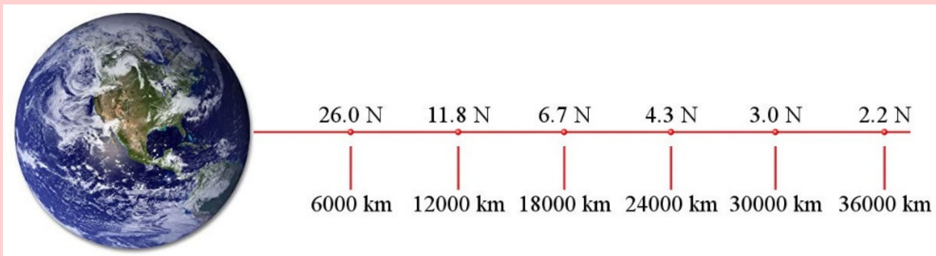
From de centre of mass
(More details in chapter 11)

(Note that r is the distance between the centres of the planets.)

For those interested, here are some historical anecdotes about the discovery of the law of gravitation.
https://physique.merici.ca/mechanics/notesG-eng.pdf

Common mistake: Thinking That There is no Gravity in Space

The law of gravitation states that the force of gravitation decreases with distance. As we move away from the Earth, the gravitational force made by the Earth decreases. However, the force is still quite great even when outside the atmosphere. The following image shows the gravitational force made by the Earth on a 10 kg object at different distances from the Earth.



Common Mistake: Thinking that when Two Objects Attract Each Other with a Gravitational Force, the Less Massive Object Exerts a Smaller Force on the Other Object

Intuitively, people suspect that the Earth exerts a force on the Moon and that the Moon also exerts a force on Earth. However, many would say that the Earth exerts a much greater force on the Moon than the Moon exerts on the Earth because it is more massive. Admire the answers of some people in this video.
http://www.youtube.com/watch?v=8bTdMmNZm2M
Of course, all this is false; the force must have the same magnitude according to Newton's third law.

Example 6.6.1

A 350 kg meteorite is 10,000 km from the centre of the Earth. What is the magnitude of the gravitational force made by the Earth on the meteorite knowing that the Earth has a mass of 5,974 x 10^{24} kg?

The magnitude of the force is

$$
\begin{aligned}
F_g &= G\frac{m_1 m_2}{r^2} \\
&= 6.674\times 10^{-11} \tfrac{Nm^2}{kg^2} \cdot \frac{350kg \cdot 5.974\times 10^{24} kg}{\left(10\,000\times 10^3 m\right)^2} \\
&= 1395N
\end{aligned}
$$

The Gravitational Field

The gravitational force on an object of mass m made by a planet (or star) of mass M_p

$$F_g = G\frac{M_p m}{r^2} = m\left(G\frac{M_p}{r^2}\right)$$

can be written in the following form.

Gravitational Force (General Formula)

$$F_g = mg$$

where g is defined as the magnitude of the gravitational field made by the planet (or star).

Magnitude of Gravitational Field Made by a Planet With a Mass M_p

$$g = \frac{GM_p}{r^2}$$

In fact, the calculation of gravitational force on m mass was separated into two steps:

1) The gravitational field made by the planet or planets is calculated at the position where the mass m is.

2) Gravitational force on the mass m is calculated with $F_g = mg$.

Example 6.6.2

A 350 kg meteorite is 10,000 km from the centre of the Earth. The mass of the Earth is 5,974 x 10^{24} kg.

a) What is the magnitude of the gravitational field made by the Earth at the location of the meteorite?

The magnitude of the field is

$$\begin{aligned} g &= G\frac{M_p}{r^2} \\ &= 6.674\times 10^{-11}\,\tfrac{Nm^2}{kg^2}\cdot\frac{5.974\times 10^{24}kg}{\left(10\,000\times 10^3 m\right)^2} \\ &= 3.987\,\tfrac{N}{kg} \end{aligned}$$

b) What is the magnitude of the gravitational force made by the Earth on the meteorite?

The magnitude of the force is

$$\begin{aligned} F_g &= mg \\ &= 350kg \cdot 3.987 \tfrac{N}{kg} \\ &= 1395N \end{aligned}$$

Why separate the calculation in this way since one can calculate the force directly (as was done in the previous example)? It's a good thing to do since once we know the magnitude of the gravitational field in one place, the gravitational force can be simply calculated with $F_g = mg$ for different masses without having to redo the big calculation with the mass and radius of the planet (as we have done since chapter 4).

These formulas can be applied to find the gravitational force made on an object by any planet (or star).

Example 6.6.3

A 10 kg stone is on the surface of the Moon. The mass of the Moon is 7.36 x 10^{22} kg, and the radius of the Moon is 1740 km.

a) What is the magnitude of the gravitational field on the surface of the Moon?

The magnitude of the field is

$$\begin{aligned} g &= G\frac{M_{Moon}}{R_{Moon}^2} \\ &= 6.674\times 10^{-11} \tfrac{Nm^2}{kg^2} \cdot \frac{7.36\times 10^{22} kg}{\left(1740\times 10^3 m\right)^2} \\ &= 1.62 \tfrac{N}{kg} \end{aligned}$$

b) What is the magnitude is the gravitational force made by the Moon on the stone?

The force is

$$\begin{aligned} F_g &= 10kg \cdot 1.62 \tfrac{N}{kg} \\ &= 16.2N \end{aligned}$$

The weight near the surface of the Moon is about 6 times smaller than the weight near the surface of the Earth because g is only 1.6 N/kg at the surface of the Moon. On the Moon,

the mass of the object is the same as on Earth, but the weight is approximately 6 times smaller. You can see what happens with a smaller gravity on the Moon in this video.
http://www.youtube.com/watch?v=HKdwcLytloU

The Force of Gravity Near the Surface of the Earth

In Chapter 4, we had a different version of the law of gravitation

$$F_g = m \cdot 9.8 \tfrac{N}{kg}$$

We have this formula because the force is $F_g = mg$ and the magnitude of the gravitational field on the Earth's surface is

$$\begin{aligned} g &= G\frac{M_E}{R_E^2} \\ &= 6.674\times 10^{-11} \tfrac{Nm^2}{kg^2} \cdot \frac{5.974\times 10^{24} kg}{\left(6378\times 10^3 m\right)^2} \\ &= 9.8 \tfrac{N}{kg} \end{aligned}$$

The value of g is 9.8 N/kg or 9.8 m/s² only near the Earth's surface. Away from the Earth's surface, g will take on another value. Therefore, g is not really a constant as one who had done only chapters 1 to 5 might think.

Even if $F = m \cdot 9{,}8$ N/kg is very different from the law of general gravitation, this formula is easily found from the general formula. The formula $F = m \cdot 9{,}8$ N/kg is just one specific case of the general formula. For the scientists of Newton's time, this is a real revelation. Before Newton, it was not obvious that the force that keeps the Moon in orbit around the Earth had the same origin as the force that draws us towards the ground. Newton began to understand that it was the same force in 1666 (after seeing an apple fall to the ground, it seems). It was not until 20 years later that Newton proved it. It is a beautiful unification of physics on Earth and in the heavens.

The field decreases as we move away from the Earth's surface. For example, at an altitude of 100 km above the Earth's surface, the value of g is

$$\begin{aligned} g &= G\frac{M_p}{r^2} \\ &= 6.674\times 10^{-11} \tfrac{Nm^2}{kg^2} \cdot \frac{5.974\times 10^{24} kg}{\left(6478\times 10^3 m\right)^2} \\ &= 9.5 \tfrac{N}{kg} \end{aligned}$$

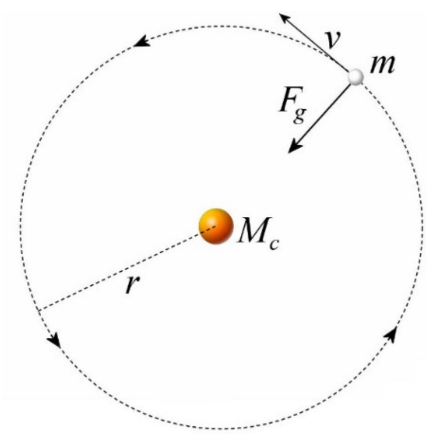
(r is not 100 km as it is the distance between the object at an altitude of 100 km and the centre of the Earth, i.e. 6378 km + 100 km.)

It can be seen that g decreases quite quickly with altitude. Basically, the value of g decreases at a rate of 0.0031 N/kg per kilometre near the Earth's surface. An object should not reach an altitude of more than a few kilometres to assume that g is a constant worth 9.8 N/kg.

Speed and Period on a Circular Orbit

Now, let's go back to the problem of orbits, the very one that set Newton on the path to the discovery of the law of gravity (although we're only going to explore circular orbits while Newton has made a much more complete solution).

For an object in a circular orbit around a planet or a star, the force of gravity is the centripetal force.



This means that

$$\frac{GM_c m}{r^2} = m\frac{v^2}{r}$$

The mass of the planet or star at the centre of the orbit is called M_c, for *central mass*.

The object in orbit must have a very precise orbital speed to follow a circular orbit.

If the previous equation is solved for v, the result is

Speed of an Object on a Circular Orbit

$$v = \sqrt{\frac{GM_c}{r}}$$

The time required to go around the orbit once is called the period T. This period is

$$T = \frac{\text{distance}}{\text{speed}} = \frac{2\pi r}{\sqrt{\frac{GM_c}{r}}}$$

Simplified, this is

Period of an Object on a Circular Orbit

$$T = 2\pi\sqrt{\frac{r^3}{GM_c}}$$

This formula is often called Kepler's third law since Kepler discovered in 1618 that $T^2 \propto r^3$ for planets orbiting around the Sun.

Example 6.6.4

Knowing that the Earth revolves around the Sun with a period of 365.2566 days and that the radius of the Earth's orbit is 149,600,000 km, calculate the mass of the Sun.

The mass is found with

$$T = 2\pi\sqrt{\frac{r^3}{GM_c}}$$

$$365.2566 \cdot 24 \cdot 60 \cdot 60 s = 2\pi\sqrt{\frac{\left(1.496\times10^{11} m\right)^3}{6.674\times10^{-11}\,\frac{Nm^2}{kg^2}\cdot M_c}}$$

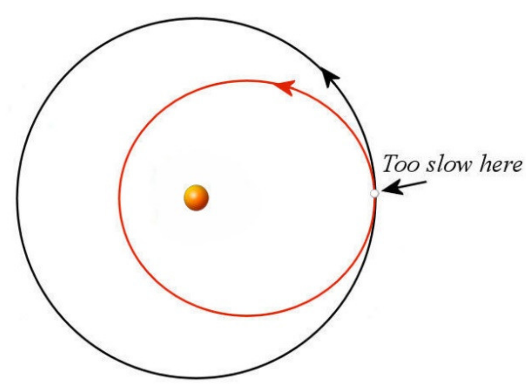
$$M_c = 1.9885\times10^{30} kg$$

The law of gravitation is used to find the mass of all the planets, stars and galaxies in the universe.

What Happens if the Object in Orbit Does Not Have the Right Speed?

The object must have a very precise speed for its orbit to be circular. However, no harm is done if the speed is not exactly equal to the speed required to make a circular orbit. If the velocity of the object is too small, we have

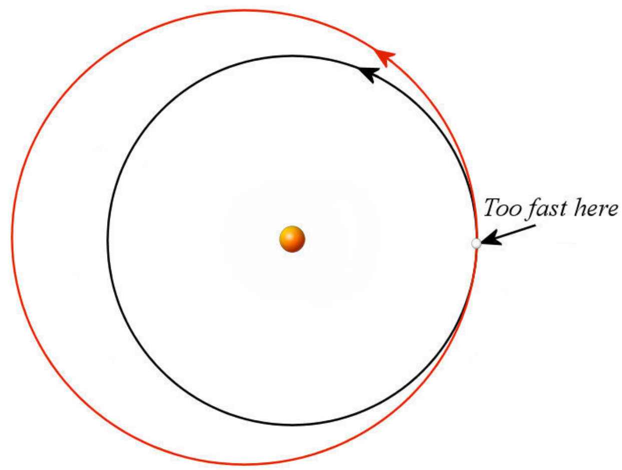
$$\frac{GM_c m}{r^2} > m\frac{v^2}{r}$$



This means that there is an excess of centripetal force. In this case, the object makes a circular motion while moving towards the centre of the orbit. This excess force does not necessarily mean that the object will crash on the central mass because the object will gain speed as it approaches the central mass and the gravitational force will increase, thereby changing the relation between the centripetal and gravitational forces. Later, the gravitational force becomes smaller than the centripetal force and the object moves away from Earth again to eventually follow an elliptical orbit.

If the speed is too high, we have

$$\frac{GM_c m}{r^2} < m\frac{v^2}{r}$$



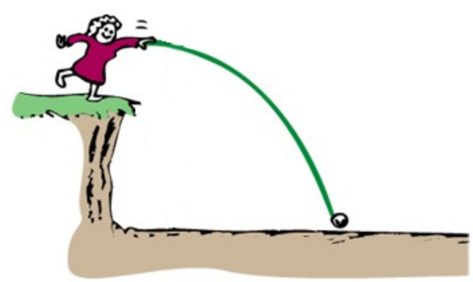
This means that there is a lack of centripetal force. In this case, the object makes a circular motion while moving away from the centre of the orbit. This lack of force does not necessarily mean that the object will get lost in space because the object will lose speed as it moves away from the central mass and the gravitational force will decrease, thereby changing the relation between the centripetal and gravitational forces. Later, the gravitational force becomes greater than the centripetal force and the object approaches the Earth again to eventually follow an elliptical orbit.

The resulting trajectories will be examined in more details in chapter 14.

Objects in Orbit Are in Free-Fall

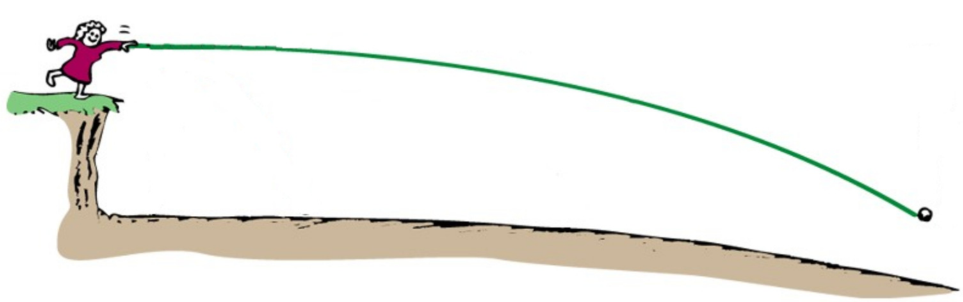
This is not obvious but objects in orbit are free-falling. To prove this, suppose that balls are launched horizontally from the top of a very high cliff, but at different speeds. There are no drag forces in these situations.

The first ball is thrown with a relatively small speed. This is a projectile that falls with a trajectory bending towards the ground resembling the path shown in this diagram. As all projectiles, this ball is free-falling.



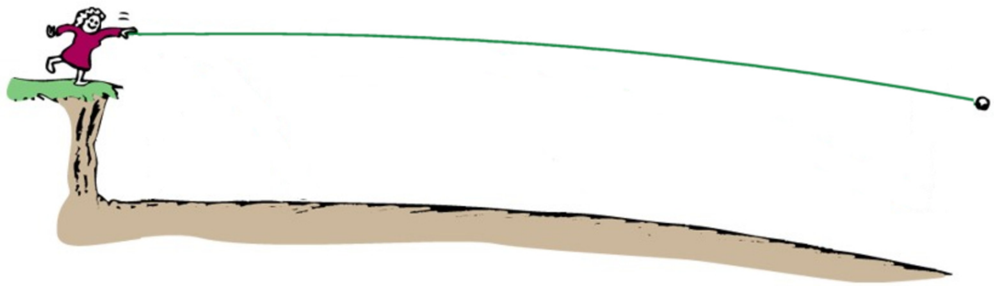
slid.es/tofergregg/gravity-and-fluid-dynamics/fullscreen#/22

If the ball is thrown with more speed, we still have a free-falling ball. The path then looks like the path shown in this diagram (where the curvature of the Earth is much exaggerated).



The ball was thrown with so much speed that the curvature of the Earth begins to be significant. The ball still curved towards the ground, but the ground also curves. In this case, the ball eventually hit the ground.

If the speed is further increased up to the speed required for a circular orbit, the following situation is achieved.

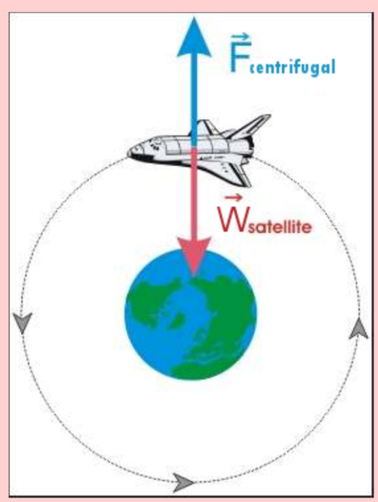


The trajectory of this free-falling object still curves towards the ground because of the force of gravity. However, the ground also curves so that the object is always at the same distance from the ground. Thus, the object, even if it is free-falling, never hit the ground and moves continuously on its circular orbit.

So, why doesn't the Moon fall towards the Earth if it is attracted by the force of gravitation? Just as the ball from the last example, the Moon is indeed free-falling, but it never hits the Earth because gravity only curves the trajectory of the Moon so that it does not approach the Earth.

The following clip gives you the same explanation.
http://www.youtube.com/watch?v=MpiknSRTmT4

Common Mistake: Thinking That There is a Balance between Gravity and a Centrifugal Force on an Object in Orbit.



Often, it is said that there is a balance between centrifugal and gravitational force for objects in a circular orbit, as shown in the diagram to the right.

This cannot be correct since the centrifugal force does not exist. Moreover, if these two forces were to cancel each other, the sum of the forces would be zero and the object would move in a straight line. A circular path cannot be obtained if the sum of the forces is zero because this would be in obvious contradiction with Newton's first law.

xp.hauduroy.free.fr/Mise_en_orbite.html

Synchronous Satellites

It is possible to place a satellite at just the right distance from Earth so that its period is 24 hours (actually 23 h 56 min 4 s). The satellite then rotates around the Earth at exactly the same rate as the Earth rotates on itself and thus always remains above the same spot on Earth. In other words, the satellite always sees the same side of the Earth. These satellites are important because their positions can easily be found since they are always in the same direction in the sky. Once an antenna is adjusted to point towards the satellite, it is always in the right direction. If the satellite were orbiting faster or slower, it would be necessary to constantly change the direction of the antenna to receive the signal. This little video gives you an animated version of this explanation.
https://www.youtube.com/watch?v=sj7zsGkpZxg

Example 6.6.5

How far from the surface of the Earth should synchronous satellites be placed and what should be their speed knowing that the mass of the Earth is 5.974 x 10^{24} kg?

Using the period formula, the distance can be found.

$$T = 2\pi\sqrt{\frac{r^3}{GM_c}}$$

$$24 \cdot 60 \cdot 60 s = 2\pi\sqrt{\frac{r^3}{6.674\times10^{-11} \frac{Nm^2}{kg^2} \cdot 5.974\times10^{24} kg}}$$

$$r = 4.2245\times10^7 m = 42,245 km$$

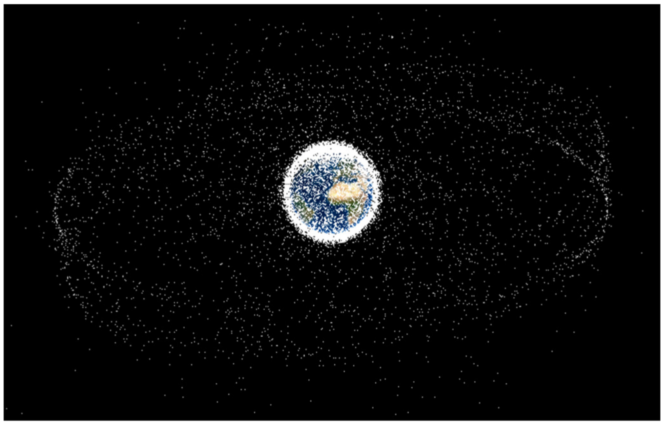
All synchronous satellites have this orbital radius. As the Earth has a radius of 6378 km, they are thus 35,866 km away from the Earth's surface.

The orbital speed of these satellites is

$$\begin{aligned} v &= \sqrt{\frac{GM_c}{r}} \\ &= \sqrt{\frac{6.674\times10^{-11} \frac{Nm^2}{kg^2} \cdot 5.974\times10^{24} kg}{4.2245\times10^7 m}} \\ &= 3,072 \tfrac{m}{s} = 11,060 \tfrac{km}{h} \end{aligned}$$

All synchronous satellites have this speed.

There are, therefore, two very busy locations around the Earth to place satellites. Most satellites are placed close to the Earth (which is cheaper) or in synchronous orbit. The following diagram showing the position of satellites around the Earth illustrates this. Most of them are very near or at a distance where the satellites have a 24-hour period.



maps.esri.com/rc/sat2/index.html

SUMMARY OF EQUATIONS

Equations of Circular Motion

Radial direction (towards the centre or away from the centre)

$$\sum F = ma_c$$

Tangential direction (in the direction of velocity or opposite to the direction of velocity

$$\sum F = ma_T$$

Third axis

In these notes, the acceleration is always zero in the direction of this axis.

$$\sum F = 0$$

Uniform Circular Motion

$a_c = \frac{v^2}{r}$ *or* $a_c = \frac{4\pi^2 r}{T^2}$ If the axis points towards the centre.

$a_c = -\frac{v^2}{r}$ *or* $a_c = -\frac{4\pi^2 r}{T^2}$ If the axis points away from the centre.

$a_T = 0$

Non-Uniform Circular Motion

$a_c = \frac{v^2}{r}$ If the axis points towards the centre.

$a_c = -\frac{v^2}{r}$ If the axis points away from the centre.

$a_T \neq 0$

Law of Gravitation (General Formula)

1) Magnitude of the force

$$F_g = G\frac{m_1 m_2}{r^2}$$

where $G = 6.674 \times 10^{-11}$ Nm²/kg²

2) Direction of the force

Attraction of the two masses one towards the other.

3) Application point of the force

From de centre of mass

Gravitational Force (General Formula)

$$F_g = mg$$

Magnitude of Gravitational Field Made by a Planet With a Mass M_p

$$g = \frac{GM_p}{r^2}$$

Speed of an Object on a Circular Orbit

$$v = \sqrt{\frac{GM_c}{r}}$$

Period of an Object on a Circular Orbit

$$T = 2\pi\sqrt{\frac{r^3}{GM_c}}$$

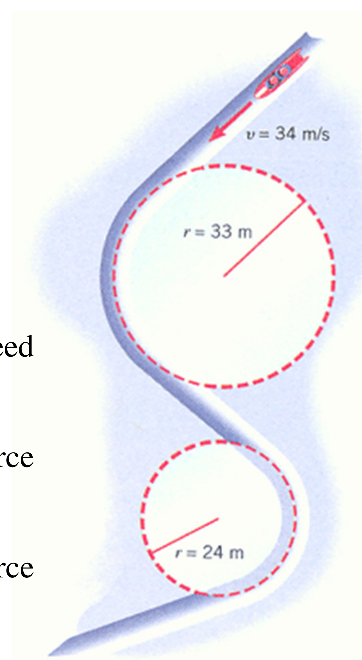
EXERCISES

6.3 Uniform Circular Motion

1. This 200 kg sled moves along a track with a steady speed of 34 m/s.

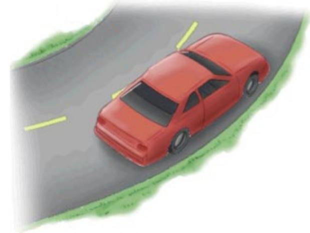
   a) What is the magnitude of the centripetal force when the sled is the bend with a 33 m radius?

   b) What is the magnitude of the centripetal force when the sled is the bend with a 24 m radius?



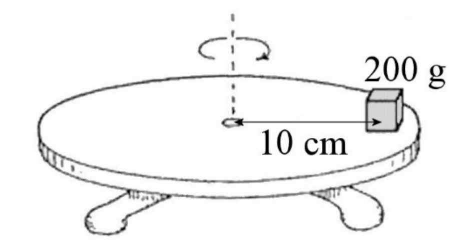
whs.wsd.wednet.edu/Faculty/Busse/MathHomePage/busseclasses/apphysics/studyguides/chapter5/apphysicsch5_2006.php

2. A car going at a steady speed of 120 km/h takes a curve with a 100 m radius. The road is not banked. What is the minimum value that the coefficient of static friction between the tires and the road must have so that the car takes this curve without slipping?

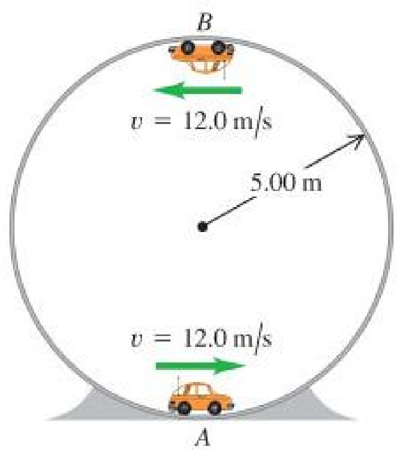


www.ux1.eiu.edu/~addavis/1350/06CirMtn/FlatCurve.html

3. A small 200 g block is placed on a turntable, 10 cm from the axis of rotation. What is the rotation period if the block is travelling at the maximum speed that it can have without slipping off the turntable if the coefficient of static friction between the block and the table is 0.6?

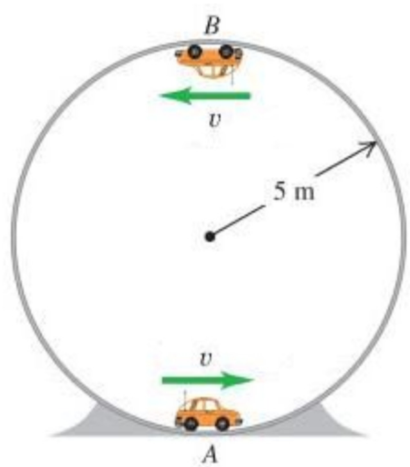


4. In the situation shown in the diagram, what are the normal forces exerted on the 1000 kg car at points A and B?

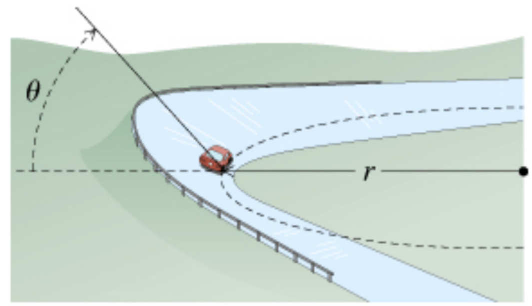


www.chegg.com/homework-help/questions-and-answers/physics-archive-2011-october-07

5. How fast must this car go in order to stay in contact with the track at the top of its trajectory (point B)?



6. A car passes on a banked road ($\theta = 20°$) having a radius of $r = 80$ m. The road is so icy that there is absolutely no friction between the road and the tires. How fast must this car go to take this curve without slipping?



7. A 1200 kg car passes on a banked road ($\theta = 30°$) having a radius of $r = 80$ m.

   a) What is the force of friction exerted on the car if the car's speed is 100 m/s?

   b) What is the force of friction exerted on the car if the car's speed is 10 m/s?

8. Here is a 1200 kg car running on a vertical wall.

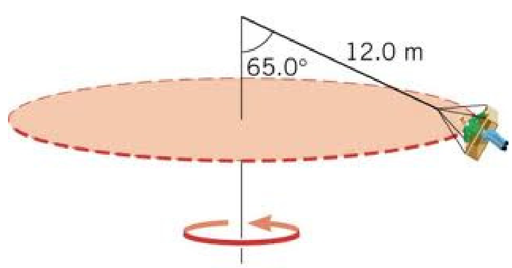
   The coefficient of static friction between the tires and the wall is 0.8, and the radius of the track is 5 m.

www.mazda.com.au/community/news/2012/4/mazda2-conquers-the-wall-of-death

   a) What is the minimum speed that the car must have so that it does not slip?

   b) What is the magnitude of the normal force exerted on the car if the car's speed is equal to the minimum speed?

9. The person in this ride has a mass of 60 kg.

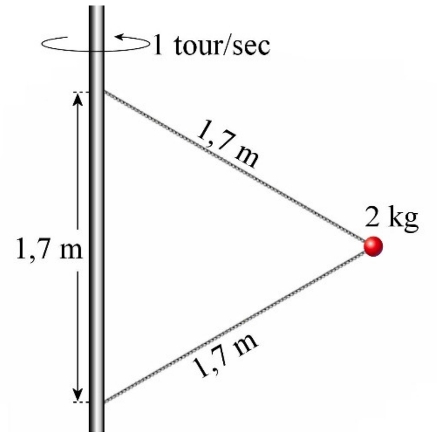


mail.rdcrd.ab.ca/~smolesky/Physics35/1) KinCirc/Day1.html

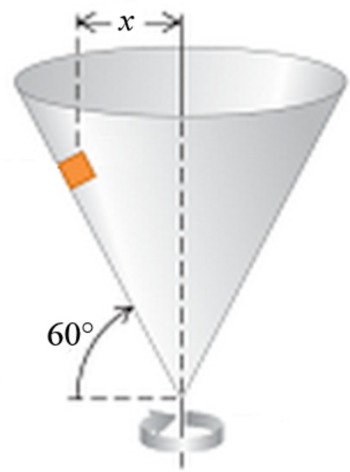
   a) What is the tension of the rope?

   b) How long does it take for this person to go around the ride once?

10. What are the tensions of the strings in the following system?



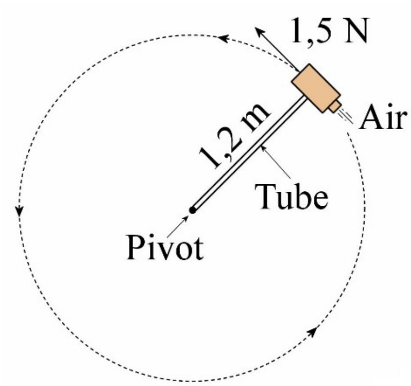
11. When this cone turns at the rate of 2 revolutions per second, the small block does not slide on the cone. If there is no friction between the block and the cone, what is the value of *x*?



www.chegg.com/homework-help/questions-and-answers/small-block-mass-m-placed-inside-mv-cried-cone-rotating-vertical-axis-time-one-revolution--q7837211

6.5 Non-Uniform Circular Motion

12. A small tube takes compressed air to a 3 kg block so that the compressed air propels the block and gives it a tangential acceleration. The block is placed on a table, and there is no friction between the block and the table. Here's this situation, viewed from the top. Initially, the block is at rest and the air stream exerted a 1.5 N force on the block. Neglect the mass of the tube.



a) What is the acceleration of the block 2 seconds after the start of the motion?

b) What is the tension in the tube 2 seconds after the start of the motion?

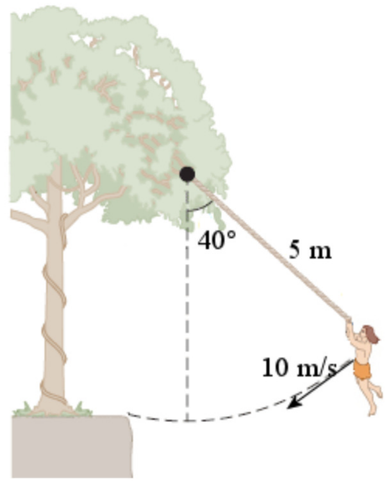
13. Here's Gontran, whose mass is 65 kg, playing Tarzan.

a) What is the magnitude of Gontran's centripetal acceleration?

b) What is the magnitude of Gontran's tangential acceleration?

c) What is the magnitude of Gontran's acceleration?

d) What is the tension of the rope?



www.chegg.com/homework-help/questions-and-answers/jane-mass-460-kg-needs-swing-river-having-width-d-filled-person-eating-crocodiles-save-tar-q2225989

6.6 Gravitation and Circular Orbits

Use the following data for these exercises.

Earth	Mass = 5.972×10^{24} kg Radius = 6371 km
Moon	Mass = 7.34×10^{22} kg Radius = 1737 km Distance between the Earth and the Moon = 384,400 km
Mars	Mass = 6.4185×10^{23} kg Radius = 3386 km

14. What is the gravitational field 1000 km above Earth's surface?

15. A 70 kg person is on the surface of Mars.

a) What is the gravitational field at the surface of Mars?

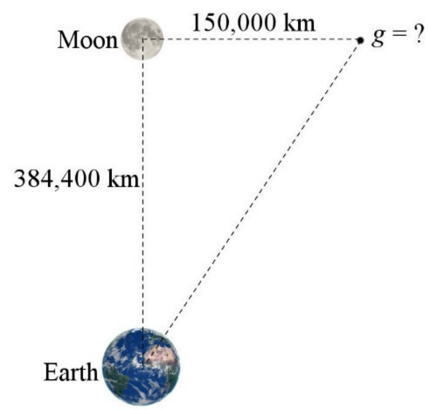
b) What would be the weight of the person on the surface of Mars?

c) This weight represents what percentage of the weight of the person on Earth?

To answer the next two questions, you should know that the field is the vector sum of the fields made by each planet. The gravitational field vector is always directed towards the planet that makes the field.

16. How far from the Earth does the gravitational field vanish between the Earth and the Moon?

17. What is the magnitude of the gravitational field at the position shown in the diagram?



18. Calculate the mass of the Earth knowing that the Moon revolves around it with a 27.32-day period on an orbit whose radius is 384 400 km.

19. Io orbits Jupiter with a 1.796-day period on an orbit whose radius is 421,700 km. Ganymede also orbits around Jupiter but on an orbit whose radius is 1,070,400 km.

a) What is the period of Ganymede?

b) What is the orbital speed of Ganymede?

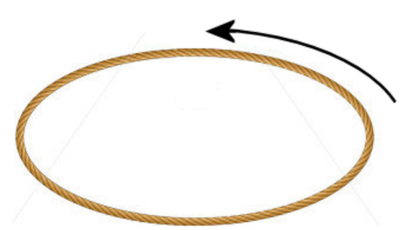
20. The Apollo capsule was orbiting the Moon at an altitude of 100 km above the surface. Knowing that the Moon has a mass of 7.35×10^{22} kg and a radius of 1737 km, how long did it take for the Apollo capsule to go around the Moon once?

21. How far from the surface of the Earth should a satellite be placed so that it goes around the Earth in 2 days? (Mass of the Earth = 5.97×10^{24} kg, radius of the Earth = 6378 km.)

Challenges

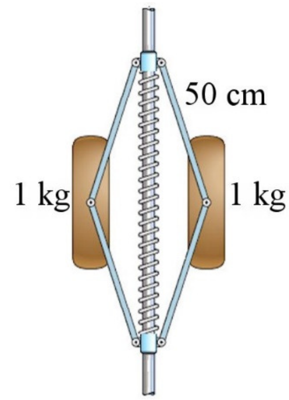
(Questions more difficult than the exam questions.)

22. A rope loop having a 50 cm radius and a 6 kg mass is spinning at a rate of 4 revolutions per second. What is the tension of the rope?



www.shutterstock.com/pic.mhtml?utm_medium=Affiliate&utm_campaign=Publicdomainvectors&tpl=241614-42119&utm_source=241614&irgwc=1&id=176570504

23. In the situation shown in the diagram, the spring constant is 2000 N/m and the length of the spring when it is neither stretched nor compressed is 80 cm. What is the length of the spring if this system rotates at 10 revolutions per second and if it is in a spaceship far away from all stars and planets (so there is no gravity)? (There is no friction.)



www.chegg.com/homework-help/mechanical-tachometer-measures-rotational-speed-n-shaft-hori-chapter-6-problem-159p-solution-9781118213551-exc

ANSWERS

6.3 Uniform Circular Motion

1. a) 7006 N b) 9633 N
2. 1.134
3. 0.8194 s
4. Lowest point: 38,600 N higher point: 19,000 N
5. 7 m/s
6. 16.89 m/s
7. a) 124,023 N downhill b) 4581 N uphill
8. a) 7.826 m/s b) 14,700 N
9. a) 1391 N b) 4.52 s
10. 47.51 N (lower string) and 86,71 N (upper string)
11. 10.75 cm

6.5 Non-Uniform Circular Motion

12. a) 0.9718 m/s² b) 2.5 N
13. a) 20 m/s² b) 6.299 m/s² c) 20.969 m/s² d) 1788 N

6.6 Gravitation and Circular Orbits

14. 7.336 N/kg
15. a) 3.736 N/kg b) 261.5 N c) 38.1%
16. 346,037 km from the Earth's centre
17. 2.429×10^{-3} N/kg
18. 6.03×10^{24} kg
19. a) 7.263 j b) 10.72 km/s
20. 1.962 h
21. 60,673 km

Challenges

22. 301.6 N
23. 40.263 cm