# 1 KINEMATICS

***Alphonse and Bertrand are involved in a car race. Initially, their cars are at rest at the starting line. Alphonse's car has an acceleration of 5 m/s² until it reaches a maximum velocity of 30 m/s. Bertrand's car has a 3 m/s² acceleration until it reaches a maximum velocity of 42 m/s. Where and when will Bertrand's car overtake Alphonse's car?***

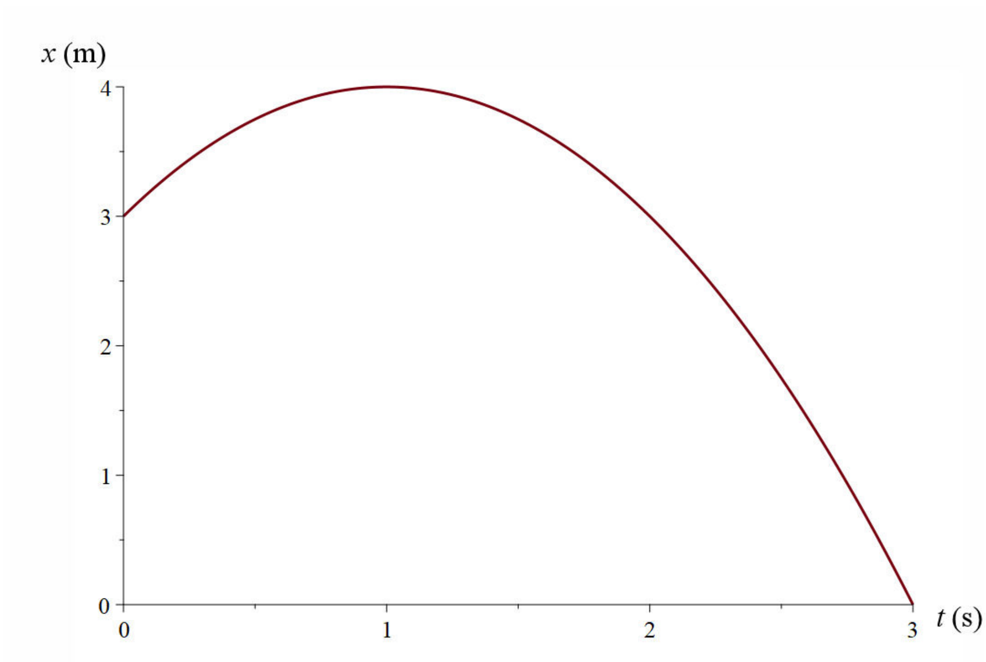

***www.dragracingonline.com/raceresults/2008/x_7-spectacular-3.html***

**Discover the answer to this question in this chapter.**

# 1.1 WHAT IS KINEMATICS?

Kinematics is a branch of physics that describes the motion of objects. To achieve this, the position as a function of time can be given with a formula such as $x = (3 + 2t - t^2)$ m, for example.

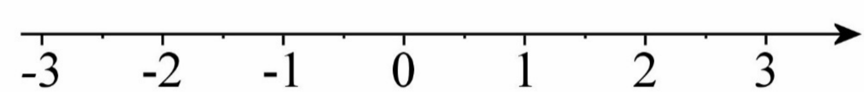
There are actually many ways to describe a motion. As an example, a formula for the velocity as a function of time or as a function of position can be given. A graph of the position or velocity as a function of time can be provided instead. The following graph shows the position as a function of time for the formula given above.



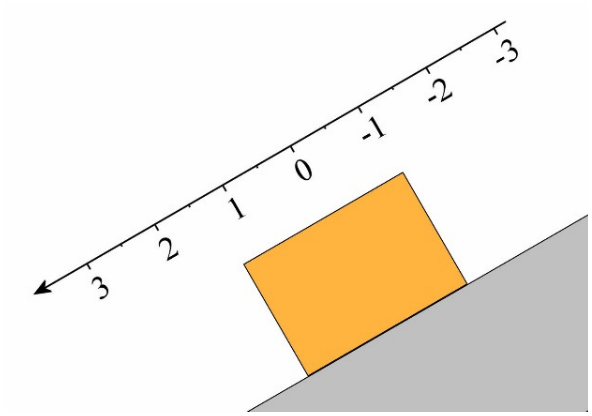

Kinematics is one of the oldest branches of physics since the equations used to describe the motion of an object with a constant acceleration are known since the $14^{th}$ century.

# 1.2 POSITION, DISTANCE AND DISPLACEMENT

In this chapter, one-dimensional motion (i.e., a motion of objects along a straight line) is examined. To give the position along a line, an axis is used.




The $x$ values increase towards the right but an $x$-axis with values increasing towards the left can also be used. The axis may have any orientation. For example, a vertical axis is used to describe a free-fall motion. Then, the position values may increase upwards or downwards depending on your choice of orientation. In this case, $y$ can be used to note the position, even though it would be quite correct to continue to use $x$.




The axis could also be tilted to describe the position of an object moving along an incline.

It simply remains to decide where the origin $x = 0$ is. Very often, the origin $x = 0$ is set at the initial position of the object.

The **displacement** of an object is simply the change in position of the object. If an object is initially at position $x_1$ and is later at position $x_2$, then the displacement is

**Displacement**

$$\Delta x = x_2 - x_1$$

The **distance** travelled by an object is the total length of the path travelled between two positions. Thus, if an object is thrown upwards to a height of 20 m and then returns to the ground, the distance travelled by the object is 40 m while the displacement is zero since the object returned to its starting position. To calculate the displacement, only the initial and final positions matter. Whatever happened between these two moments is irrelevant.

# 1.3 AVERAGE VELOCITY

## Definition of Average Velocity

Average velocity is defined by the displacement divided by the time elapsed during this displacement.

**Average Velocity**

$$\overline{v} = \frac{\Delta x}{\Delta t}$$

This must not be confused with the average speed, which is

$$\text{Average speed} = \frac{\text{distance}}{\Delta t}$$

## Example 1.3.1

An object moves on the $x$-axis. It first goes from $x = 0$ m to $x = 50$ m in 5 seconds and then goes from $x = 50$ m to $x = -10$ m in 15 seconds.

a) What is the displacement of the object?

The displacement is

$$\begin{aligned}\Delta x &= x_2 - x_1 \\ &= -10m - 0m \\ &= -10m\end{aligned}$$

b) What is the distance travelled by the object?

The object has travelled 50 m to the right and then 60 m to the left. The distance travelled is therefore 110 m.

c) What is the average velocity of the object?

The average velocity is

$$\begin{aligned}\overline{v} &= \frac{\Delta x}{\Delta t} \\ &= \frac{-10m}{20s} \\ &= -0.5 \tfrac{m}{s}\end{aligned}$$

d) What is the average speed of the object?

The average speed is

$$\begin{aligned}\text{average speed} &= \frac{\text{distance}}{\Delta t} \\ &= \frac{110m}{20s} \\ &= 5.5 \tfrac{m}{s}\end{aligned}$$

## Example 1.3.2

Conrad travels from Quebec to Montreal by car. He travels the 250 km distance by going at 100 km/h for the first 125 km, and then at 80 km/h for the last 125 km. What is his average velocity?

It is tempting to say 90 km/h, but this is not the right answer. Let's do this properly by calculating

$$\overline{v} = \frac{\Delta x}{\Delta t}$$

Thus, the total displacement and the total duration of the journey must be found.

It's easy to calculate the total displacement.

$$\begin{aligned}\Delta x &= 125km + 125km \\ &= 250km\end{aligned}$$

However, a bit of calculation is needed to obtain the time taken by Conrad to go from Quebec to Montreal. By taking the equation for the average velocity for the first part of the trip, the duration of this part is

$$\begin{aligned}\overline{v}_1 &= \frac{\Delta x_1}{\Delta t_1} \\ 100\tfrac{km}{h} &= \frac{125km}{\Delta t_1} \\ \Delta t_1 &= 1.25h\end{aligned}$$

Using the same equation, the second part lasts

$$\begin{aligned}\overline{v}_2 &= \frac{\Delta x_2}{\Delta t_2} \\ 80\tfrac{km}{h} &= \frac{125km}{\Delta t_2} \\ \Delta t_2 &= 1.5625h\end{aligned}$$

Therefore, the total duration of the trip is

$$\begin{aligned}\Delta t &= 1.25h + 1.5625h \\ &= 2.8125h\end{aligned}$$

Therefore, average velocity is

$$\begin{aligned}\overline{v} &= \frac{\Delta x}{\Delta t} \\ &= \frac{250km}{2.8125h} \\ &= 88.89\tfrac{km}{h}\end{aligned}$$

The previous two examples show that m/s and km/h can both be used as units for speed and velocity. In fact, several units are possible, provided that it is always a unit of distance divided by a unit of time. Thus, mm/day would be quite acceptable as a unit of velocity.

But how do you go from one unit to another? Here's the conversion technic to change km/h in m/s.

$$45\tfrac{km}{h} = 45\frac{\cancel{km}}{\cancel{h}} \times \frac{1000m}{1\cancel{km}} \times \frac{1\cancel{h}}{3600s} = 12.5\tfrac{m}{s}$$

The first multiplication changes the *km* in *m*. This multiplication is actually 1 since 1000 m and 1 km are the same distances. The *km* are at the bottom of the division to eliminate the *km* from the starting unit. The second multiplication changes the *h* in *s*. Again, this multiplication is 1 since 1 h and 3600 s are the same times. Now, the *h* are at the top of the fraction to eliminate the *h* from the starting unit. Note that, with the units cancelling each other, only *m/s* remains.

## Example 1.3.3

The position of an object is given by the formula $x = 3m + 2\tfrac{m}{s}\cdot t - 1\tfrac{m}{s^2}\cdot t^2$. What is the average velocity between $t = 0$ s and $t = 1$ s?

The average velocity is

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

The positions at $t = 0$ s and $t = 1$ s must be found to calculate $\Delta x$.

At $t = 0$ s, the position is

$$\begin{aligned} x_1 &= 3m + 2\tfrac{m}{s}\cdot 0s - 1\tfrac{m}{s^2}\cdot (0s)^2 \\ &= 3m \end{aligned}$$

At $t = 1$ s, the position is

$$\begin{aligned} x_2 &= 3m + 2\tfrac{m}{s}\cdot 1s - 1\tfrac{m}{s^2}\cdot (1s)^2 \\ &= 4m \end{aligned}$$

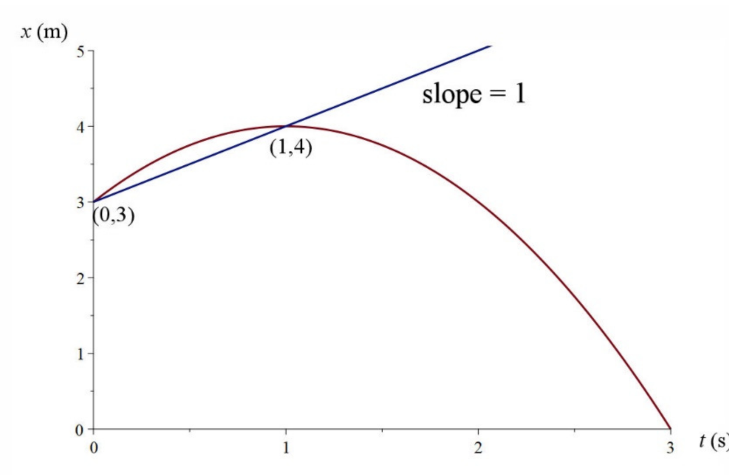
Therefore, average velocity is

$$\begin{aligned} \bar{v} &= \frac{\Delta x}{\Delta t} \\ &= \frac{x_2 - x_1}{\Delta t} \\ &= \frac{4m - 3m}{1s} \\ &= 1\tfrac{m}{s} \end{aligned}$$

## Average Velocity on the Graph of Position

On a position-versus-time graph, the average velocity

$$\overline{v} = \frac{\Delta x}{\Delta t}$$




is the slope of the line connecting the points corresponding to the times between which the average velocity is sought. Let's use the data from the last example to illustrate this.

As the average velocity between $t = 0$ s and $t = 1$ s is sought, the points for the position at these times must be used on the graph. Those points are (0,3) and (1,4). The average velocity is the slope of the straight line connecting these two points.

# 1.4 INSTANTANEOUS VELOCITY

## Definition of Instantaneous Velocity

Throughout a motion, the velocity may vary. In a previous example, the average velocity was 88.89 km/h but the velocity changed during the journey. It is also possible to go to Montreal by having a continuously varying velocity, e.g. an always-increasing velocity. But then, how can the velocity at a particular instant of time be known?

To achieve this, the velocity is calculated in the same way as the average velocity is calculated but by considering the displacement during a very short time, so that the velocity does not have any time to change. But what time duration is sufficiently small? Is a second short enough? For a car, this may seem correct at first. But the velocity of a car may change very quickly during an intense braking and even faster in an accident. Therefore, an even shorter time should be used. Perhaps a billionth of a second will do? For a car, it's probably good enough, but it's probably too long for particles, like electrons, whose velocity can change very quickly. How short is short enough?

In fact, there is no need to search for this very small time that could adapt to any situation since this very short time already exists in mathematics: it is an infinitesimally small time period. Using this value, instantaneous velocity is

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

which is

**Instantaneous Velocity**

$$v = \frac{dx}{dt}$$

For simplicity, we always mean "instantaneous velocity" when we talk about velocity.

## Example 1.4.1

The position of an object is given by the formula $x = 3m + 2\frac{m}{s} \cdot t - 1\frac{m}{s^2} \cdot t^2$. What is the velocity at $t = 2$ s?

Since velocity is the derivative of position, the velocity is

$$\begin{aligned} v &= \frac{dx}{dt} \\ &= \frac{d\left(3m + 2\frac{m}{s} \cdot t - 1\frac{m}{s^2} \cdot t^2\right)}{dt} \\ &= 2\frac{m}{s} - 2\frac{m}{s^2} \cdot t \end{aligned}$$

Therefore, the velocity at $t = 2$ s is

$$\begin{aligned} v &= 2\frac{m}{s} - 2\frac{m}{s^2} \cdot 2s \\ &= -2\frac{m}{s} \end{aligned}$$

## Speed

Speed is the magnitude of the velocity. In one dimension, this means that speed is simply the absolute value of velocity.

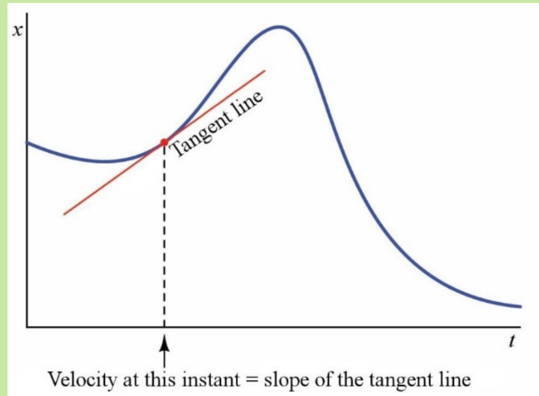
$$speed = |v|$$

## Instantaneous Velocity on the Graph of Position

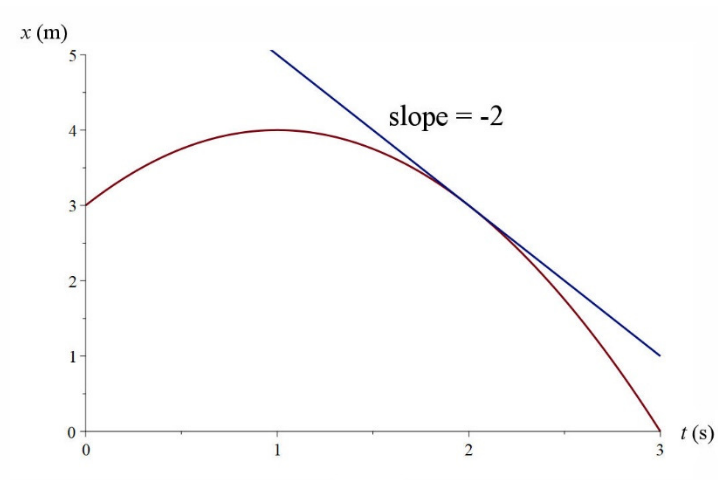
Instantaneous velocity is the derivative of position. As the derivative is the slope, we have

On a graph of the **position** of an object as a function of time, the slope is the velocity of the object.

control.com/textbook/calculus/how-derivatives-and-integrals-relate-to-one-another/




Here is the graphical representation of the previous example. The velocity is the slope of the tangent line at this instant.



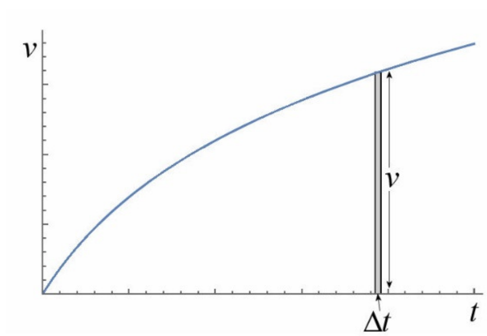

## Displacement on the Graph of Velocity

The displacement of an object can be found from the velocity of an object. Since

$$v = \frac{\Delta x}{\Delta t}$$

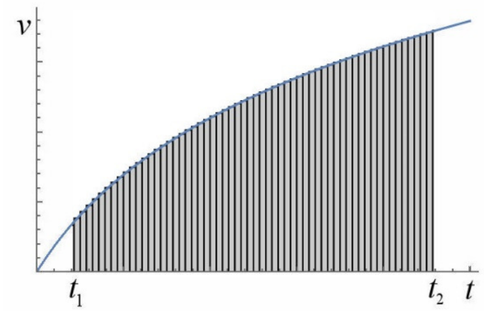
where $\Delta t$ is very small, the displacement during a small time $\Delta t$ can be calculated. This displacement is

$$\Delta x = v\Delta t$$



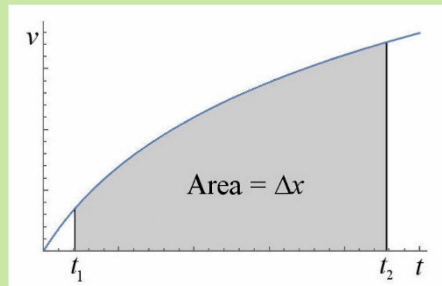

On a graph, $v\Delta t$ is the area of a very thin rectangle, as on the graph to the right.

The area of this small rectangle is only the displacement during a small time interval $\Delta t$. If the total displacement between time $t_1$ and $t_2$ time is sought, then all the small displacement, so all the areas of the small rectangles between these two times, must be added.

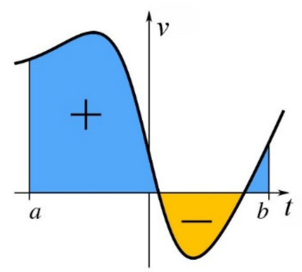



The sum of the areas of all these rectangles gives the area under the curve. (The area doesn't look exactly the same since there are small bits of rectangles that protrude or are missing, but these little parts are not actually there since the rectangles are very very thin.)

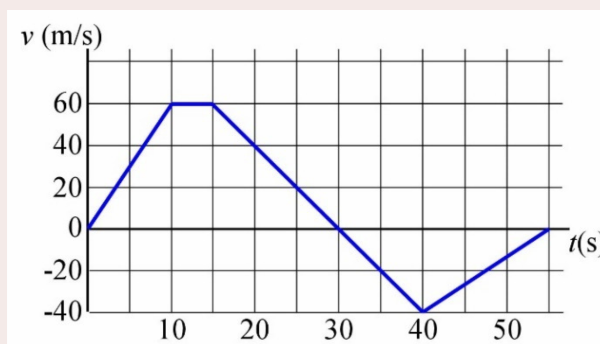
On a graph of the **velocity** of an object as a function of time, the area under the curve is equal to the displacement.




Caution: when the area is under the time axis, the area is negative.

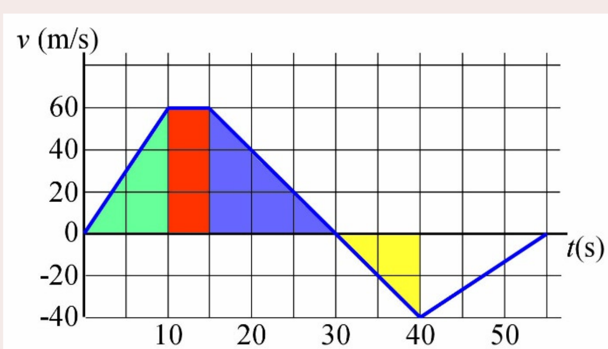



## Example 1.4.1




Here is the velocity-versus-time graph of an object. What is the displacement between $t = 0$ s and $t = 40$ s?

To find the displacement, the area under the curve is calculated between $t = 0$ s and $t = 40$ s.




To achieve this, the area is divided into several parts.

First, there is a triangle (in green). The area is 300 m.

Then, there is a rectangle (in red). The area is 300 m.

Then there is a triangle (in blue). The area is 450 m.

Finally, there is another triangle (in yellow). The area is 200 m.

(We stop there because we were asked to find the displacement between 0 s and **40 s.**)

The total displacement is then 300 m + 300 m + 450 m – 200 m = 850 m.

The last value is negative because the area is under the time axis. It makes sense to have then a negative displacement since the velocity is negative during that time.

Note that if all the areas are taken as being positive even when they are under the time axis, the distance travelled by the object is obtained.

# 1.5 MOTION WITH CONSTANT VELOCITY

## Equation of Motion

Let's consider what happens at a constant velocity. As it is known that

$$v = \frac{dx}{dt}$$

the position as a function of time can be found by wondering what should have been derived to obtain the constant $v$. Obviously, it is

$$x = vt + cst$$

The value of the constant can be found by setting that the position at $t = 0$ s is the initial position (written as $x_0$). Using this information, the equation becomes

$$x_0 = v \cdot 0 + cst$$

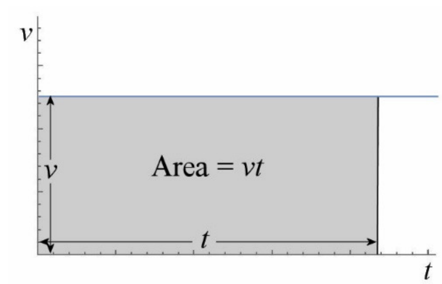
which means that $cst = x_0$ . The position is then

**Equation of Motion at Constant Velocity**

$$x = x_0 + vt$$

This equation can also be obtained with a graph. For a constant-speed motion, the velocity graph is a horizontal line.

The displacement between $t = 0$ and time $t$ is given by the area shown on the figure on the right.



As the area is the area of a rectangle, it is

$$\Delta x = vt$$
$$x - x_0 = vt$$
$$x = x_0 + vt$$

This is the same equation as the one obtained previously.

## Example 1.5.1

How long does it take to travel 200 km at a steady velocity of 80 km/h?

Using $x_0$ = 0 km and $x$ = 200 km, the time duration is found with

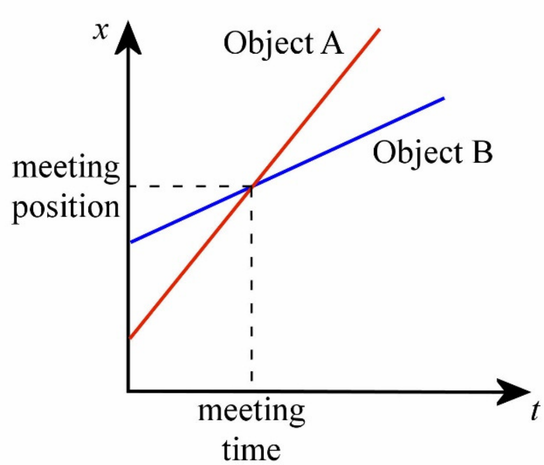
$$x = x_0 + vt$$
$$200km = 0km + 80\tfrac{km}{h} \cdot t$$
$$t = \frac{200km}{80\frac{km}{h}}$$
$$t = 2.5h$$

Note that in this formula $x_0$ is always the position at $t$ = 0 and $x$ is always the position at time $t$.

## When Will Two Objects Be at the Same Place?

Sometimes, the moment when two objects are at the same position is sought. This could be because we want to find when two objects collide or when an object catches up with another object. The trick to solve this kind of problem is very simple: when an object catches up with another or when there is a collision, the two objects are at the same place, which means that the equation $x_A = x_B$ must be solved.

Note that graphically, this amounts to find the point of intersection of the straight lines of the position as a function of time of the two objects.




Let's see what this means for objects having a constant velocity. The two objects are at a distance $L$ from each other and move at different velocities. They will be at the same place when $x_1 = x_2$.


fr.depositphotos.com/2577683/stock-illustration-Car.html

Object 1 position's is

$$x_1 = 0 + v_1 t$$

(The origin $x = 0$ was set at the initial position of object 1.) Object 2 position's (which is always the one what the largest value of $x$) is

$$x_2 = L + v_2 t$$

When the two objects meet, they are at the same location. This means that

$$x_1 = x_2$$
$$v_1 t = L + v_2 t$$
$$(v_1 - v_2)t = L$$

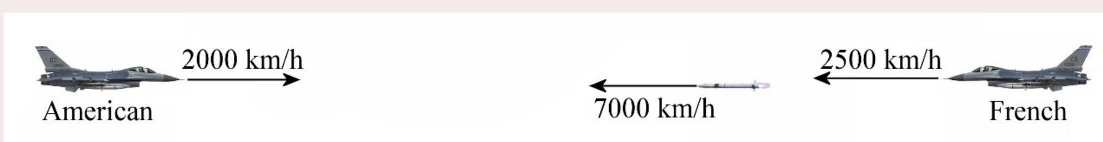
which gives

**Moment When Two Objects, Initially at a Distance *L* from Each Other, Are at the Same Place if they Move at Constant Velocities.**

$$t = \frac{L}{v_1 - v_2}$$

## Example 1.5.2

A French plane and an American plane are heading towards each other. The speed of the French aircraft is 2500 km/h and the speed of the American aircraft is 2000 km/h. When they are 6000 m apart, the French plane sends a missile towards the American plane.




How long will it take for the missile to arrive at the American plane if the speed of the missile is 7000 km/h?

The American aircraft is object 1 and as the missile is object 2. Thus

$$v_1 = 2000 \tfrac{km}{h}$$
$$v_2 = -7000 \tfrac{km}{h}$$

In metres per second, these velocities are

$$v_1 = 555.6 \tfrac{m}{s}$$
$$v_2 = -1944.4 \tfrac{m}{s}$$

Therefore, the time is

$$\begin{aligned} t &= \frac{L}{v_1 - v_2} \\ &= \frac{6000m}{555.6 \tfrac{m}{s} - -1944.4 \tfrac{m}{s}} \\ &= 2.4s \end{aligned}$$

## What if the Velocity Changes?

Even if the velocity of an object changes, it is possible to solve the problem with the equations of motion at constant velocity if the velocity changes by step. This means that the velocity is constant for a while and then suddenly changes to another constant value. There can be as many changes as one wants.

In this case, the problem must be divided into parts. The first part is the motion with the constant velocity there is at the outset of the motion, the second part is the one with the second constant velocity and so on… The values of the position at the end of the first part then become the initial position for the second part.

## Example 1.5.3

A car is going at 35 m/s for 200 seconds, and then at 20 m/s for 100 seconds. What is the displacement of the car during these 300 seconds?

As the velocity changes, the problem must be divided into two parts with a constant velocity. In the first part, the car is travelling at 35 m/s for 200 s. Let the starting position be the origin of the axis so that $x_0$ = 0 m. The position at the end of this first part is

$$\begin{aligned} x &= x_0 + vt \\ &= 0m + 35 \tfrac{m}{s} \cdot 200s \\ &= 7000m \end{aligned}$$

In the second part, the car moves at 20 m/s for 100 seconds and the initial position is the final position of the first part ($x_0$ = 7000 m). The position at the end of this phase is

$$\begin{aligned} x &= x_0 + vt \\ &= 7000m + 20\tfrac{m}{s} \cdot 100s \\ &= 9000m \end{aligned}$$

The displacement of the car is then

$$\begin{aligned} \Delta x &= 9000m - 0m \\ &= 9000m \end{aligned}$$

# 1.6 ACCELERATION

## Average Acceleration

Acceleration indicates if the velocity of an object changes. It specifies by how much the velocity changes during each unit of time. Average acceleration is defined by

**Average Acceleration**

$$\overline{a} = \frac{\Delta v}{\Delta t}$$

The SI unit for acceleration is $\frac{m/s}{s}$, which can be written as m/s².

An average acceleration of 3 m/s² means that, on average, the velocity increases by 3 m/s each second.

## Example 1.6.1

Here's the velocity of an object at two instants.

At $t$ = 0 s, $v$ = 0 m/s.
At $t$ = 2 s, $v$ = 10 m/s towards the right.

What is its average acceleration between these two instants?

The average acceleration is

$$\begin{aligned} \overline{a} &= \frac{\Delta v}{\Delta t} \\ &= \frac{10\frac{m}{s} - 0\frac{m}{s}}{2s} \\ &= 5\tfrac{m}{s^2} \end{aligned}$$

The following example shows that we need to pay attention to the sign of the velocity when calculating the average acceleration.

## Example 1.6.2

Here is the velocity of an object at two instants.

At $t$ = 0 s, $v$ = 10 m/s towards the left.
At $t$ = 20 s, $v$ = 50 m/s towards the left.

What is its average acceleration between these two instants?

The average acceleration is

$$\begin{aligned}\overline{a} &= \frac{\Delta v}{\Delta t} \\ &= \frac{-50\frac{m}{s} - -10\frac{m}{s}}{20s} \\ &= -2\frac{m}{s^2}\end{aligned}$$

In this last example, the velocities are negative since the object is moving towards the left. (This implies that an axis directed towards the right was used.) In fact, you can decide, for every problem, the direction of the axis (the positive direction). If the velocity is in this direction, it is positive, and if it is in the opposite direction, it is negative.

This last example also shows that a negative acceleration does not necessarily mean that the speed of the object decreases (remember that the speed is the magnitude of the velocity). In the last example, the acceleration was negative and yet the speed changed from 10 m/s to 50 m/s. The following rule is the correct one:

If the velocity and the acceleration have identical signs, then the speed increases.

If the velocity and the acceleration have opposite signs, then the speed decreases.

To understand this rule, do not forget that acceleration is the velocity that is added at every unit of time. Thus, if an object has a velocity of -100 m/s and an acceleration of 5 m/s², then 5 m/s is added to the velocity every second. The velocity at different times is therefore

$t$ = 0 s, $v$ = -100 m/s
$t$ = 1 s, $v$ = -95 m/s
$t$ = 2 s, $v$ = -90 m/s
$t$ = 3 s, $v$ = -85 m/s
$t$ = 4 s, $v$ = -80 m/s

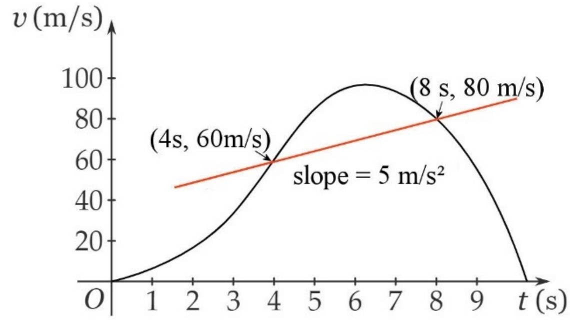
and so on. It is obvious that the speed decreases. According to the rules given above, this is what happens when the velocity and the acceleration have opposite signs.

## Average Acceleration on the Graph of Velocity

On a graph of velocity as a function of time, the average acceleration

$$\overline{a} = \frac{\Delta v}{\Delta t}$$

represents the slope of the line that connects the points that correspond to the times between which we want to know the average acceleration. For example, if the velocity changes as shown on this graph, then the average acceleration between 4 s and 8 s is 5 m/s².




www.phyley.com/average-acceleration

## Instantaneous Acceleration

Remember that average acceleration is an average value. Sometimes the velocity could increase quickly and sometimes it could increase more slowly. This is the case with a car when it starts moving from rest: its velocity increases rapidly at first and more slowly thereafter. An instantaneous acceleration can then be defined to provide information on the rate of change of velocity at a specific instant of time. The trick is the same as the one used for the instantaneous velocity: this acceleration must be calculated using the change of velocity during a very short time so that it has no time to change. Instantaneous acceleration is then

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

which is

**Instantaneous Acceleration**

$$a = \frac{dv}{dt}$$

From now on, we mean "instantaneous acceleration" when we simply talk about acceleration.

## Example 1.6.3

The position of an object is given by the formula $x = 3m - 1\frac{m}{s} \cdot t + 2\frac{m}{s^2} \cdot t^2 + 1\frac{m}{s^3} \cdot t^3$. What is its acceleration at $t = 2$ s?

Acceleration is the derivative of velocity. Let's start by finding the velocity formula, which is the derivative of the formula for the position.

$$\begin{aligned} v &= \frac{dx}{dt} \\ &= \frac{d\left(3m - 1\frac{m}{s}\cdot t + 2\frac{m}{s^2}\cdot t^2 + 1\frac{m}{s^3}\cdot t^3\right)}{dt} \\ &= -1\frac{m}{s} + 4\frac{m}{s^2}\cdot t + 3\frac{m}{s^3}\cdot t^2 \end{aligned}$$

The acceleration can then be found by deriving this velocity.

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{d\left(-1\frac{m}{s} + 4\frac{m}{s^2}\cdot t + 3\frac{m}{s^3}\cdot t^2\right)}{dt} \\ &= 4\frac{m}{s^2} + 6\frac{m}{s^3}\cdot t \end{aligned}$$

At $t = 2$ s, we have

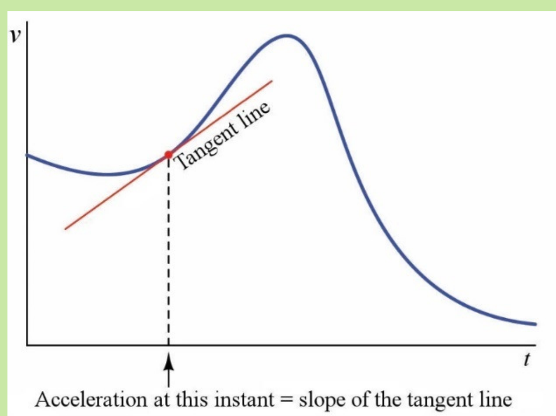
$$\begin{aligned} a &= 4\frac{m}{s^2} + 6\frac{m}{s^3}\cdot 2s \\ &= 16\frac{m}{s^2} \end{aligned}$$

## Instantaneous Acceleration on a Graph of the Velocity

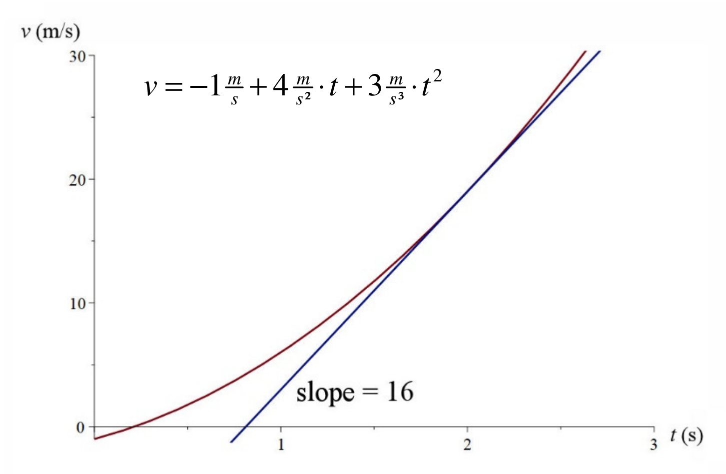
Since acceleration is the derivative of velocity, the acceleration is the slope on a velocity-versus-time graph.

On a graph of the **velocity** of an object as a function of time, the slope is the acceleration of the object.

control.com/textbook/calculus/how-derivatives-and-integrals-relate-to-one-another/

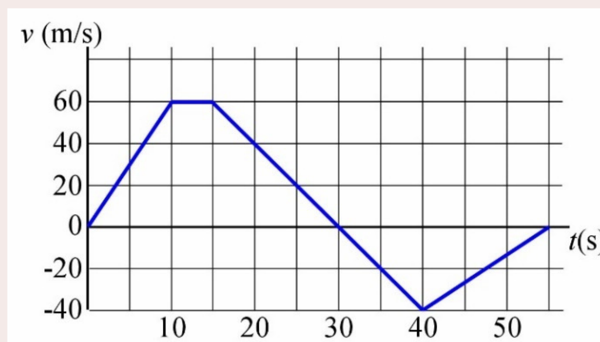


Using the data from the previous example, we see that the slope of the tangent line at $t$ = 2 s is 16 m/s².




## Example 1.6.4

Here is the velocity-versus-time graph of an object. What is the acceleration of the object at $t$ = 20 s?




To find the acceleration, we must calculate the slope at $t$ = 20 s. We will take the points (15 s, 60 m/s) and (30 s, 0 m/s) (2 points taken at random on the slope) to calculate the slope.

$$a = \frac{0\frac{m}{s} - 60\frac{m}{s}}{30s - 15s}$$
$$= -4\frac{m}{s^2}$$

## Variation of Velocity on the Graph of Acceleration

The change of velocity of an object can be found from the acceleration of an object. Since

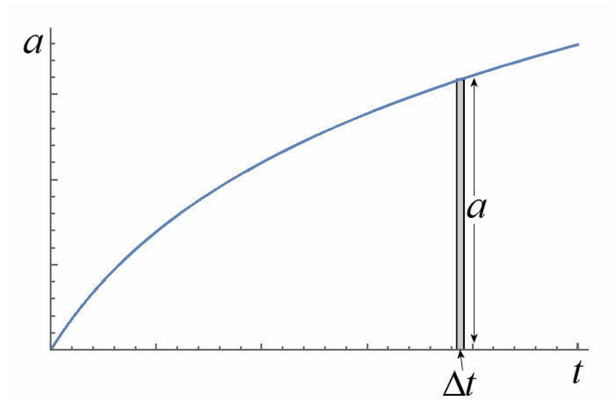
$$v = \frac{\Delta x}{\Delta t}$$

where $\Delta t$ is very small, the change of velocity during a small time $\Delta t$ can be calculated. This change of velocity is

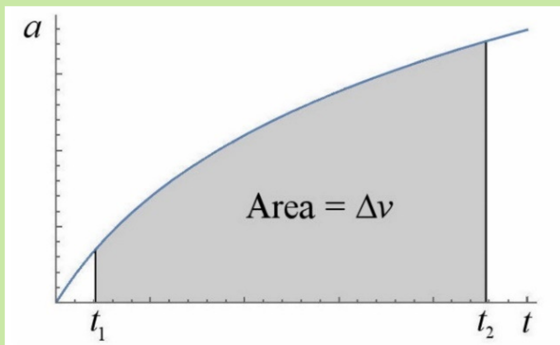
$$\Delta v = a\Delta t$$

On a graph, $a\Delta t$ is the area of a very thin rectangle, as on the graph to the right.

The area of this small rectangle is only the change of velocity during a small time interval $\Delta t$. If the total change of velocity between time $t_1$ and $t_2$ time is sought, then all the small displacement, so all the areas of the small rectangles between these two times, must be added. The sum of the areas of all these rectangles gives the area under the curve.




On a graph of the **velocity** of an object as a function of time, the area under the curve is equal to the displacement.

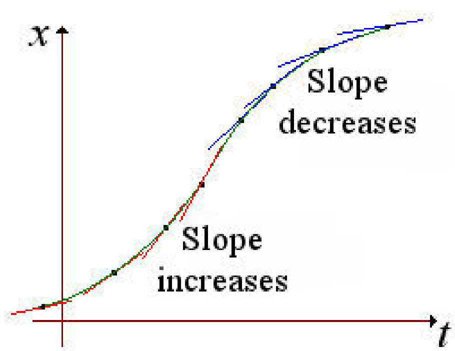



## Acceleration on the Graph of Position

On a graph of the position as a function of time, the sign of acceleration can be easily known. To obtain the sign, just look at whether the slope decreases or increases with time.

When the slope increases, the velocity increases. The acceleration is then positive.

When the slope decreases, the velocity decreases. The acceleration is then negative.

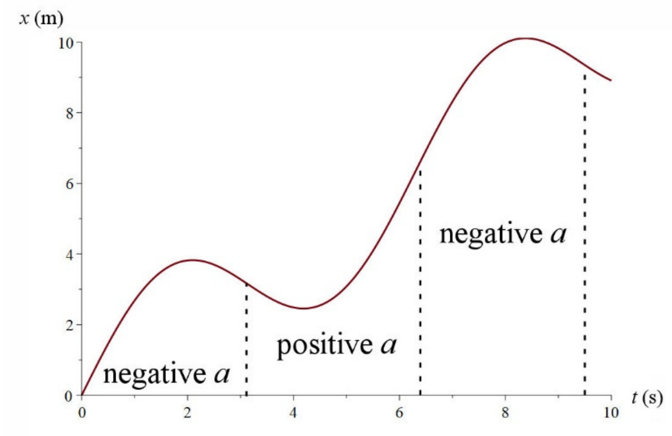



serge.mehl.free.fr/anx/Inflexion.html

It is also possible to get an idea of the acceleration of an object on a position-versus-time graph. As acceleration is the derivative of velocity and the latter is the derivative of position, acceleration is the second derivative of position.

$$a = \frac{d^2x}{dt^2}$$

Since this second derivative represents the concavity on a graph, the concavity on a position-versus-time graph is the acceleration of the object. Obviously, it is harder to get an idea of the exact value of the acceleration with concavity, but the sign can easily be determined as the graph on the right shows.




## Variation of Acceleration

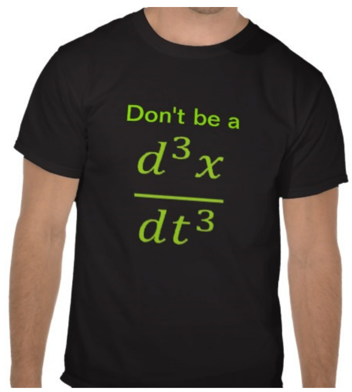
It is also possible to have a changing acceleration. The change in acceleration can then be measured with the average rate of change of acceleration. This average rate of change is the "average jerk".

$$\bar{j} = \frac{a_2 - a_1}{\Delta t} = \frac{\Delta a}{\Delta t}$$

And the instantaneous rate of change is the "instantaneous jerk".

$$j = \frac{da}{dt}$$

You can now understand jokes like the one on this T-shirt, a joke very few people get ☺.




www.zazzle.ca/plaisanterie_de_calcul_physique_t_shirts-235003554617890293?lang=fr

It was also suggested that the rate of change of the jerk is the snap, that the rate of change of the snap is the crackle and that the rate of change of the crackle is the pop!

# 1.7 MOTION WITH CONSTANT ACCELERATION

## The Equations of Motion

Let's consider what happens with a constant acceleration. Since

$$a = \frac{dv}{dt}$$

the velocity as a function of time can be found by wondering what must be derived to get the constant $a$. Obviously, it is

$$v = at + cst$$

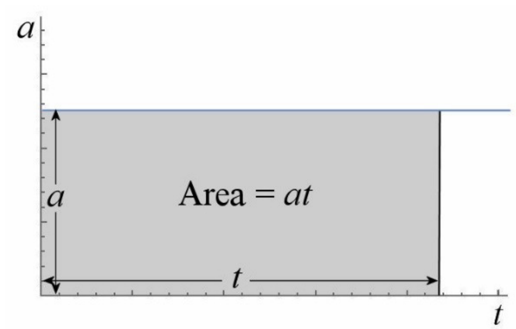
The value of the constant can be found by setting that the velocity at $t$ = 0 s is the initial velocity (written as $v_0$). Using this information, the equation is now

$$v_0 = a \cdot 0 + cst$$

This means that $cst = v_0$ and that the velocity is then

$$v = v_0 + at$$

This equation can also be obtained with a graph. For a constant-acceleration motion, the graph of the acceleration is a horizontal line.




As the area is the area of a rectangle, we have $\Delta v = at$. With this equation, we obtain

$$v - v_0 = at$$

This is the same equation as the one obtained previously.

The position as a function of time can also be found with the definition of velocity.

$$v = \frac{dx}{dt}$$

$$v_0 + at = \frac{dx}{dt}$$

The position can be found by searching what must be derived to get $v_0 + at$. Obviously, it is

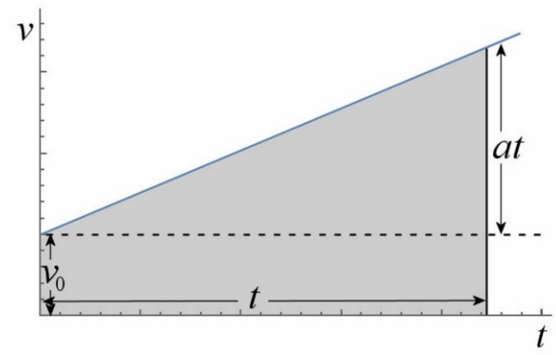
$$x = v_0 t + \frac{1}{2} a t^2 + cst$$

Again, the value of the constant can be found by setting that the position at $t$ = 0 s is the initial position $x_0$. The position equation then becomes

$$x_0 = v_0 \cdot 0 + \frac{1}{2} \cdot a \cdot 0^2 + cst$$

The constant is therefore $x_0$. This means that the position equation is

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$




The position as a function of time can also be found with the graph of the velocity. As the acceleration is constant, the graph of the velocity is a graph with a constant slope.

The increase in height on the graph is equal to *at* since this variation is equal to $\Delta v$ and we know that $\Delta v = at$.

The displacement is equal to the grey area on the graph. This area has been divided into two parts: a rectangle and a triangle. The area is therefore equal to

$$\begin{aligned}\Delta x &= \text{area of the rectangle} + \text{area of the triangle}\\ &= v_0 t + \frac{1}{2} t \cdot at \\ &= v_0 t + \frac{1}{2} a t^2\end{aligned}$$

This is the same equation as the one obtained previously.

With the position and the velocity as a function of time, any problem can be solved. But two other equations that are often very useful can also be found. They are not essential but they will allow us to solve problems faster. The first one is an equation linking the velocity and the position, without resorting to time. Solving for $t$ in the velocity equation

$$v = v_0 + at$$

we obtain

$$t = \frac{v - v_0}{a}$$

Substituting then into the position equation

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

gives

$$x = x_0 + v_0 \left( \frac{v - v_0}{a} \right) + \frac{1}{2} a \left( \frac{v - v_0}{a} \right)^2$$

With a little algebra, this equation becomes

$$x = x_0 + \frac{v_0 v - v_0^2}{a} + \frac{v^2 - 2 v_0 v + v_0^2}{2a}$$

$$x - x_0 = \frac{v_0 v - v_0^2}{a} + \frac{v^2 - 2v_0 v + v_0^2}{2a}$$

$$2a\left(x - x_0\right) = 2v_0 v - 2v_0^2 + v^2 - 2v_0 v + v_0^2$$

$$2a\left(x - x_0\right) = v^2 - v_0^2$$

The other equation gives the position as a function of time without resorting to the acceleration. Solving for $a$ in the velocity equation

$$v = v_0 + at$$

we obtain

$$a = \frac{v - v_0}{t}$$

Substituting then in the position equation

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

gives

$$x = x_0 + v_0 t + \frac{1}{2}\left(\frac{v - v_0}{t}\right) t^2$$

Simplified, this equation is

$$x = x_0 + \frac{1}{2}\left(v_0 + v\right) t$$

Problem-solving of motion with constant acceleration can then be done with these four beautiful equations.

**Equations of Motion with Constant Acceleration (also Called Uniformly Accelerated Motion)**

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$2a\left(x - x_0\right) = v^2 - v_0^2$$

$$x = x_0 + \frac{1}{2}\left(v_0 + v\right) t$$

Warning: remember that these four equations are only valid for a motion with **constant acceleration**. Do not use these equations if the acceleration changes…

In most cases, problem-solving with constant acceleration is straightforward because one of the four equations directly allows you to find the answer you are looking for. Simply look at the four equations one after the other, and identify the unknowns in each equation. Most of the time, there will be an equation for which the only unknown is the answer you're looking for.

### Common Mistake: Wrong Sign for $v$ or $a$

Be sure to define clearly a positive direction. If a vector (velocity or acceleration) is in the same direction as your positive direction, it is positive. Conversely, if a vector (velocity or acceleration) is in the opposite direction to your positive direction, it is negative.

## Example 1.7.1

A plane goes from 0 m/s to 100 m/s in 8 seconds with a constant acceleration.

a) What is the acceleration of the plane?

This first equation gives us

$$\begin{aligned} v &= v_0 + at \\ 100\tfrac{m}{s} &= 0\tfrac{m}{s} + a \cdot 8s \\ a &= 12.5\tfrac{m}{s^2} \end{aligned}$$

b) How far has travelled the plane in 8 seconds?

The second equation gives us

$$\begin{aligned} x &= x_0 + v_0 t + \frac{1}{2}at^2 \\ &= 0 + 0 \cdot 8s + \frac{1}{2} \cdot 12.5\tfrac{m}{s^2} \cdot (8s)^2 \\ &= 400m \end{aligned}$$

Note that the third or the fourth equations could have been used to solve this part of the problem since the only unknown in each of these three equations is the position.

Note that is this formula $x_0$ and $v_0$ are always the position and the velocity at $t = 0$ while $x$ and $v$ are always the position and the velocity at time $t$.

## Example 1.7.2

A car has a speed of 50 m/s and a constant acceleration. 100 m farther away, the speed of the car is 60 m/s. How much time did it take for the car to travel these 100 m?

The time is found with

$$x = x_0 + \frac{1}{2}(v_0 + v)t$$
$$100m = 0m + \frac{1}{2}\cdot\left(50\tfrac{m}{s} + 60\tfrac{m}{s}\right)\cdot t$$
$$t = 1.818s$$

## Example 1.7.3

A Boeing 747 must have a speed of 278 km/h to take off. Knowing that the Boeing's acceleration is 2.8 m/s², how long does the runway have to be for the Boeing to take off?

The distance travelled on the track is

$$2a(x - x_0) = v^2 - v_0^2$$
$$2\cdot 2.8\tfrac{m}{s^2}\cdot(x - 0m) = \left(77.22\tfrac{m}{s}\right)^2 - \left(0\tfrac{m}{s}\right)^2$$
$$x = 1064.9m$$

There's no need to focus more on these rather simple cases. Let's look at some more complex problems which are not directly solved by one equation.

## Example 1.7.4

A car passes by a telephone pole with a velocity of 90 km/h and an acceleration of 3 m/s². How far from the pole was it 2 seconds earlier?

<u>1st Solution</u>

The velocity at $t$ = 2 s (25 m/s) and the acceleration are known. Let's put the axis origin $x$ = 0 at the pole and try to calculate $x_0$. The unknowns in the four equations are

| | |
|---|---|
| $v = v_0 + at$ | Unknown: $v_0$ |
| $x = x_0 + v_0 t + \frac{1}{2}at^2$ | Unknowns: $v_0$ and $x_0$ |
| $2a(x - x_0) = v^2 - v_0^2$ | Unknowns: $v_0$ and $x_0$ |
| $x = x_0 + \frac{1}{2}(v + v_0)t$ | Unknowns: $v_0$ and $x_0$ |

It's clear that $x_0$ is never the only unknown in an equation. On the other hand, a solving strategy to this problem can easily be found. $v_0$ must be found first with the first equation and then $x_0$ can be found with any of the other three equations since the only remaining unknown in theses equation will be $x_0$.

According to the first equation, $v_0$ is

$$\begin{aligned} v &= v_0 + at \\ 25\tfrac{m}{s} &= v_0 + 3\tfrac{m}{s^2}\cdot 2s \\ v_0 &= 19\tfrac{m}{s} \end{aligned}$$

The initial position can then be found with the second equation (the third or the fourth equation could also have been used).

$$\begin{aligned} x &= x_0 + v_0 t + \frac{1}{2}at^2 \\ 0 &= x_0 + 19\tfrac{m}{s}\cdot 2s + \frac{1}{2}\cdot 3\tfrac{m}{s^2}\cdot (2s)^2 \\ x_0 &= -44m \end{aligned}$$

The car was therefore 44 m from the pole 2 seconds earlier.

2nd Solution

This problem can also be solved with a single equation if the time $t = 0$ s is set at the moment the car is by the pole. The problem can then be solved by calculating the position of the car at $t = -2$ s. Still using the $x = 0$ at the pole, this problem is solved directly with the second equation.

$$\begin{aligned} x &= x_0 + v_0 t + \frac{1}{2}at^2 \\ &= 0 + 25\tfrac{m}{s}\cdot(-2s) + \frac{1}{2}\cdot 3\tfrac{m}{s^2}\cdot(-2s)^2 \\ &= -44m \end{aligned}$$

This last solution shows that $x_0$ is not specifically the initial position. Properly speaking, it is the position at $t = 0$ s. As it is possible to have a motion before $t = 0$ s (which is arbitrary), $x_0$ is not always the initial position. Similarly, $v_0$ is not necessarily the initial velocity. It's simply the velocity at $t = 0$ s. Obviously, if the motion begins at $t = 0$ s, $x_0$ and $v_0$ are indeed the initial position and the initial velocity.

## What if the Acceleration Changes?

It is possible to solve a problem with the equations of motion with constant acceleration if the acceleration changes by step, which means that it is constant for a while and then it suddenly changes to another constant value. There can be many changes of acceleration like that.

In this case, the problem is divided into parts. The first part is the motion with a constant acceleration there is at the outset, the second part is the one with the second constant acceleration and so on… The values of position and velocity at the end of the first part become the initial values for the second part. By applying the equations of the motion with constant acceleration in each part, the conditions of application of these formulas are met since the acceleration is indeed constant in each of the parts.

## Example 1.7.5

A car initially at rest has an acceleration of 6 m/s² for 5 seconds, and then an acceleration of 2 m/s² for 4 seconds and finally an acceleration of -15 m/s² for 2 seconds. What is the displacement of the car during these 11 seconds and what is the velocity of the car at the end of this motion?

1st Part: $a$ = 6 m/s², $x_0$ = 0 m and $v_0$ = 0 m/s

The velocity at the end of this part is

$$\begin{aligned} v &= v_0 + at \\ &= 0 + 6\tfrac{m}{s^2} \cdot 5s \\ &= 30\tfrac{m}{s} \end{aligned}$$

The position at the end of this part is

$$\begin{aligned} x &= x_0 + v_0 t + \frac{1}{2}at^2 \\ &= 0 + 0 \cdot 5s + \frac{1}{2} \cdot 6\tfrac{m}{s^2} \cdot (5s)^2 \\ &= 75m \end{aligned}$$

2nd Part: $a$ = 2 m/s², $x_0$ = 75 m and $v_0$ = 30 m/s

(Note: the initial values of the position and the velocity of this part are the values at the end of the first part.)

The velocity at the end of this part is

$$\begin{aligned} v &= v_0 + at \\ &= 30\tfrac{m}{s} + 2\tfrac{m}{s^2} \cdot 4s \\ &= 38\tfrac{m}{s} \end{aligned}$$

The position at the end of this part is

$$\begin{aligned} x &= x_0 + v_0 t + \frac{1}{2}at^2 \\ &= 75m + 30\tfrac{m}{s} \cdot 4s + \frac{1}{2} \cdot 2\tfrac{m}{s^2} \cdot (4s)^2 \\ &= 211m \end{aligned}$$

<u>3rd Part:</u> $a$ = -15 m/s², $x_0$ = 211 m and $v_0$ = 38 m/s

The velocity at the end of this part is

$$\begin{aligned} v &= v_0 + at \\ &= 38\tfrac{m}{s} + \left(-15\tfrac{m}{s^2}\right) \cdot 2s \\ &= 8\tfrac{m}{s} \end{aligned}$$

The position at the end of this part is

$$\begin{aligned} x &= x_0 + v_0 t + \frac{1}{2}at^2 \\ &= 211m + 38\tfrac{m}{s} \cdot 2s + \frac{1}{2} \cdot \left(-15\tfrac{m}{s^2}\right) \cdot (2s)^2 \\ &= 257m \end{aligned}$$

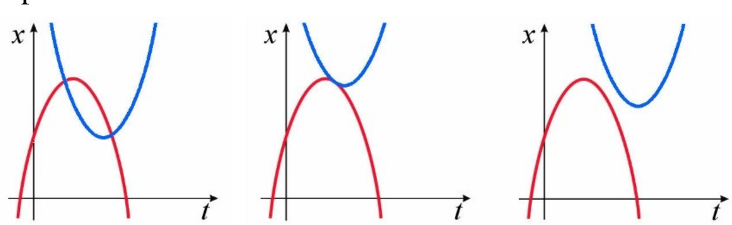
The total displacement is then

$$\begin{aligned} \Delta x &= x_2 - x_1 \\ &= 257m - 0m \\ &= 257m \end{aligned}$$

and the final velocity is 8 m/s.

## When Will Two Objects Be at the Same Place?

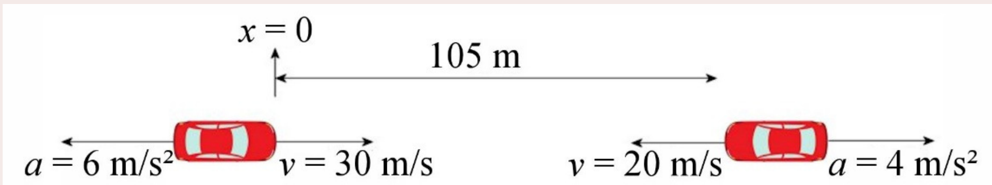
As mentioned previously, the problem is solved with the equation $x_A = x_B$. Again, this means that the points of intersection of the two functions giving the position according to the time must be found. If acceleration is constant, we're actually looking for the point of intersection of two parabolas. In this case, there may be 2 points of intersection, 1 point of intersection or no point of intersection.




2 solutions 1 solution 0 solutions

## Example 1.7.6

Two cars are moving towards each other. The car on the left moves towards the right with a velocity of 30 m/s and the car on the right moves towards the left with a velocity of 20 m/s. When they are 105 m from each other, they start to slow down simultaneously. The car to the left slows down at 6 m/s² while the car on the right slows down at 4 m/s². Will they hit each other and, if so, where and when?




fr.depositphotos.com/2577683/stock-illustration-Car.html

There will be a collision if $x_A = x_B$.

For car $A$ (the one on the left), we have $x_0$ = 0 m, $v_0$ = 30 m/s and $a$ = -6 m/s². Its position as a function of time is therefore

$$\begin{aligned} x_A &= x_0 + v_0 t + \frac{1}{2} a t^2 \\ &= 0m + 30\tfrac{m}{s} \cdot t + \frac{1}{2} \cdot \left(-6\tfrac{m}{s^2}\right) \cdot t^2 \\ &= 30\tfrac{m}{s} \cdot t - 3\tfrac{m}{s^2} \cdot t^2 \end{aligned}$$

For car $B$ (the one on the right), we have $x_0$ = 105 m, $v_0$ = -20 m/s and $a$ = 4 m/s². Its position as a function of time is therefore

$$\begin{aligned} x_B &= x_0 + v_0 t + \frac{1}{2} a t^2 \\ &= 105m + \left(-20\tfrac{m}{s}\right) \cdot t + \frac{1}{2} \cdot 4\tfrac{m}{s^2} \cdot t^2 \\ &= 105m - 20\tfrac{m}{s} \cdot t + 2\tfrac{m}{s^2} \cdot t^2 \end{aligned}$$

If the two cars collide, then

$$\begin{gathered} x_A = x_B \\ 30\tfrac{m}{s} \cdot t - 3\tfrac{m}{s^2} \cdot t^2 = 105m - 20\tfrac{m}{s} \cdot t + 2\tfrac{m}{s^2} \cdot t^2 \\ 105m - 50\tfrac{m}{s} \cdot t + 5\tfrac{m}{s^2} \cdot t^2 = 0 \end{gathered}$$

This quadratic equation can be solved with

$$t = \frac{50\tfrac{m}{s} \pm \sqrt{\left(50\tfrac{m}{s}\right)^2 - 4 \cdot 5\tfrac{m}{s^2} \cdot 105m}}{10\tfrac{m}{s^2}}$$

$$t = \frac{50 \pm 20}{10} s$$

The two solutions are then $t$ = 3 s and $t$ = 7 s. Since there is a solution, there is a collision. If there had been no solution, as it sometimes happens with quadratic equations, it would have meant that there is no collision.

But then, why are there 2 solutions? Will the cars hit each other twice? Actually, they will only hit each other only once, at $t$ = 3 s. The second solution is not good because it is clear that the acceleration of the cars will change when they collide. Thus, the formulas of the position as a function of time made previously are not valid for $t$ > 3 s since the accelerations change at that time.

The location of the collision can be found by using one of the two position formulas. This position is

$$\begin{aligned} x_A &= 30\tfrac{m}{s} \cdot t - 3\tfrac{m}{s^2} \cdot t^2 \\ &= 30\tfrac{m}{s} \cdot 3s - 3\tfrac{m}{s^2} \cdot (3s)^2 \\ &= 63m \end{aligned}$$

(You can check that $x_B$ gives the same position.)

The velocity of the two cars when they hit each other can also be found.

$$v_A = v_0 + at = 30\tfrac{m}{s} + \left(-6\tfrac{m}{s^2}\right) \cdot 3s = 12\tfrac{m}{s}$$
$$v_B = v_0 + at = -20\tfrac{m}{s} + 4\tfrac{m}{s^2} \cdot 3s = -8\tfrac{m}{s}$$

## Example 1.7.7

Alphonse and Bertrand are involved in a car race. Initially, their cars are at rest at the starting line. Alphonse's car has an acceleration of 5 m/s² until it reaches a maximum velocity of 30 m/s. Bertrand's car has a 3 m/s² acceleration until it reaches a maximum velocity of 42 m/s. Where and when will Bertrand's car overtake Alphonse's car?

As the accelerations change in this problem, this problem must be divided into parts.

1st Part: both cars are accelerating

$a_A$ = 5 m/s², $a_B$ = 3 m/s²

As this part ends when Alphonse's car (car $A$) reaches its maximum velocity, the duration of this phase is

$$v_A = v_0 + at$$
$$30\tfrac{m}{s} = 0\tfrac{m}{s} + 5\tfrac{m}{s^2} \cdot t$$
$$t = 6s$$

At the end of this part, the positions of the cars are

$$x_A = x_0 + v_0 t + \frac{1}{2}at^2 = 0 + 0 + \frac{1}{2} \cdot 5\tfrac{m}{s^2} \cdot (6s)^2 = 90m$$

$$x_B = x_0 + v_0 t + \frac{1}{2}at^2 = 0 + 0 + \frac{1}{2} \cdot 3\tfrac{m}{s^2} \cdot (6s)^2 = 54m$$

and the velocities are

$$v_A = 30\tfrac{m}{s}$$

$$v_B = v_0 + at = 0\tfrac{m}{s} + 3\tfrac{m}{s^2} \cdot 6s = 18\tfrac{m}{s}$$

<u>2nd Part: Car *A* is at its maximum velocity; car B accelerates on.</u>

$$a_A = 0 \text{ m/s}^2,\ a_B = 3 \text{ m/s}^2$$

(Note: the initial values of the positions and the velocities for this part are the values at the end of the first part.)

As this part ends when Bertrand's car (car *B*) reaches its maximum velocity, the duration of this phase is

$$v_B = v_0 + at$$
$$42\tfrac{m}{s} = 18\tfrac{m}{s} + 3\tfrac{m}{s^2} \cdot t$$
$$t = 8s$$

At the end of this phase, the positions of the cars are

$$x_A = x_0 + v_0 t + \frac{1}{2}at^2 = 90m + 30\tfrac{m}{s} \cdot 8s + 0 = 330m$$

$$x_B = x_0 + v_0 t + \frac{1}{2}at^2 = 54m + 18\tfrac{m}{s} \cdot 8s + \frac{1}{2} \cdot 3\tfrac{m}{s^2} \cdot (8s)^2 = 294m$$

(Car *B* still has not caught up with car *A*. Had Bertrand passed Alphonse, we would then proceed in the same way as we will do in the third part to find exactly where and when Bertrand would have caught up with Alphonse.)

The velocities at the end of the second part are

$$v_A = 30\tfrac{m}{s} \qquad\qquad v_B = 42\tfrac{m}{s}$$

<u>3rd Part: Both cars are moving at their maximal velocities.</u>

$$a_A = 0 \text{ m/s}^2, \; a_B = 0 \text{ m/s}^2$$

This part never ends. As car $B$ is going faster than car $A$, it will surely catch it at some point. When Bertrand catches up with Alphonse, we have $x_A = x_B$.

Positions of the cars as a function of time are

$$\begin{aligned} x_A &= x_0 + v_0 t + \frac{1}{2}at^2 \\ &= 330m + 30\tfrac{m}{s}\cdot t + 0 \end{aligned} \qquad\qquad \begin{aligned} x_B &= x_0 + v_0 t + \frac{1}{2}at^2 \\ &= 294m + 42\tfrac{m}{s}\cdot t + 0 \end{aligned}$$

If they are at the same location, then $x_A = x_B$

$$\begin{aligned} x_A &= x_B \\ 330m + 30\tfrac{m}{s}\cdot t &= 294m + 42\tfrac{m}{s}\cdot t \\ 36m &= 12\tfrac{m}{s}\cdot t \\ t &= 3s \end{aligned}$$

Therefore, the position is

$$\begin{aligned} x_A &= 330m + 30\tfrac{m}{s}\cdot 3s \\ &= 420m \end{aligned}$$

($x_B$ would have given the same answer) and the total time from the start is

$$6 \text{ s} + 8\text{s} + 3\text{s} = 17 \text{ s}.$$

(Note: the times must be added since the time starts anew at the beginning of each part. However, we directly get the position since we always kept our origin $x = 0$ m at the same place, i.e. at the starting line.)

## How to Obtain the Equations of Motion from the Positions at Specific Times

Sometimes, the position of an object at a specific moment has to be found from the positions of the object at 3 other times. The following example shows how to solve this kind of problem.

## Example 1.7.8

Here are the positions of an object at three instants in time.

$x = 2$ m at $t = 1$ s
$x = 3$ m at $t = 2$ s
$x = 11$ m at $t = 4$ s

Where is the object at $t = 10$ s?

The equation of motion

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

gives these 3 equations at the 3 given times.

$$2m = x_0 + (1s) v_0 + \frac{1}{2} \cdot a \cdot (1s)^2$$

$$3m = x_0 + (2s) v_0 + \frac{1}{2} \cdot a \cdot (2s)^2$$

$$11m = x_0 + (4s) v_0 + \frac{1}{2} \cdot a \cdot (4s)^2$$

Those give us 3 equations with 3 unknowns.

$$2m = x_0 + (1s) v_0 + (0.5s^2) a$$

$$3m = x_0 + (2s) v_0 + (2s^2) a$$

$$11m = x_0 + (4s) v_0 + (8s^2) a$$

With three equations and three unknowns, any method can be used to solve this system of equations (Gauss-Jordan, for example…). Leaving aside the details of this mathematical manipulation, the solution is

$x_0 = 3$ m, $v_0 = -2$ m/s and $a = 2$ m/s²

The position at $t = 10$ s is then

$$\begin{aligned} x &= x_0 + v_0 t + \frac{1}{2} a t^2 \\ &= 3m + \left(-2\tfrac{m}{s}\right) \cdot 10s + \frac{1}{2} \cdot 2\tfrac{m}{s^2} \cdot (10s)^2 \\ &= 83m \end{aligned}$$

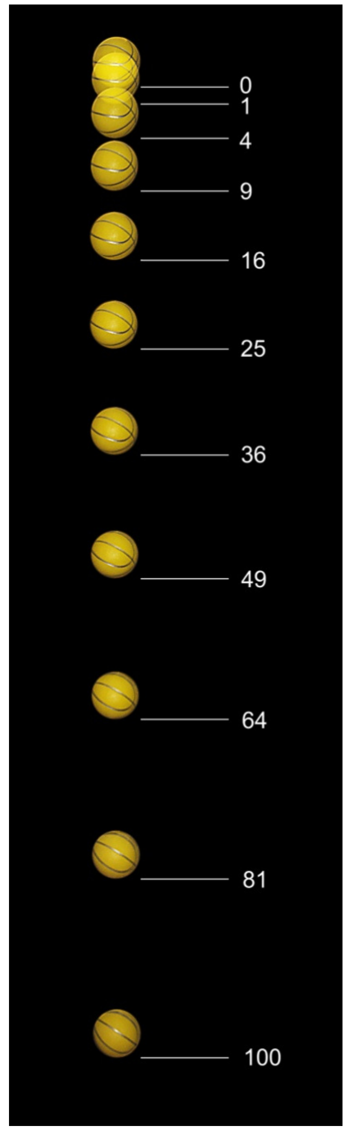
This solution seems short, but it is actually quite long when you have to solve the system of equations. It is much easier to solve, however, if one of the positions is the position at $t = 0$. The, the equation for $t = 0$ directly gives the value of $x_0$. Then, only two equations and two unknowns remain, and $v_0$ and $a$ can easily be found with them.

# 1.8 FREE FALL

## Gravitational Acceleration

When we talk about free fall, we are talking about an object on which only the gravitational force is exerted. This means that there is no friction that acts on the object. Note that an object thrown upwards is in free fall during its entire motion (upwards and downwards).

The motion of a free-falling object can be studied by using, for example, a strobe to photograph the object at regular intervals to easily measure its positions (as on the picture to the right).




en.wikipedia.org/wiki/Equations_for_a_falling_body

With the positions of the object as a function of time, it's easy to conclude that the free-fall motion is a constant acceleration motion and that all objects, regardless of their mass, have an acceleration of 9.8 m/s² downwards.

$$a_{gravitational} = 9.8 \tfrac{m}{s^2} \text{ downwards}$$

The symbol $g$ is used to represent the magnitude of this acceleration.

$$g = 9.8 \tfrac{m}{s^2}$$

## Mass Does not Matter

It can never be repeated enough: all the objects fall with the same acceleration in free fall, no matter their mass. Thus, if two objects of different mass are dropped simultaneously from rest into vacuum, they will arrive on the ground at the same time.

This may sound surprising, but it's probably because you have never seen an object fall in vacuum. To know the effect of gravity alone, air friction must be completely eliminated. This can be done by dropping objects in a vacuum chamber. In this video, a feather and a bowling ball reach the ground at the same time when they fall in a vacuum chamber.
http://physique.merici.ca/mecanique/Chute-vide.wmv
The demonstration can also be done on the Moon, where there's no air. Apollo 15 astronaut Dave Scott did this experiment in 1971.

http://www.youtube.com/watch?v=03SPBXALJZI
However, the objects fall with a lower acceleration on the Moon. The force of gravity being weaker on the surface of the Moon, the gravitational acceleration is only 1.6 m/s².

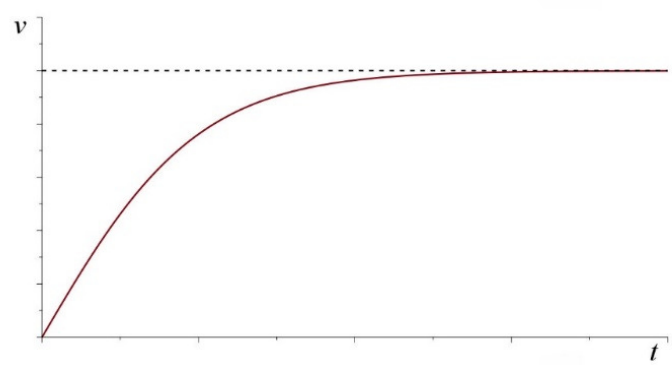
These clips clearly show that the acceleration of objects is the same, regardless of the mass of the object, in free fall.

## How Was it Discovered Free-Fall is a Motion With the Same Constant Acceleration?

It took about 2000 years of studies to reach the conclusion that all objects, whatever their mass, fall with the same acceleration in vacuum and that they fall with a constant acceleration. The problem is that it is very difficult to discover that acceleration is constant for falling objects when there is no way to eliminate friction.

*Effect of Mass*

When there is air friction, the falling speed varies in the way shown on this graph (which will see again in Chapter 5).




The graph shows that the speed of objects falling in air tends towards a maximum speed (horizontal dashed line). Since this maximum speed depends on the mass and the shape of the object, the motion of falling objects differs according to their mass and shape. This effect can be seen in this clip in which a stove and a pillow are dropped for the same height.
http://www.youtube.com/watch?v=RGVcKYpo9EM
It is not easy to deduce that mass does not matter from such an observation!

This experiment strongly suggests that more massive objects fall faster than lighter objects. Even today, many people will tell you that heavier objects fall faster than light objects, which is usually true for a fall in air, but not always (we will see in Chapter 5 that a lighter object could fall faster than a heavier object). Thus, it is not surprising to find that almost all scholars up to the 17th century, beginning with Aristotle in the 4th century BC, thought that more massive objects fall faster than lighter objects. Aristotle even goes so far as to say that an object 3 times more massive than another will take three times less time to reach the ground if we drop the objects at the same time from the same height.

However, a few have found that fall times can be virtually identical for objects made in some circumstances even if the masses are very different. This is what can be seen in these videos of falling pumpkins.
http://www.youtube.com/watch?v=gVAJcd4JXyE
http://www.youtube.com/watch?v=r2-h_bpUSqM
Philoponus is the first, in the 6th century, to claim that, sometimes, there is practically no difference between arrival times when two objects of different mass fall simultaneously from the same height. A few others opposed Aristotle in the next 1000 years, but they were not very numerous.

In the 16th century, Giambattista Benedetti was one of those who dared to contradict Aristotle. His theories (which were wrong according to current physics) led him to conclude, in 1552, that objects of the same density should fall at the same speed in air even if they have different masses. He was therefore able to explain why the fall times are sometimes different (objects of different densities) and sometimes identical (objects of the same density). In 1586, the Dutchman Simon Stevin verified this assertion by dropping from the top of a tower about 10 meters high two objects with identical densities, but different masses. He then observed that the arrival times are indeed identical (or almost). The fact that it took 34 years between the time the idea was formulated and the time a first experimental verification was carried out is really telling about the attitude of the scientists of the time. There were not many who thought that experimentation was an essential step in verifying claims at the time. (Note that the idea that the fall should depend only on density had quickly spread in Europe. The long delay is therefore not due to a slow diffusion of ideas.)

Galileo will go even further. According to him, the fall in vacuum is identical for all objects, even if they have different densities. He begins with a reasoning that shows that we arrive at a contradiction if we assume that more massive objects fall faster. Assuming that more massive objects fall with a greater acceleration than less massive objects, what would happen if a 10 kg object is tied to a 1 kg object? First, it can be inferred that this 11 kg object should fall with a grater acceleration than the 10 kg object since it is more massive. But then, it can also be argued that this 11 kg object should fall with a smaller acceleration than the 10 kg object since the 10 kg object has to drag a 1 kg object falling with a smaller acceleration! The only way out of this contradiction is to assume that the speed of fall is the same for all masses. (Note that the reasoning is valid only if the speed of fall depends solely on the mass.)

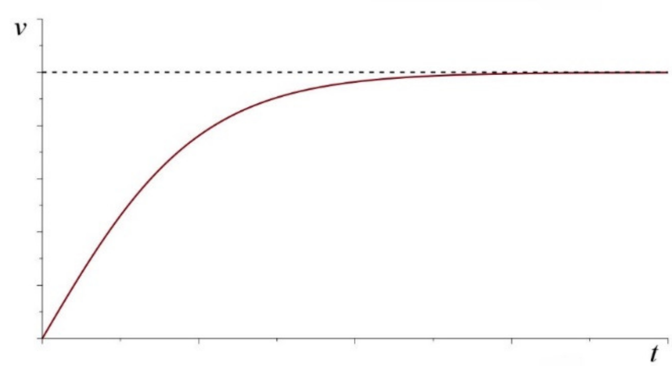
Galileo could not prove his theory experimentally by dropping objects in vacuum since the vacuum pump had not been invented yet. However, Galileo manages to show that there is a link between free fall and the movement of pendulums (which is normal since the 2 phenomena are related to gravitation). He shows that if the fall times are identical for 2 objects dropped from the same height in vacuum, then the oscillation times of the pendulum should be the same if these 2 objects are fixed at the end of strings of the same length. This allows for more precise checks than letting objects fall. In a fall, friction quickly becomes important, and the fall times are very difficult to measure with great accuracy. With very long pendulums and small oscillations, the problem of friction is eliminated because the speed of the object is not very great, and the air friction does not have much effect. The periods of oscillation can also be measured much more accurately by measuring the time of many oscillations.

Galileo published his conclusions in 1638. Some contested, but his ideas prevailed after a few decades. The measurements clearly showed that the oscillation times of a pendulum are identical when the object at the end of the pendulum is changed, regardless of its mass and composition. Newton himself says he did this experiment with many types of materials and found that the period was always the same. Newton was so convinced that the times of falls were identical in vacuum that he made it one of the starting points of his mechanics in 1687. In Newton's mechanics, all objects fall identically in vacuum, regardless of their mass. The observation of the first falls in vacuum (coin and feather in a tube) made by Boyle in 1660 has surely convinced several others.

But there was still a small doubt. Could it be that the fall is not exactly the same for all objects in a vacuum? In fact, they continued to verify that the fall is identical with increasingly precise experiments. By 1900, it had been verified that the fall is identical with an accuracy of $10^{-6}$ %. Today, accuracy reaches $10^{-13}$ %. As recently as 2016, the Microscope satellite was sent into space to verify the equality of fall. According to Einstein's equivalence principle, there should be no difference at all (this principle is the starting point of his theory of general relativity). On the other hand, there may be a small difference according to other theories. For example, string theory predicts that gravitation could be a little different depending on the composition of the object.

*Constant Acceleration*

The graph of the speed also shows that the fall of an object in air is not a constant acceleration motion (it is clear that the acceleration, which is the slope, is not constant). It's not easy to discover that acceleration is constant when it's not really constant!




However, the acceleration is approximately constant at 9.8 m/s² for all objects at the beginning of the motion (we can see that the slope is approximately constant at the beginning). Then, the speed is small, and the frictional force is not very large compared to the weight of the object. The acceleration is therefore approximately constant if the drop distance is not too great.

Of course, it is hard to discover that acceleration is constant when nobody know what acceleration is. But even after the concept of acceleration was clarified and the formulas of constant acceleration motion were obtained in the 14th century, it still took almost 300 years before Galileo discovered that acceleration was constant. Why was it so long?

Everyone knows back then that the speed increases with the fall. It is said that speed increases with fall time or speed increases with fall distance. You don't have to be a super genius to come to that conclusion. Sometimes, in the Middle Ages, some are more precise and say that speed is proportional to time ($v \propto t$) or proportional to the distance of fall ($v \propto y$). These two possibilities are quite different. The first corresponds to a constant acceleration motion, but not the second (in a constant acceleration motion, we have instead $v \propto \sqrt{y}$). Very often, they think that both possibilities are true since no one has the mathematical tools to understand that these two possibilities are very different.

No one is experimenting and no one thinks it's relevant to do so. Rather, they try to deduce what the motion will be from the theories that explain the cause of the motion (which we will see in Chapter 3) and then they do not check if it corresponds to the real motion. It must be said that the experiment would not have been easy to do, because they have, at that time, only this equation of constant acceleration motion.

$$y = \frac{1}{2}\left(v_0 + v\right)t$$

Obviously, the equation was not given in this form. Just to give you an idea, here's how Oresme gave this law in the mid-14th century.

*Every quality, if it is uniformly difform, is of the same quantity as would be the quality of the same or equal subject that is uniform according to the degree of the middle point of the same subject.*

It is not easy for us to find the formula in this sentence. The problem is that the formula only refers to speeds and they didn't really have a way to measure instantaneous speed at that time.

Thus, the idea of a constant acceleration fall ($v \propto t$) was already present at least since the 14th century but it had not been formulated from observations. It was only one possibility among others, and it was considered simply because it was a simple solution, just like $v \propto y$. Most of the time, they accept both possibilities at the same time without understanding that there is a contradiction between the two but, sometimes some have come to favor one solution over another. For example, Albert of Saxony (14th century) thinks that $v \propto y$ is the right solution while Domingo de Soto (1551) thinks that $v \propto t$ is the right solution. As this motion is a constantly accelerating motion, Domingo de Soto could pass for a visionary, but he is not really one. He does not provide any arguments and it is not known how he came to prefer this solution. Certainly, it was not an experiment that brought him to this law. He has simply come to prefer this law, probably for philosophical reasons, in the same way that others have come to prefer $v \propto y$.

At the beginning of the 17th century, the issue is still unresolved. Galileo, in his early years (around 1590), still accepted $v \propto y$ and $v \propto t$ at the same time. However, Galileo changed everything a few years later. In 1603-1604, the latter decided to experimentally verify how bodies fell. That was really a whole new way of doing things!

Galileo will first show that $v \propto y$ and $v \propto t$ are completely different motion. Also, he manages to show that $v \propto t$ means that the distance travelled is

$$y \propto t^2$$

This law makes it much easier to check whether a falling object is doing a constant acceleration motion since it is much easier to measure the distance traveled than the speed of a falling object. He finally shows (although the evidence is not so convincing) that if the vertical fall is done with a constant acceleration, then a fall along an inclined plane should also be a constant acceleration. The lower acceleration on the plane then slows down the entire fall process and facilitates the experiment.

The only problem that remains is that Galileo had no simple way to measure time. Galileo measured it with his pulse, with the amount of water flowing from a basin and with small obstacles placed on the track that make a small noise when the slipping object passes. A variant of this method can be seen in the picture on the right. This method involves placing small bells that are struck by the ball rolling along the inclined plane. The position of the bells is adjusted so that the ringing is done at regular intervals. Galileo ensured the regularity of the sounds emitted by singing a military march during the experiment. This experimental setup can be seen in action in this video (filmed at Woolsthorpe Manor, birthplace of Isaac Newton).
https://www.youtube.com/watch?v=eUbv78PHaro

The results of the experiment clearly showed that the distance traveled by the object increases with the square of time ($y \propto t^2$) and that the motion is a constant accelerated motion.

Galileo published his discoveries several years later (1632 and 1638), but his ideas were not unanimously accepted. Many accept the idea of a constant acceleration, but others dispute it mainly because they think that the falling motion cannot begin with zero speed. For them, the speed must suddenly change from zero to a certain speed when the object is released. Others, including Descartes, make mistakes when they try to calculate the position of an object that has a constant acceleration. It's easy for us, but it was not easy for them to deal with the continual changes in speed when the concept of instantaneous speed is not clearly defined. Galileo had done this calculation perfectly to arrive at $y \propto t^2$, but many others made mistakes. Slowly, these divergent opinions faded, but it will take several decades for them to disappear.

Newton dispelled all doubts in 1687. Free fall is automatically a constant-accelerating motion in Newton's theory. In fact, Newton shows that the acceleration is not quite constant since the gravitational force decreases a little with altitude. However, the acceleration variations are small near the Earth's surface and the acceleration is virtually constant.

## Free-Fall Examples

Most of the time, solutions to free-fall problems can be quickly obtained by using the equations for constant acceleration motion with an acceleration of 9.8 m/s².

### Common Mistake: Wrong Sign for $a$.

Acceleration is always downwards in free fall. Some students think that the acceleration is upwards when the object goes up and downwards when the object goes down. It's wrong. While going up, the velocity is upwards and the acceleration is downwards. As the velocity and the acceleration are in opposite directions, the object slows down. While going down, the velocity and the acceleration are both downwards. The velocity then increases since that is what happens when the acceleration and the velocity are in the same direction.

### Common Mistake: Thinking that $a$ = 0 at the Highest Point.

The acceleration is always 9.8 m/s² downwards during free fall, including at the highest point on the trajectory. At this point, the velocity is zero, but not the acceleration. If the acceleration were to be zero, the velocity of the object would remain constant. As the velocity at this point is zero, the object would always remain stationary at the highest point.

### Common Mistake: Thinking that $v$ = 0 When the Object Hits the Ground.

If asked to find when an object is going to hit the ground, some students use $v = 0$ at the ground if this problem can be solved with an equation involving velocity. It is true that the velocity is zero after the collision with the ground but the velocity just before the contact with the ground must be used in the equation. As soon as the object touches the ground, the acceleration changes and the equations of free fall are no longer valid.

## Example 1.8.1

An object is thrown upwards from the ground with a velocity of 20 m/s.

a) What is the maximum height reached by the object?

To resolve this problem, an axis directed upwards is used with the origin $y$ = 0 m at the ground.

Here $a$ = -9.8 m/s² (it is negative because an upwards axis was chosen and the acceleration is downwards), $v_0$= 20 m/s and $y_0$ = 0 m. The maximum height is reached when the velocity is zero. (The object goes up when the velocity is positive and goes down when the velocity is negative. Therefore $v$ = 0 at the highest point.) Therefore, $y$ when $v$ = 0 is sought. This problem is solved directly with the following equation.

$$2a(y-y_0)=v^2-v_0^2$$
$$2\cdot\left(-9.8\tfrac{m}{s^2}\right)\cdot\left(y-0m\right)=0-\left(20\tfrac{m}{s}\right)^2$$
$$y=20.4m$$

b) How long does it take for the object to reach the maximum height?

This problem is solved with the equation

$$v=v_0+at$$
$$0=20\tfrac{m}{s}+\left(-9.8\tfrac{m}{s^2}\right)\cdot t$$
$$t=2.04s$$

c) What is the velocity of the object when it hits the ground?

When the object hits the ground, it returns to $y$ = 0. This problem is then solved with the equation

$$2a(y-y_0)=v^2-v_0^2$$
$$2\cdot\left(-9.8\tfrac{m}{s^2}\right)\cdot\left(0m-0m\right)=v^2-\left(20\tfrac{m}{s}\right)^2$$
$$v=\pm20\tfrac{m}{s}$$

Some students solve this problem in two parts: the ascent and the descent. Although it is not wrong to do this division, it is useless since it can be solved in one part because the acceleration is always constant in this motion.

d) How long does it take for the object to come back to the ground?

This problem is solved directly with the following equation.

$$v = v_0 + at$$
$$-20\tfrac{m}{s} = 20\tfrac{m}{s} + \left(-9.8\tfrac{m}{s^2}\right)\cdot t$$
$$t = 4.08s$$

(Observe that $v$ = 0 was not used for the velocity on the ground. The velocity just before the contact with the ground had to be used.

e) What is the displacement between 1 s and 1.5 s after the start of the motion?

To solve this problem, the position at $t$ = 1 s and $t$ = 1.5 s must be known.

At $t$ = 1 s, the position is

$$\begin{aligned} y_1 &= y_0 + v_0 t + \frac{1}{2}at^2 \\ &= 0m + 20\tfrac{m}{s}\cdot 1s + \frac{1}{2}\cdot\left(-9.8\tfrac{m}{s^2}\right)\cdot(1s)^2 \\ &= 15.1m \end{aligned}$$

At $t$ = 1.5 s, the position is

$$\begin{aligned} y_2 &= y_0 + v_0 t + \frac{1}{2}at^2 \\ &= 0m + 20\tfrac{m}{s}\cdot 1.5s + \frac{1}{2}\cdot\left(-9.8\tfrac{m}{s^2}\right)\cdot(1.5s)^2 \\ &= 18.98m \end{aligned}$$

Thus, the displacement is

$$\begin{aligned} \Delta y &= y_2 - y_1 \\ &= 18.98m - 15.1m \\ &= 3.88m \end{aligned}$$

f) When is the object 10 m above ground?

This problem can be solved with

$$y = y_0 + v_0 t + \frac{1}{2}at^2$$
$$10m = 0m + 20\tfrac{m}{s}\cdot t + \frac{1}{2}\cdot\left(-9.8\tfrac{m}{s^2}\right)\cdot t^2$$
$$t = 0.583s \quad \text{and} \quad 3.498s \quad \text{(solutions to a quadratic equation)}$$

There are two answers, and those two answers are correct since the object is 10 m above ground once going upwards and once going downwards.

Most problems are like this: one of the equations for the motion at constant acceleration directly solves the problem. However, there are some problems which are not as simple as that. Here's an example.

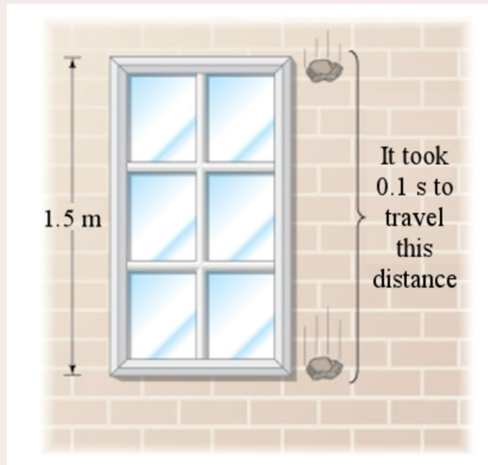
## Example 1.8.2

A stone thrown vertically from the top of the roof of a building takes 0.1 s to travel past a window 1.5 m tall. The top of the window is 10 m below the roof. How fast was the stone thrown?

Some data are missing for solving this problem directly. But since some information concerning the passage of the stone in front of the window is given (the time to pass from the top of the windows to the bottom of the window), it may be worthwhile to divide the problem into two parts.

1) From the roof to the top of the window.

2) From the top of the window to the bottom of the window.




www.chegg.com/homework-help/questions-and-answers/falling-stone-takes-033-s-travel-past-window-22-m-tall-figure-height-top-window-stone-fall-q3912013

We'll start with the second part. A $y$-axis directed downwards is used and the origin $y = 0$ m located at the roof. For this part, we know:

The acceleration ($a = 9.8$ m/s². It's positive since the $y$-axis is downwards.)
The initial position ($y_0 = 10$ m)
The final position ($y = 11.5$ m)
The duration of the displacement (0.1 s)

The initial velocity of this part, i.e. the velocity of the stone at the top of the window, can now be found. The velocity is calculated with the equation

$$y = y_0 + v_0 t + \frac{1}{2} a t^2$$

$$11.5m = 10m + v_0 \cdot 0{,}1s + \frac{1}{2} \cdot 9.8 \tfrac{m}{s^2} \cdot (0.1s)^2$$

$$v_0 = 14.51 \tfrac{m}{s}$$

For the first part of the motion (from the roof to the top of the window), we now know:

The acceleration ($a = 9.8$ m/s²)

The initial position ($y_0 = 0$ m)
The final position ($y = 10$ m)
The final velocity ($v = 14.51$ m/s)

The initial velocity is then found with

$$2a(y-y_0)=v^2-v_0^2$$
$$2\cdot 9.8\tfrac{m}{s^2}\cdot(10m-0m)=\left(14.51\tfrac{m}{s}\right)^2-{v_0}^2$$
$$v_0=\pm 3.813\tfrac{m}{s}$$

The stone has been launched with a velocity of 3.813 m/s upwards or downwards. Both answers are correct because a stone thrown upwards at 3.813 m/s comes back to the roof with the same velocity but directed downwards. This is the same thing as if it had been thrown straight downwards at 3.813 m/s.

# 1.9 MOTION WITH A NON-CONSTANT ACCELERATION

What if the acceleration is always changing? In that case, the equations of motion at constant acceleration must be forgotten and the basic definitions of velocity and acceleration have to be used.

$$v=\frac{dx}{dt} \qquad\qquad a=\frac{dv}{dt}$$

From these definitions, the velocity of an object can be found from the equation of the position as a function of time. Just derive the formula of the position to get it. The acceleration as a function of time can also be found by deriving the velocity equation. But is it possible to do the opposite, that is, to find the position if the formula for the velocity is known? It's simple; just do the opposite of the derivative!

## Example 1.9.1

The velocity of an object is given by $v=9\tfrac{m}{s^3}\cdot t^2$. Where is the object at $t = 5$ s if it is at $x = 10$ m when $t = 0$ s?

Since velocity is the derivative of position, $9t^2$ must be the derivative of the position. Which function has been derived to get $9t^2$? The answer is $3t^3$ (derive it and you'll see that you'll obtain $9t^2$). However, this is not the only solution because $3t^3 + 3$ would also give the same derivative. Actually, the most general solution is

$$x=3\tfrac{m}{s^3}\cdot t^3+\text{constant}$$

Regardless of the value of the constant, the derivative is always $9t^2$.

But what is the value of this unknown constant? It is possible to find its value if the position at some moment in time is known. These data are called *initial conditions*. Here, the object is said to be at $x = 10$ m when $t = 0$ s. Substituting these values in the equation, it becomes

$$\begin{aligned} x &= 3\tfrac{m}{s^3} t^3 + \text{constant} \\ 10m &= 3\tfrac{m}{s^3} \cdot 0^3 + \text{constant} \\ 10m &= 0 + \text{constant} \\ \text{constant} &= 10m \end{aligned}$$

Thus, the position formula is

$$x = 3\tfrac{m}{s^3} \cdot t^3 + 10m$$

At $t = 5$ s, the object is thus

$$\begin{aligned} x &= 3\tfrac{m}{s^3} \cdot (5s)^3 + 10m \\ &= 385m \end{aligned}$$

Formally, doing the opposite of the derivative is called *doing the integral*. Therefore, $3t^3$ + constant is the integral of $9t^2$. The correct notation for this procedure is

$$\int 9\tfrac{m}{s^3} \cdot t^2 dt = 3\tfrac{m}{s^3} \cdot t^3 + \text{constant}$$

This seems a little complicated, but you will understand this notation in calculus. Doing the integral of a function is usually harder than finding the derivative of a function (as you'll discover in calculus).

Thus, with the correct notation, the equations are

**Position from Velocity**

$$x = \int v dt$$

**Velocity from Acceleration**

$$v = \int a dt$$

You are probably not yet very familiar with integrals and this is why we will limit ourselves to functions of positions $x$, velocities $v$ and accelerations $a$ that are only polynomials of $t$. So, you just have to know the integral of $t$ with an exponent. This integral is

$$\int t^n dt = \frac{t^{n+1}}{n+1}$$

This means that the exponent is increased by 1 and you then divide by the new exponent obtained. For example, the integral of $t^{20}$ is $\frac{1}{21}t^{21}$.

## Example 1.9.2

The acceleration of an object is given by $a = 6\frac{m}{s^3} \cdot t$. At $t = 0$ s, the position of the object is $x = 0$ m, and its velocity is $v = 20$ m/s. What are the velocity and the position at $t = 5$ s?

Let's start by calculating the velocity with an integral. The velocity is

$$v = \int a dt$$
$$v = \int 6\tfrac{m}{s^3} \cdot t dt$$
$$v = 3\tfrac{m}{s^3} \cdot t^2 + C$$

Using the initial conditions, the value of the constant can be found.

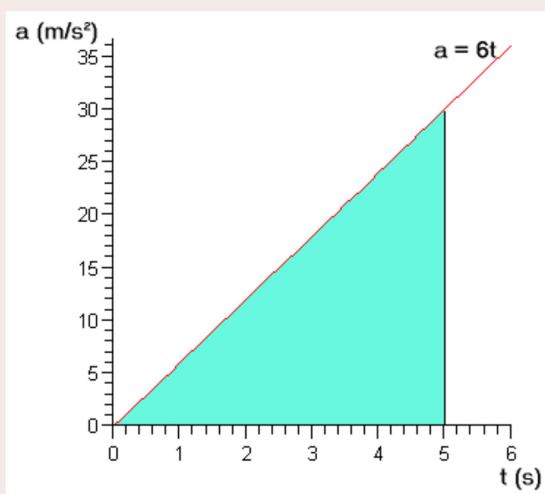
$$v = 3\tfrac{m}{s^3} \cdot t^2 + C$$
$$20\tfrac{m}{s} = 0 + C$$
$$C = 20\tfrac{m}{s}$$

Therefore, the formula $v = 3\frac{m}{s^3} \cdot t^2 + 20\frac{m}{s}$ gives the velocity.

Therefore, the velocity at $t = 5$ s is

$$\begin{aligned} v &= 3\tfrac{m}{s^3} \cdot (5s)^2 + 20\tfrac{m}{s} \\ &= 95\tfrac{m}{s} \end{aligned}$$

This calculation could also have been done with the graphical method by calculating the blue area on this graph.




This area of the triangle is 75 m/s. Note that this area is the change of velocity. But since the initial velocity is 20 m/s, the velocity is 95 m/s.

The position is then found with the integral

$$x = \int v dt$$
$$x = \int \left(3\tfrac{m}{s^3} \cdot t^2 + 20\tfrac{m}{s}\right) dt$$
$$x = 1\tfrac{m}{s^3} \cdot t^3 + 20\tfrac{m}{s} \cdot t + C$$

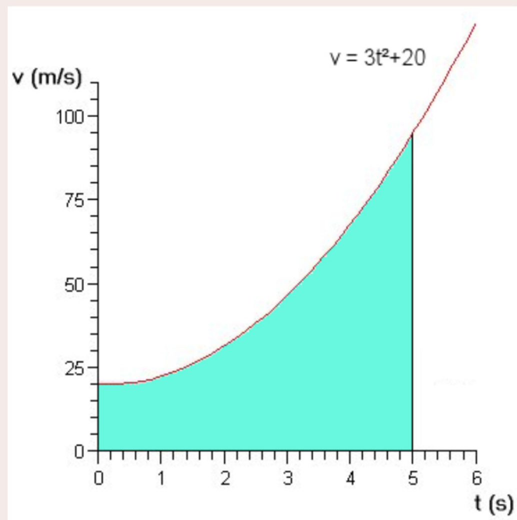
Using the initial conditions, the value of the constant can be found.

$$x = 1\tfrac{m}{s^3} \cdot t^3 + 20\tfrac{m}{s} \cdot t + C$$
$$0 = 0 + 0 + C$$
$$C = 0$$

The formula $x = 1\tfrac{m}{s^3} \cdot t^3 + 20\tfrac{m}{s} \cdot t$ is, therefore, the formula of the position.

Thus, the position at $t$ = 5 s is

$$x = 1\tfrac{m}{s^3} \cdot (5s)^3 + 20\tfrac{m}{s} \cdot 5s$$
$$= 225m$$




The graphical method could also be used to determine the displacement. To do this, the area under the curve on the graph on the right must be calculated.

The difficulty here is that we have no formula to calculate an area of that shape. Don't be sad. Actually, you know how to calculate the area now. The area was calculated using an integral to obtain 225 m. This means that you can calculate the area under curves using integrals. You will explore this beautiful idea further in calculus.

Note again that this area gives the displacement, therefore, the change in position. As the initial position is 0 m, the position after 5 seconds is 225 m. If the initial position had been 10 m, the final position would have been 235 m (225 m more than the initial position).

## A Small Variation

In the following example, the velocity is given as a function of position, not as a function of time. This example shows how to solve this kind of problem. (This goes beyond what you need to know in this course. It's just there to show you that it's possible to solve this kind of problem.)

## Example 1.9.3

The velocity of an object is given by $v = \frac{x}{2s}$. At $t$ = 0 s, the position is $x$ = 100 m. What are the velocity and the position at $t$ = 5 s?

According to the definition of velocity, this means that

$$v = \frac{x}{2s}$$
$$\frac{dx}{dt} = \frac{x}{2s}$$
$$\frac{dx}{x} = \frac{1}{2s}dt$$

With an integral, this becomes

$$\int \frac{dx}{x} = \int \frac{1}{2s}dt$$
$$\ln(x) = \frac{t}{2s} + C$$

Using the initial conditions, the value of the constant can be found

$$\ln(100m) = \frac{0s}{2s} + C$$
$$C = \ln(100m)$$

The position as a function of time is then

$$\ln(x) = \frac{t}{2s} + \ln(100m)$$
$$\ln(x) - \ln(100m) = \frac{t}{2s}$$
$$\ln\frac{x}{100m} = \frac{t}{2s}$$
$$\frac{x}{100m} = e^{t/2s}$$
$$x = 100m \cdot e^{t/2s}$$

Thus, at $t$ = 5 s, the position is

$$\begin{aligned} x &= 100m \cdot e^{5s/2s} \\ &= 1218m \end{aligned}$$

and the velocity is

$$\begin{aligned} v &= \frac{x}{2s} \\ &= \frac{1218m}{2s} \\ &= 609\tfrac{m}{s} \end{aligned}$$

# SUMMARY OF EQUATIONS

**Displacement**

$$\Delta x = x_2 - x_1$$

**Average Velocity**

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

**Instantaneous Velocity**

$$v = \frac{dx}{dt}$$

**Equation of Motion at Constant Velocity**

$$x = x_0 + vt$$

**Moment when Two Objects, Initially at a Distance *L* from Each Other, Are in the Same Place if they Move at Constant Velocities.**

$$t = \frac{L}{v_1 - v_2}$$

**Average Acceleration**

$$\bar{a} = \frac{\Delta v}{\Delta t}$$

**Instantaneous Acceleration**

$$a = \frac{dv}{dt}$$

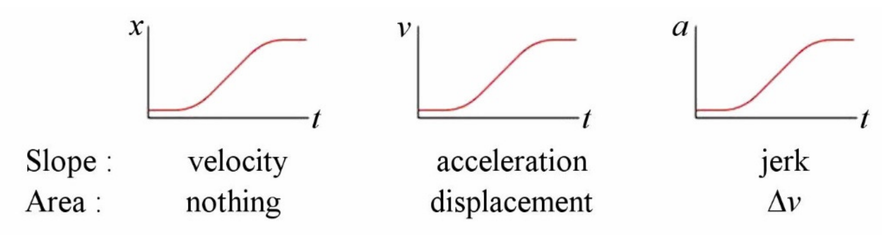
**Position from Velocity**

$$x = \int v dt$$

**Velocity from Acceleration**

$$v = \int a\,dt$$

**Graphical Representations**




**Equations of Motion with Constant Acceleration (also Called Uniformly Accelerated Motion)**

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

$$2a\left(x - x_0\right) = v^2 - v_0^2$$

$$x = x_0 + \frac{1}{2}\left(v_0 + v\right)t$$

**Free-Fall**

$$a_{\text{gravitational}} = 9.8 m/s^2 \text{ downwards}$$

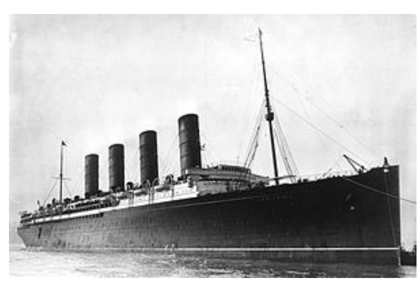
$$g = 9.8 \tfrac{m}{s^2}$$

# EXERCISES

## 1.3 Average Velocity

1. What was the average velocity of the Apollo 11 capsule, knowing it went to the Moon, at a distance of 384,400 km, in 72 hours and 49 minutes?

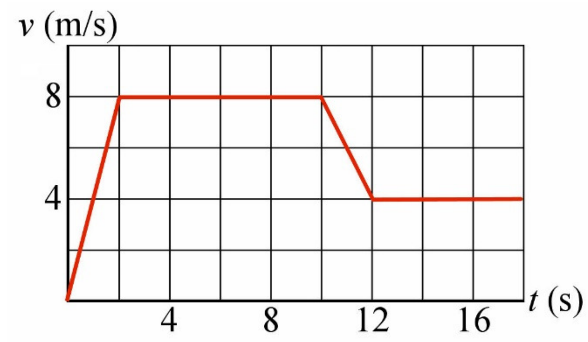
2. In 1907, the liner *Lusitania* beat the record for the fastest crossing of the Atlantic to win the Blue Ribbon. She then beat the ship *Deutschland,* which held the Blue Ribbon since 1903 when she crosses the Atlantic in 5 days, 11 hours and 54 minutes for an average velocity of 23.15 knots. The Lusitania did better with an average velocity of 23.99 knots. Knowing that 1 knot = 1.853 km/h, by how much time did the *Lusitania* beat the *Deutschland*?



en.wikipedia.org/wiki/RMS_Lusitania

## 1.4 Instantaneous Velocity

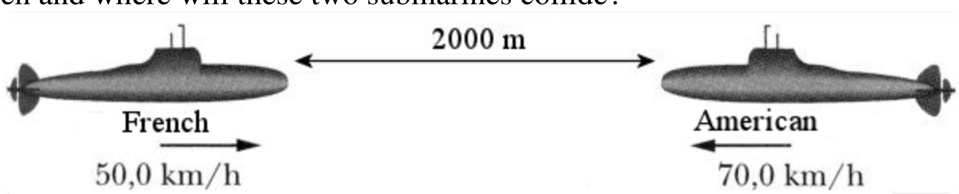
3. The velocity of an angry panda as a function of time is given by this graph. What is the displacement of the panda between $t$ = 2 s and $t$ = 14 s?




## 1.5 Motion With Constant Velocity

4. Richard took a walk in a straight line with a constant velocity of 5 km/h. How much time lasted his walk if its displacement was 350 m?

5. When and where will these two submarines collide?




Halliday, Resnick, Walker, Ondes, optique et physique moderne, Chenelière/McGraw-Hill, 2004

6. While on vacation near a river, little Nicole catches a fish with her bare hands, a feat that was seen as an act of war by a family of grizzly bears feasting close by. The bears, angry, start running after little Nicole. Terrified, Nicole runs at 15 km/h towards the family car, which is 100 m away, while the bears, which are initially 30 m from Nicole, set off in pursuit with a 25 km/h velocity. Will the bears catch Nicole?


www.shoppedornot.com/shopped-or-not/girl-running-from-bear/

7. Going from Quebec to Boston, Dieudonné goes at 110 km/h for 4 hours and at 130 km/h for 2 hours.

   a) What is the total displacement?
   b) What is the average velocity?

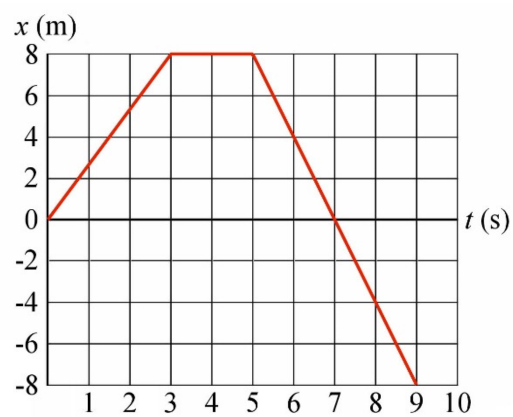
8. Phil moves at 30 m/s for 80 s and then retraces his steps with a 20 m/s speed for 15 s.

   a) What is Phil's total displacement?
   b) What is the distance travelled by Phil?
   c) What is Phil's average velocity?
   d) What is Phil's average speed?

9. An earthquake generates two different types of waves that propagate in the ground. The primary waves travel at 8 km/s, and the secondary waves travel at 5 km/s. Thus, an observer at some distance from the epicentre of the earthquake receives the primary waves first and then the secondary waves. How far is the observer from the epicentre of the earthquake if the secondary waves arrive 40 seconds after the primary waves?

10. The position of an object as a function of time is given by the following graph.

    a) What is the displacement of the object between $t = 0$ s and $t = 9$ s?
    b) What is the distance travelled between $t = 0$ s and $t = 9$ s?
    c) What is the average velocity between $t = 3$ s and $t = 9$ s?
    d) What is the velocity at $t = 1$ s?
    e) What is the velocity at $t = 8$ s?




11. The position of an object is given by the formula $x = 4\frac{m}{s^2} \cdot t^2 - 5\frac{m}{s} \cdot t + 10m$ .

    a) What is the average velocity between $t = 0$ s and $t = 2$ s?
    b) What is the velocity at $t = 2$ s?

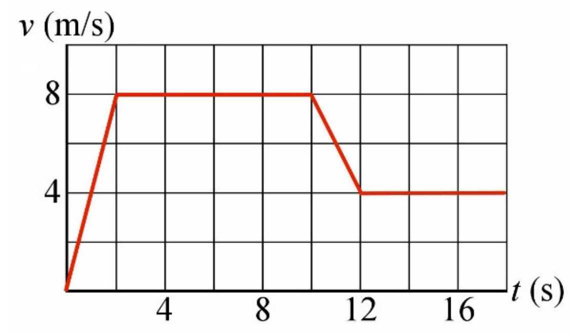
## 1.6 Acceleration

12. A turbo Kia Optima starting from rest accelerates to 100 km/h in 6.1 seconds. What is the average acceleration of the car?

13. An Acura TSX going at 120 km/h stops in 3.6 s with an intense braking. What is the average acceleration of the car as it brakes?

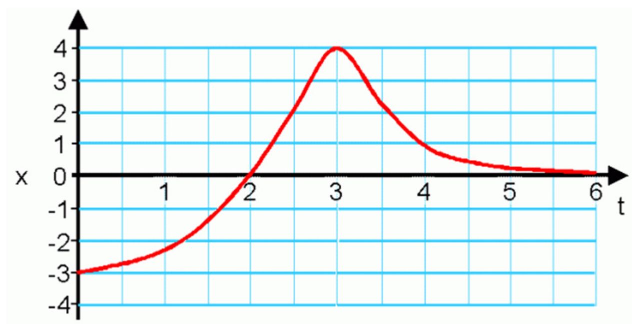
14. The velocity of an object as a function of time is given by the following graph.

a) What is the average acceleration between $t = 0$ s and $t = 4$ s?
b) What is the average acceleration between $t = 10$ s and $t = 12$ s?
c) What is the average acceleration between $t = 4$ s and $t = 8$ s?
d) What is the acceleration at $t = 1$ s?
e) What is the acceleration at $t = 14$ s?




15. The position of an object as a function of time is given by the following graph.

a) What is the sign (positive, negative or zero) of the acceleration at $t = 1$ s?
b) What is the sign (positive, negative or zero) of the acceleration at $t = 3$ s?
c) What is the sign (positive, negative or zero) of the acceleration at $t = 4$ s?




www.kwantlen.ca/science/physics/faculty/mcoombes/webtests/xtgraphquiz/xtGraphQuiz.htm

16. The position of an object is given by the formula $x = 3\frac{m}{s^3} \cdot t^3 - 8\frac{m}{s^2} \cdot t^2 + 2\frac{m}{s} \cdot t - 6m$.

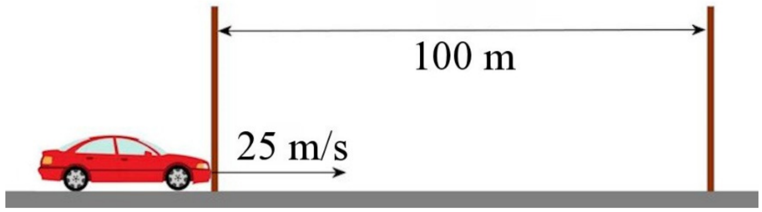
a) What is the average acceleration between $t = 0$ s and $t = 1$ s?
b) What is the acceleration at $t = 2$ s?
c) What is the jerk at $t = 1$ s?

## 1.7 Motion with Constant Acceleration

17. A BMW starts from rest with an acceleration of 5 m/s² for 6 s.

a) What is the distance travelled by the car during this time?
b) What is the velocity of the car after 6 seconds?

18. There are two poles 100 m apart besides the road. Ulysses travels on this road in a car having a steady acceleration. When Ulysses goes by the first pole, he has a velocity of 25 m/s. When Ulysses passes by the second pole, his velocity is now 15 m/s.




a) How long did it take for Ulysses to travel from one pole to the other?
b) What is the acceleration of the car?
c) Suppose there is a third pole 100 metres to the right of the second pole. What will the velocity of the car be when it passes by this third pole if the acceleration remains constant?

19. Near the end of a 1600 m race (in a straight line), Mahamadou gives a final effort to finish the race. While he is 200 m from the finish line and has a velocity of 5 m/s, his extra efforts allow him to have a constant acceleration up to the end of the race so that he travels the last 200 m in 25 seconds.

a) What is Mahamadou's acceleration in the last 200 m?
b) What is Mahamadou's velocity at the finish line?

20. At a bowling competition, Oliver throws his bowling ball with a velocity of 10 m/s. As the ball heads towards the pins, friction makes the ball slow down at the rate of 0.2 m/s², so that the ball hits the pins with a velocity of 9.65 m/s. What is the distance travelled by the bowling ball?

21. An accelerating car passes on a 30 m long bridge in 1.2 s. At the far end of the bridge, the velocity of the car is 20 m/s. What was the velocity of the car at the near end of the bridge?

22. The velocity of a car braking with a constant acceleration passes from 30 m/s to 24 m/s over a distance of 32 m. What is the stopping distance of the car if it continues to brake with the same acceleration and if the initial velocity was 42 m/s?

23. Going through the la Verandrye Park at night at 108 km/h, Marie-Pascale suddenly sees a moose in the middle of the road, a 100 m in front of her car. Her car then continues moving at 108 km/h for 0.5 second, the time it takes for Marie-Pascale to react and apply the brakes, and then slows down with an acceleration of 4 m/s². Will she hit the moose?

24. Here are the accelerations of the Didier and Gilles during a 100 m race.

Didier: accelerates at 5 m/s² for 1.8 second then decelerates at 0.1 m/s² up to the end.

Gilles: accelerates at 6 m/s² for 1.7 second and then decelerates at 0.24 m/s² up to the end.

Obviously, they were both at rest at the start of the race. Who won the race and by how much time?

25. Two rockets are initially at rest one besides the other. The first rocket starts moving with a constant acceleration of 5 m/s² while the other rocket starts to move only two seconds later with a steady acceleration of 6 m/s². Where and when will the second rocket catch up with the first rocket?

26. Here are the positions of an object at three moments in time.

$x = 5$ m at $t = 0$ s
$x = 5$ m at $t = 1$ s
$x = 9$ m at $t = 2$ s

Where will the object be at $t = 5$ s?

## 1.8 Free Fall

27. Here is a video showing a curious free-fall activity with pumpkins.
http://www.youtube.com/watch?v=tbNKVmWj1K4
The pumpkins are dropped from a height of 12 m.

a) What is the velocity of the pumpkin when it hits the car?
b) How long did the free fall of the pumpkin last?

28. Arthur throws a stone upwards with a velocity of 28 m/s from the edge of a cliff 80 metres high.

a) How high will the stone go?
b) When will the stone be 25 m above its starting point?
c) When will the stone be 25 m below its starting point?
d) What is the velocity of the stone when it is 20 m above its point of departure?
e) When does the speed equal 10 m/s?


onlinephys.com/kinematics1Dc.html

f) How high above its starting point is the stone when the speed is 12 m/s?
g) What is the velocity of the stone when it reaches the bottom of the cliff?
h) What is the total flight time of the stone?

29. Tony receives a balloon moving at 24 m/s filled with water on the head. The balloon was launched by Tryphon from a window 10 m above Tony's head. What was the initial velocity of the balloon?

30. Julian throws a ball straight up. How fast was the ball launched if it rises to a height of 80 m above its starting point?

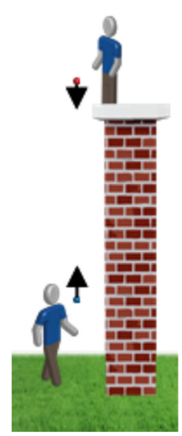
31. Hubert kicks a ball directly upwards with his foot. What was the initial velocity of the ball if it returns to Hubert's foot after a 12-second flight?

32. An object is thrown directly upwards with a velocity $v_0$. After ascending 5 m, the velocity of the object is now only 30% of its initial velocity $v_0$. What is the value of $v_0$?

33. A rocket, initially at rest on the ground, takes off vertically. While its engine works, the rocket accelerated at 4 m/s² upwards. After 20 s, the engine stops and the rocket is in free fall.

a) How high will the rocket go?
b) How much time will it take for the rocket to come back to the ground?

34. Kim dropped a ball, without giving it an initial velocity, from the top of the CN Tower at a height of 400 m. 1 second later, Leon throws a ball downwards from the same spot with a velocity of 12 m/s. Will Leon's ball overtake Kim's ball and, if so, how high above ground and after how much time?



35. Johnny, at the top of a tower, throws a ball downwards at 5 m/s while Frederique, on the ground, throws a ball upwards with a speed of 15 m/s. Both balls start simultaneously. Initially, Frederique's ball is 1 m from the ground and Johnny's ball is 51 m above the ground.

a) When will the balls collide (if they do)?
b) At what height above ground will the balls collide?
c) What is the velocity of each ball when they collide?

## 1.9 Motion with Non-Constant Acceleration

36. The velocity of a remote-controlled car is given by the formula

$$v = 2\tfrac{m}{s^4} \cdot t^3 - 6\tfrac{m}{s^3} \cdot t^2 + 4\tfrac{m}{s^2} \cdot t + 2\tfrac{m}{s}$$

a) What is the position of the car at $t$ = 2 s if it was initially at $x$ = 5 m?
b) What is the acceleration of the car at $t$ = 2 s ?

37. The initial velocity of an object is 4 m/s, and its acceleration is given by the formula

$$a = 36\tfrac{m}{s^4} \cdot t^2 + 10\tfrac{m}{s^2}$$

a) What is the velocity of the object at $t$ = 1 s.
b) What is the displacement of the object between $t$ = 0 s and $t$ = 4 s.

## Challenges

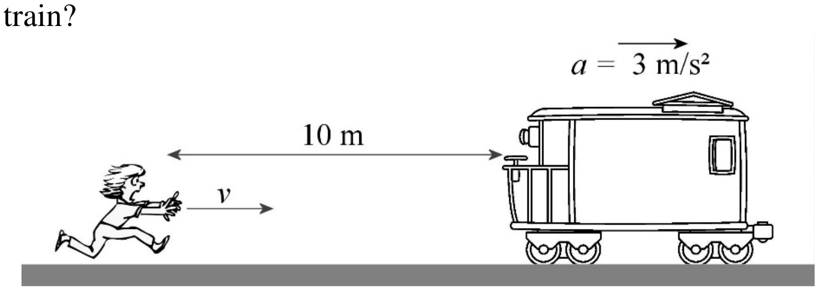
(Questions more difficult than the exam questions.)

38. On March 30, 2012, Larry Dixon travelled 400 m in 4.503 seconds with his dragster. Let's simplify a little by assuming that he had a constant acceleration until he reaches a maximum speed of 534 km/h. which it maintained up to the finish line.

a) What was the duration of the acceleration phase?
b) What was the acceleration of the dragster at the beginning of its motion?

www.pinterest.com/rlowen31/top-fuel-dragster/

39. Sophie wants to board a train. To get there, she runs on the track in the direction of the rear of the train. When she is 10 m from the train, it leaves with a constant acceleration of 3 m/s². What is the minimum speed that Sophie must maintain to catch the train?




www.clipartkid.com/train-cars-cliparts/ and www.clipartkid.com/running-work-scared-cliparts/

# ANSWERS

## 1.3 Average Velocity

1. 1466.4 m/s = 5279 km/h
2. 4 h 37 min

## 1.4 Instantaneous Velocity

3. 84 m

## 1.5 Motion With Constant Velocity

4. 4 min 12 s
5. In 60 seconds. The collision occurs 833 m to the right of the starting position of the French submarine.
6. The bears will catch her before she arrives at the car and small Nicole must kindly give back the fish to the grizzly bears.
7. a) 700 km b) 116.7 km/h
8. a) 2100 m b) 2700 m c) 22.1 m/s d) 28.42 m/s
9. 533 km
10. a) -8 m b) 24 m c) -2.67 m/s d) 2.67 m/s e) -4 m/s
11. a) 3 m/s b) 11 m/s

## 1.6 Acceleration

12. 4.554 m/s²
13. – 9.26 m/s²
14. a) 2 m/s² b) -2 m/s² c) 0 m/s² d) 4 m/s² e) 0 m/s²
15. a) positive b) negative c) positive
16. a) -7 m/s² b) 20 m/s² c) 18 m/s³

## 1.7 Motion With Constant Acceleration

17. a) 90 m b) 30 m/s
18. a) 5 s b) -2 m/s² c) The car will stop before reaching the third pole.
19. a) 0.24 m/s² b) 11 m/s
20. 17.19 m
21. 30 m/s
22. 174.2 m
23. She hits the moose

24. Gilles wins by 0.796 s
25. 22.954 s after the start of the first rocket, 1317 m from the start.
26. 45 m

## 1.8 Free Fall

27. a) 15.34 m/s b) 1.565 s
28. a) 40 m above the top of the cliff b) 1.108 s and 4.607 s c) 6.499 s
    d) ±19.8 m/s e) 1.837 s and 3.878 s f) 32.65 m g) -48.50 m/s h) 7.806 s
29. ±19.49 m/s
30. 39.6 m/s
31. 58.8 m/s
32. 10.38 m/s
33. a) 1127 m b) 43.326 s
34. Leon's ball catches up with Kim's ball 2.227 s after Leon's ball departure, 348.97 m above ground.
35. a) 2.5 s b) 7.875 m above ground c) Johnny's ball velocity is 29.5 m/s downwards, and Frederique's ball velocity is 9.5 m/s downwards.

## 1.9 Motion with Non-Constant Acceleration

36. a) 9m b) 4 m/s²
37. a) 26 m/s b) 864 m

## Challenges

38. a) 3.613 s b) 41.06 m/s²
39. 7.746 m/s