A ladder rests against a smooth wall. There is no friction between the wall and the ladder, but there is a frictional force between the floor and the ladder. What is the minimum angle $\theta$ required between the ladder and the ground so that the ladder does not slip and falls to the ground?

Discover the answer to this question in this chapter.
13.1 Static Equilibrium

Statics is the study of objects at rest. There is no acceleration and no angular acceleration for these objects or for any components of these objects. Statics may seem easy, but it can become quite challenging if an object as complex as the Quebec Bridge is considered. In this case, statics is used to determine the force exerted on each of the beams of the bridge. Obviously, this part of physics is very important for engineers who have to design buildings or bridges. By knowing the forces exerted on the components, it is possible to determine what should be the composition and the size of these components so that they can withstand the force exerted.

The equations for statics are very simple and are not new. If the acceleration and angular acceleration are zero, then the following conditions must be met:

\[
\begin{align*}
\sum F_x &= 0 \\
\sum F_y &= 0 \\
\sum \tau &= 0
\end{align*}
\]

(Only the \(x\) and \(y\)-components of the forces will be considered here. There will be no \(z\)-component of the force in these notes.)

Force between Two Objects in Contact

Two objects in contact but not fastened together

The forces between two objects when they are simply in contact have already been described. These are the normal force and the friction force.

These are the forces exerted when the two objects are not fixed to each other. This means that there are no nails, no screws, no welds or any other fasteners like that. The objects are simply touching each other.
**Pivots**

Sometimes, the two objects are linked together by a pivot. This pivot connects the two objects together but allows the two objects to rotate freely. The pivot can then exert a force (which is a normal force actually), but no torque.

The normal force exerted by the pivot can be in any direction. This force between objects is then resolved into two components. It is customary to call these two components $H$ for the horizontal component, and $V$ for the vertical component.

In previous chapters, the normal force always had to be positive because the direction of the force was known and taken into account. However, it is sometimes very difficult to predict the direction of the normal force here as it can be in any direction with a pivot. Therefore, it will be assumed that the components are positive, but it will then be possible to obtain negative values for the $H$ and $V$ components. If a negative value is obtained, then the component is in the direction opposite to the assumed direction.

**Two Objects Fixed Together**

In this case, the two objects can be fixed together by screws, nails, a weld or any other method. Then, the fastening may exert a force that prevents the object from moving. Once again, this force is resolved into horizontal and vertical components, denoted $H$ and $V$. The fastening can also prevent the object from rotating, which means that there may be a torque exerted by the fastening.

Here, too, the $H$ and $V$ components can be negative. As the force between the two objects can be directed in any direction, the components can be positive or negative. The torque can also be negative since it can prevent the rotation of the object in any direction.
Resolution Method

1) Find all the forces acting on the object.

   a) Gravitational forces

      There is a force of gravity on all objects unless its mass is neglected.

   b) Contact forces

      b1) If the object touches another object without being fixed together, then the following forces are exerted:

         - A normal force, which is a repulsive force between bodies.
         - A force of friction (unless specified otherwise).

      b2) If there is a pivot, then the following forces are exerted:

         - A horizontal component $H$ of the normal force.
         - A vertical component $V$ of the normal force.

      b3) If the objects are fastened to each other, then the following forces are exerted:

         - A horizontal component $H$ of the normal force.
         - A vertical component $V$ of the normal force.
         - A torque exerted by the fixture.

   c) Forces made by strings or rods

      All strings exert a tension force.
      All rods exert a tension or a compression force.

2) Resolve these forces into $x$ and $y$-components with

   $F_x = F \cos \theta \quad F_y = F \sin \theta$

3) Calculate the torque exerted by each of these forces, taking care to give the right sign to each torque.
4) Write the static equilibrium equations.

\[ \sum F_x = 0 \]
\[ \sum F_y = 0 \]
\[ \sum \tau = 0 \]

As there’s no acceleration, the axes can be chosen freely (remember, an axis had to be in the direction of the acceleration). In statics, any orientation can be used for the axes, provided that \( x \) is perpendicular to \( y \).

5) Solve these equations to find the unknowns.

It is suggested to make steps 2 and 3 using a table like this.

<table>
<thead>
<tr>
<th>Forces</th>
<th>( x )</th>
<th>( y )</th>
<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>0</td>
<td>-294 N</td>
<td>0 Nm</td>
</tr>
<tr>
<td>String 1</td>
<td>(-T_1)</td>
<td>0</td>
<td>( T_1 ) ( 2 ) m ( \sin 30° )</td>
</tr>
<tr>
<td>String 2</td>
<td>( T_2 ) ( \cos 45° )</td>
<td>( T_2 ) ( \sin 45° )</td>
<td>(-T_2 ) ( 3 ) m ( \sin 45° )</td>
</tr>
</tbody>
</table>

It will then be easy to do step 4. Just add all the items in the \( x \) column to obtain the sum of the \( x \)-component of the forces, add all the items in the \( y \) column to obtain the sum of the \( y \)-component of the forces, and add all the items in the \( \tau \) column to obtain the sum of the torques.

**When There Is No Axis of Rotation**

The sum of torques seems to imply that there must be an axis of rotation or a pivot but sometimes there is none. If a ladder is leaning on a wall and the sum of torques must be done, where is the axis of rotation? Actually, any point can be taken as the axis of rotation in statics. However, some points are better than others. Often the best choice is to set the axis at the point where there are the largest number of unknown forces acting. This simplifies the equations to solve since the torque made by these forces then vanishes.
Examples

Example 13.1.1

In the following situation, a diving board is fastened to the left support by a pivot and simply rest on the right support. There is no friction between the right support and the board. What is the force made on the diving board by each support?

The forces acting on the board are:

1) The weight of the 5 kg board (49N downwards)
2) The force exerted by the left support ($H$ and $V$ components since this is a pivot)
3) The force exerted by the right support (An upwards normal)
4) The force exerted by the person (a downwards normal force equal to the weight of the person (588 N))

The axis of rotation can be chosen to be anywhere. It is placed on the left end of the board, where there are the largest number of unknown forces acting.
The table of force is

<table>
<thead>
<tr>
<th>Forces</th>
<th>$x$</th>
<th>$y$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight of the</td>
<td>0</td>
<td>-49 N</td>
<td>$49 \text{ N} \times 1.5 \text{ m} \times \sin 90^\circ$</td>
</tr>
<tr>
<td>board</td>
<td></td>
<td></td>
<td>$= 73.5 \text{ Nm}$</td>
</tr>
<tr>
<td>Support 1</td>
<td>$H$</td>
<td>$V$</td>
<td>$0 \text{ Nm}$</td>
</tr>
<tr>
<td>Support 2</td>
<td>0</td>
<td>$F_N$</td>
<td>$-F_N \times 1 \text{ m} \times \sin 90^\circ$</td>
</tr>
<tr>
<td>Diver</td>
<td>0</td>
<td>-588 N</td>
<td>$588 \text{ N} \times 3 \text{ m} \times \sin 90^\circ$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$= 1764 \text{ Nm}$</td>
</tr>
</tbody>
</table>

The two equations for the sum of forces are

\[
\sum F_x = H = 0
\]

\[
\sum F_y = -49N + V + F_N - 588N = 0
\]

Before making the sum of torque, check whether the problem can be solved with only the force equations (it sometimes happens). This is not the case here because there are three unknowns and only two equations. The sum of torques is then needed.

\[
\sum \tau = 73.5Nm - F_N \cdot 1m + 1764Nm = 0
\]

This equation can be solved for the normal force to obtain

\[
F_N = 1837.5N
\]

Substituting this value into the sum of the $y$-components, $V$ can be found.

\[
-49N + V + F_N - 588N = 0
\]

\[
-49N + V + 1837.5N - 588N = 0
\]

\[
V = -1200.5N
\]

Finally, it was already known that $H = 0$ from the equation for the sum of the $x$-component of the forces. Therefore, the forces are as follows.

![1837.5 N](image1.png)

![1200.5 N](image2.png)
It can be concluded that the left support must be fastened to the board because the support pulls on the board. If the board were simply resting on the support, the board would lift because a normal force cannot exert an attractive force between two objects. The pivot must also be able to provide the 1200.5 N needed, otherwise the pivot would break. Therefore, a large enough pivot must be used so that it can exert such a force without breaking. The board does not need to be fastened to the right support because it simply pushes on the board, something a normal force can easily do.

**Example 13.1.2**

In the following situation, what is the tension of the rope and what is the force exerted by the pivot on the beam (magnitude and direction)?

The forces on the beam are:

1) The weight (245 N downwards)
2) The tension force of rope 1
3) The tension force of rope 2 (equal to the weight of the 100 kg mass (980 N))
4) The force exerted by the pivot \( (H \text{ and } V) \)

The direction of the tension force of rope 1 is shown in the figure to the left.

Therefore, the table of forces is
<table>
<thead>
<tr>
<th>Forces</th>
<th>x</th>
<th>y</th>
<th>τ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight of the beam</td>
<td>0</td>
<td>-245 N</td>
<td>-245 N x 1 m x sin70°</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>= -230.2 Nm</td>
</tr>
<tr>
<td>Pivot</td>
<td>H</td>
<td>V</td>
<td>0 Nm</td>
</tr>
<tr>
<td>Tension 1</td>
<td>T cos 150°</td>
<td>T sin 150°</td>
<td>T x 2 m x sin50°</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>= T x 1.532 m</td>
</tr>
<tr>
<td>Tension 2</td>
<td>0</td>
<td>-980 N</td>
<td>-980 N x 2 m x sin70°</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>= -1841.8 Nm</td>
</tr>
</tbody>
</table>

The equations are then

\[
\sum F_x = H + T \cos(150°) = 0
\]

\[
\sum F_y = -245N + V + T \sin(150°) - 980N = 0
\]

\[
\sum \tau = -230.2Nm + T \times 1.532m - 1841.8Nm = 0
\]

There is a single unknown in the third equation. This equation can be solved for \(T\) to obtain

\[
T = 1352.4N
\]

This value can then be substituted into the two forces equations.

\[
H + T \cos(150°) = 0
\]

\[
H + 1352.4N \cos(150°) = 0
\]

\[
H = 1171.2N
\]

\[
-245N + V + T \sin(150°) - 980N = 0
\]

\[
-245N + V + 1352.4N \sin(150°) - 980N = 0
\]

\[
V = 548.8N
\]

The magnitude and the direction of the force exerted by the pivot can then be found. From the component, the magnitude and direction are

\[
F_{pivot} = \sqrt{H^2 + V^2} = 1293.4N
\]

\[
\theta = \arctan \frac{V}{H} = 25.1°
\]
Example 13.1.3

A ladder rests against a smooth wall. There is no friction between the wall and the ladder, but there is a frictional force between the floor and the ladder. What is the minimum angle $\theta$ required between the ground and the ladder so that the ladder does not slip and falls to the ground?

The forces on the ladder are:

1) The weight of the ladder (98 N downwards)
2) The tension force of the rope tied to the box (equal to the weight of the box (245 N))
3) The normal force exerted by the wall (towards the right)
4) The normal force exerted by the ground (upwards)
5) The friction force exerted by the ground

The direction of the friction force is not known. It can be towards the right or towards the left. We’ll suppose here that this force is towards the left. If a positive value is obtained, this is the correct direction. If a negative value is obtained, then the force is instead directed towards the right.

Taking the point of contact with the ground as the axis of rotation (because this is where there are the largest number of unknown forces exerted), the table of forces is:

<table>
<thead>
<tr>
<th>Forces</th>
<th>$x$</th>
<th>$y$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight of the ladder</td>
<td>0</td>
<td>-98 N</td>
<td>-98 N $\times$ 1.5 m $\times$ sin($90^\circ$ - $\theta$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>= -147 Nm $\times$ sin($90^\circ$ - $\theta$)</td>
</tr>
<tr>
<td>Tension</td>
<td>0</td>
<td>-245 N</td>
<td>-245 N $\times$ 2 m $\times$ sin($90^\circ$ - $\theta$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>= -490 Nm $\times$ sin($90^\circ$ - $\theta$)</td>
</tr>
<tr>
<td>Normal force wall</td>
<td>$F_{N1}$</td>
<td>0</td>
<td>$F_{N1}$ $\times$ 3 m $\times$ sin($\theta$)</td>
</tr>
<tr>
<td>Normal force ground</td>
<td>0</td>
<td>$F_{N2}$</td>
<td>0 Nm</td>
</tr>
<tr>
<td>Friction force ground</td>
<td>-$F_f$</td>
<td>0</td>
<td>0 Nm</td>
</tr>
</tbody>
</table>
The equations are then

\[
\sum F_x = F_{N1} - F_f = 0 \\
\sum F_y = -98N - 245N + F_{N2} = 0 \\
\sum \tau = -147Nm \cdot \sin(90^\circ - \theta) - 490Nm \cdot \sin(90^\circ - \theta) + F_{N1} \cdot 3m \cdot \sin \theta = 0
\]

Obviously, the third equation will allow us to find the angle because it is the only equation where this variable is present. This equation is

\[
-147Nm \cdot \sin(90^\circ - \theta) - 490Nm \cdot \sin(90^\circ - \theta) + F_{N1} \cdot 3m \cdot \sin \theta = 0 \\
-147Nm \cdot \cos \theta - 490Nm \cdot \cos \theta + F_{N1} \cdot 3m \cdot \sin \theta = 0 \\
637Nm \cdot \cos \theta = F_{N1} \cdot 3m \cdot \sin \theta \\
212.33N = F_{N1} \cdot \tan \theta
\]

\(F_{N1}\) must be known to calculate the angle. According to the sum of the \(x\)-component of the forces, this normal force is equal to the friction force. Therefore, the equation becomes

\[212.33N = F_f \times \tan \theta\]

As the static friction force has a maximum value equal to \(\mu F_{N2}\), the following condition must be met

\[F_f \leq \mu F_{N2}\]

This equation means that

\[F_f \tan \theta \leq \mu F_{N2} \tan \theta\]

And thus that

\[212,33N \leq \mu F_{N2} \tan \theta\]

\[212,33N \leq 0.6 \times F_{N2} \tan \theta\]

According to the sum of the \(y\)-component of the forces, this normal force is equal to 343 N. Therefore,

\[212.33N \leq 0.6 \times 343N \times \tan \theta\]

\[1.0317 \leq \tan \theta\]

\[45.89^\circ \leq \theta\]

Therefore, the minimum angle is 45.89°.
Note that if the string holding the box is tied higher up the ladder, the torque increases, which increases the minimum angle. If a person climbs the ladder, there may be an equilibrium at first. But as the person climbs, more and more friction is required at the base of the ladder. If the maximum of static friction is exceeded, the ladder may suddenly start to slip, as shown in this clip.
http://www.youtube.com/watch?v=Fa18QERtim0

Example 13.1.4

In the situation shown in the figure, determine the forces and the torque exerted by the screws on the beam.

The forces on the beam are:

1) The weight of the beam (980 N downwards)
2) The tension of the rope holding the traffic light (equal to the weight of the traffic light (490 N))
3) The force ($H$ and $V$) and the torque ($\tau$) made by the screws.

Taking the point of contact with the wall as the axis of rotation, the table of forces is
<table>
<thead>
<tr>
<th>Forces</th>
<th>$x$</th>
<th>$y$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight of the beam</td>
<td>0</td>
<td>-980 N</td>
<td>$-980 \times 2.5 \text{ m} \times \sin 90^\circ$ $= -2450 \text{ Nm}$</td>
</tr>
<tr>
<td>Weight of the light</td>
<td>0</td>
<td>-490 N</td>
<td>$-490 \times 4 \text{ m} \times \sin 90^\circ$ $= -1960 \text{ Nm}$</td>
</tr>
<tr>
<td>Screws</td>
<td>$H$</td>
<td>$V$</td>
<td>$\tau$</td>
</tr>
</tbody>
</table>

The equations are then

$$\sum F_x = H = 0$$
$$\sum F_y = -980N - 490N + V = 0$$
$$\sum \tau = -2450Nm + -1960Nm + \tau = 0$$

The first equation gives

$$H = 0$$

The second equation gives

$$-980N - 490N + V = 0$$
$$V = 1470N$$

And the third equation gives

$$-2450Nm + -1960Nm + \tau = 0$$
$$\tau = 4410Nm$$

In this last example, the forces made by screws are shown in the figure. There are upwards forces, exerting the 1470 N $V$ component of the force. The torque is obtained with horizontal forces. These two forces are of the same magnitude since $H$ must vanish. The top screws pull on the beam and the bottom screws push on the beam (actually, it is the wall that pushes on the beam with a normal force). With these forces, there is a torque that prevents the beam from rotating.
Hanging Object

The centre of mass of a hanging object can quite easily be found experimentally with the following rule:

In static equilibrium, the centre of mass is always located exactly under the attachment point.

By attaching the object at different locations and by drawing vertical lines from the attachment points each time, the position of the centre of mass can be found. The centre of mass is at the intersection of these vertical lines.

The centre of mass must be exactly under the point of attachment because the sum of torques would not be zero otherwise. Indeed, the net torque on the hanging object shown in the figure is

\[ \sum \tau = wr \sin \theta \]

As \( w \) and \( r \) are not zero, the equilibrium can be achieved only if the sine is zero. The only possible solution here is

\[ \sin 180^\circ = 0 \]
If the angle is 180°, then the centre of mass must be directly under the attachment point. (Note that $\theta = 0°$ is also a solution. This corresponds to a centre of mass exactly above the attachment point. This is actually an equilibrium position, but it is an unstable one. The slightest perturbation will destroy this equilibrium.)

**Object Resting on the Ground**

To be in equilibrium, objects on the ground must follow this rule:

*In static equilibrium, the centre of mass is always located above the area bounded by all points of support of the object.*

This figure shows what is meant by the area bounded by all points of support of the object, also called the base of support. A chair has four points of support, which delimits a square (almost). For the chair to be in equilibrium, the centre of mass must be above this square. If the chair rests on only two legs, there are only two points of support. The area bounded by the points of support becomes a line connecting the two legs touching the ground. It then becomes almost impossible for the chair to be in equilibrium in this case because, even if it were possible to put the centre of mass above this line, the slightest disturbance would move the centre of mass so that it would no longer be above the line. The chair then falls.

For a human standing on its feet, the area bounded by the points of support is represented in the figure. It includes the bottoms of both feet (in red) and the area between the feet (in pink). To be in equilibrium, the centre of mass of the person must be above this area. If one foot is lifted, the area bounded by the points of support is then reduced to the area under the other foot. The centre of mass must, therefore, be above the foot on the ground for the person to be in equilibrium.

This picture shows a can of Mountain Dew in equilibrium. The area bounded by the points of support is not very big, but they were able to place the centre of mass above this area. This requires drinking a very precise amount of liquid so that the centre of mass is located at the right place.

schoo.ls.wikia.com/wiki/Center_of_Mass
Now consider a box resting on an inclined plane. Initially, when the angle of slope is small, the centre of mass is above the surface bounded by the points of support (which is the base of the box here). Then, the box is in equilibrium. The 2nd image represents the maximum angle at which the box is still in equilibrium. The centre of mass is still above the base of the box, but barely. If the angle in increased slightly, the centre of mass is no longer above the base of the box and it falls.

Suppose now that an empty glass is in equilibrium on an incline. If water is added, the centre of mass of the glass rises and this may destroy the equilibrium if the centre of mass is no longer above the surface bounded by the points of support (here, a circle that corresponds to the foot of the glass). If this happens, the glass topples.

If the person shown on the left of the figure tries to touch her feet, her centre of mass stays above her feet and she does not fall. To be able to do this without falling, her rear end moves a little backwards when she leans so that the centre of mass remains above her feet. If her rear end is against a wall, it cannot move backwards and the centre of mass of her body will probably not stay above the base of support as she bends. She then falls.

You can admire here a series of demonstrations on equilibrium in this clip. 
http://www.youtube.com/watch?v=2VpzHJ_R55I
It can be shown quite easily that the centre of mass must be above the area bounded by the points of support. Suppose that the centre of mass is not above the area bounded by the points of support.

If the centre of mass is taken as the axis of rotation, the torque made by any normal force acting on the support area is positive (if the positive direction shown in the figure is used). It is, therefore, impossible to have a vanishing net torque because there is no negative torque to cancel these positive torques. Therefore, equilibrium is impossible here.

The result is different if there is another point of support on the other side of the centre of mass. Normal forces acting to the right of the centre of mass exert a positive torque, and normal forces acting to the left of the centre of mass exert a negative torque. It is, therefore, possible for the net torque to be zero. As normal forces acting on each side of the centre of mass are needed to have an equilibrium, it follows that the centre of mass must be between the supporting points.

13.3 Equilibrium of an Accelerating Object

Maximum Acceleration without Tipping Over

A box is placed in a truck. It will be assumed that the friction coefficient is sufficiently large so that the box does not slip. What is the maximum acceleration that the truck can have so that the box does not topple?
If the box were to topple, it would fall towards the back of the truck and pivot on his lower right corner. Thus, there would be a rotating motion. If the box does not topple, then there’s no rotation and the net torque must vanish. The greatest acceleration for which the net torque can be zero will now be sought. This means that we have to consider the extreme situation: the box is just about to topple. It is, therefore, in equilibrium on its lower right corner. The forces exerted on the box are those shown in the figure. The dimensions shown are the width of the box \((D)\) and the height of the centre of mass \((H)\).

When an object accelerates like that, the axis is not fixed (it accelerates with the object) and the centre of mass must be used as the axis of rotation. Thus, the weight does not exert any torque as the force acts on the centre of mass. On the other hand, the friction force and the normal force both exerts torques. Here, these torques can be calculated faster with the formula

\[
\tau = Fr_{\perp}
\]

Therefore, the net torque is

\[
\sum \tau = F_N \frac{D}{2} - F_f H = 0
\]

The sum is zero because the angular acceleration is zero. Remember that the box is in equilibrium on its lower right corner. If it were actually tipping over, the acceleration would not be zero.

The normal force and the friction force can be found with the sum of forces.

\[
\sum F_x = F_f = ma
\]
\[
\sum F_y = -mg + F_N = 0
\]

The sum of the \(x\)-component of the forces is equal to \(ma\) because the box must follow the motion of the truck. Since the truck accelerates, the box must have the same acceleration.

If these equations are solved for the normal force and the friction force and the values are substituted into the torque equation, the torque equation becomes

\[
F_N \frac{D}{2} - F_f H = 0
\]
This leads to

**Maximum Acceleration of Object that Accelerates without Tipping Over on a Horizontal Surface**

\[
a_{\text{max}} = \frac{gD}{2H}
\]

The maximum acceleration of the box does not depend on its mass but depends on its width and on the height of its centre of mass. If the box is wider (large \(D\)), it is more difficult to topple the box, and the acceleration can be larger. If the centre of mass is lower (small \(H\)), the maximum acceleration can also be larger. These maximum accelerations will be compared with two examples.

**Example 13.3.1**

The box in this truck is 2 m wide and 1 m high. What is the maximum acceleration of the truck for which the box will not tip over?

As the height of the centre of mass of the box is 0.5 m, the maximum acceleration is

\[
a_{\text{max}} = \frac{9.8 \text{ m/s}^2 \times 2 \text{ m}}{2 \times 0.5 \text{ m}} = 19.6 \text{ m/s}^2
\]

This is a relatively high acceleration. In fact, this box will likely slip before tipping over because a coefficient of friction of 2 between the box and the truck would be required to achieve this acceleration without slipping. As it is rather uncommon to have such a high friction coefficient, the box will probably slip before toppling.

**Example 13.3.2**

The box in this truck is 1 m wide and 2 m high. What is the maximum acceleration of the truck for which the box will not tip over?
As the height of the centre of mass of the box is 1 m, the maximum acceleration is
\[
a_{\text{max}} = \frac{9.8 \ m/\text{s}^2 \times 1 \ m}{2 \times 1 \ m} = 4.9 \ m/\text{s}^2
\]
This is much less than for the box of the previous example. This box will probably tip over before slipping (a coefficient of friction lower than 0.5 would be required for this box to slip before tipping).

**Equilibrium and the Direction of the Apparent Weight**

An interesting result is found if the direction of the apparent weight when the box is about to tip over is calculated. The components of the apparent weight are
\[
w_{\text{app} \ x} = -ma_x = -m \left( \frac{-gD}{2H} \right) = \frac{mgD}{2H}
\]
\[
w_{\text{app} \ y} = -mg
\]
Therefore, the direction of the apparent weight is
\[
\tan \theta = \frac{w_{\text{app} \ y}}{w_{\text{app} \ x}}
\]
\[
= \frac{-mg}{\frac{mgD}{2H}}
\]
\[
= \frac{-H}{\frac{1}{2}D}
\]
This result means that the apparent weight vector is directly pointing towards the corner of the box. Indeed, when the vector points towards the corner, the tangent of the angle is
\[
\tan \theta = \frac{-H}{\frac{1}{2}D}
\]
Thus, the apparent weight points towards the corner of the box at the maximum acceleration.

If the acceleration is smaller than the maximum acceleration, it can easily be shown that the apparent point is more vertical and is pointing towards the base of the box. In this case, the box does not topple.
If the acceleration is larger than the maximum acceleration, it can easily be shown that the apparent weight is less vertical. It then points towards a place outside the base of the box. In this case, the box topples.

This leads to the following conclusion.

To have an object in equilibrium, the apparent weight of the object must point towards the area bounded by all points of support of the object.

**Cars and Bikes in a Curve**

These results can be applied to driving. In a curve, a car has a centripetal acceleration. This means there is a maximum acceleration that the car should not exceed, otherwise, the car rolls over. As centripetal acceleration depends on speed, there is a maximum speed. If the car exceeds this maximum speed, it rolls over. This maximum speed can be found with the maximum acceleration equation.

\[
\begin{align*}
a &= \frac{gD}{2H} \\
\frac{v^2}{r} &= \frac{gD}{2H} \\
v &= \sqrt{\frac{grD}{2H}}
\end{align*}
\]

For a car, \(D\) is the distance between the wheels (called the *track*) and \(H\) is the height of the centre of mass. So, it is better to have a very low centre of mass compared to the track, as a formula 1 race car, for example.

Vehicles that have a high centre of mass compared to the track have a greater tendency to roll over. Trucks are obviously part of this category because they have a track similar to cars but their centre of mass is much higher. This is why warnings are posted to prevent trucks for taking tight curves too quickly. If a truck goes too fast in a curve, this is what happens.

https://www.youtube.com/watch?v=rD4KOm-f3eE
After trucks, sports utility vehicles (SUVs) are the most likely to roll over because their centre of gravity is rather high compared to their track. In these clips, three SUVs roll over.

http://www.youtube.com/watch?v=WjhNnEL26Vs
http://www.youtube.com/watch?v=kOC4PjCdHKY
http://www.youtube.com/watch?v=I60smSzHGgM

These results also explain why motorcycles must lean in a curve. Then, the apparent weight of the bike and passenger must point towards the support area, so towards the location where the wheel touches the road.

Knowing this, the leaning angle of the bike can be found. The magnitude of the $x$-component of the apparent weight of the bike is $\frac{mv^2}{r}$ and the magnitude of the $y$-component of the apparent weight is $mg$. Therefore, the angle is

$$\tan \theta = \frac{\frac{mv^2}{r}}{mg}$$

$$\tan \theta = \frac{v^2}{rg}$$

This angle can also be found with the sum of the torque. The following figure shows the forces acting on the bike.

The sum of the torques, measured from the centre of mass (as must be done), is

$$\sum \tau = F_N d \sin \theta - F_f d \cos \theta = 0$$

where $d$ is the length of the dotted line in the figure. Since the normal force is equal to the weight and the friction force is equal to the centripetal force, the equation becomes

$$mgd \sin \theta - \frac{mv^2}{r} d \cos \theta = 0$$

Solving this equation for the angle, the results is
If the bike goes faster, there is more centripetal acceleration. With more acceleration, the angle between the vertical and the apparent weight increases. This means that leaning angle of a bike in a curve increases with speed, as stated in the formula.

\[ mgd \sin \theta = \frac{mv^2}{r} d \cos \theta \]
\[ g \sin \theta = \frac{v^2}{r} \cos \theta \]
\[ \tan \theta = \frac{v^2}{rg} \]

**Common Mistake: Invoke a Centrifugal Force to Explain the Leaning Angle of Motorcycles.**

If you do a little research on the internet to find an explanation for the leaning angle of bikes, you will inevitably find an explanation like this:

*On the bike, a gravitational force and a centrifugal force act. The torque made by these two forces (measured from the point of contact with the ground) must cancel, which allows determining the angle of inclination.*

[Forum link](forum.motoline-france.com/viewtopic.php?f=64&t=1265&start=60)

Of course, this cannot be correct since there is no such thing as a centrifugal force. This explanation was even used in “Pour la Science” of July 2003 and a similar explanation was used in “Pour la Science” of February 2016 to explain why bobsleds move up the wall in curves!

(Small nuance: the explanation with the centrifugal force is correct if an accelerating frame of reference is used. With such a frame, fictitious forces appear, and one of these forces is called the centrifugal force. However, I think that the intricacies of accelerating frames must be avoided when trying to explain basic physics).
Equilibrium conditions

\[ \sum F_x = 0 \]
\[ \sum F_y = 0 \]
\[ \sum \tau = 0 \]

In static equilibrium, the centre of mass is always located exactly under the attachment point.

In static equilibrium, the centre of mass is always located above the area bounded by all points of support of the object.

Maximum Acceleration of Object that Accelerates without Tipping Over on a Horizontal Surface

\[ a_{\text{max}} = \frac{gD}{2H} \]

To have an object in equilibrium, the apparent weight of the object must point towards the area bounded by all points of support of the object.
13.1 Static Equilibrium

1. What are the tensions of the two cables supporting this beam?

2. What are the tension of the rope and the force made by the pivot on the beam (magnitude and direction) in this situation?
3. What are the tension of the rope and the force made by the pivot on the beam (magnitude and direction) in this situation?

![Diagram of a 70 kg load and a 20 kg load on a beam with a pivot at one end.]

4. What is the force exerted by each of these two workers supporting this 100 kg beam?

![Diagram of two workers, Sam and Joe, supporting a 100 kg beam. Sam is at 1.00 m and Joe is at 2.00 m from the pivot.]

www.chegg.com/homework-help/questions-and-answers/physics-archive-2010-november-09

5. Etienne supports a rod as illustrated in the figure. What is the force exerted by each of his hands?

6. If this cat moves forwards a little, the board begins to tip. What is the mass of the board?
7. What is the tension of the rope supporting this 200 kg bridge and what is the force exerted by the pivot (magnitude and direction)?

8. Gaëlle climbs along a tree trunk as shown in the figure. The trunk has a length of 5 m, and its centre of mass is 2 m from the larger end of the trunk. There is no friction between the trunk and the vertical wall while the coefficient of static friction between the trunk and the ground is 0.8. Can Gaëlle walk up to the end of the trunk without the log slipping?
9. Lying on this board, Annabelle wants to determine how far her centre of mass is from her feet.

[Image of a person lying on a board with scales showing 380 N and 280 N at either end, and a distance of 2.50 m between them.]

www.d.umn.edu/~djohns30/phys1001-1/examples/examples3.htm

a) What is Annabelle’s mass?
b) How far from her feet is Annabelle’s centre of mass?

10. Masses of 125 kg and 275 kg are suspended from each end of a 4 m lever having a negligible mass. Where should the fulcrum be placed for this system to be in equilibrium (x in the figure)?

[Image of a lever with masses of 125 kg and 275 kg at opposite ends, with x and 4-x labeled.]

mathcentral.uregina.ca/QQ/database/QQ.09.07/s/eric1.html
11. In the situation shown in the figure, determine the force (magnitude and direction) and the torque exerted by the screws on the beam.

![Beam Diagram](https://www.chegg.com/homework-help/questions-and-answers/shown-uniform-beam weighing-650-attached-wall-point-beam-is-subjected-three-forces-460-150-2-q786950)

12. A horizontal beam with a mass of 40 kg and a length of 2 m is fastened to a wall. A 30 kg block is placed on the beam at a distance $x$ from the wall.

a) Knowing that the screws can exert a maximum torque of 800 Nm, at what maximum distance from the wall can the 30 kg block be placed so that the screws do not give?

b) At this maximum distance, what are the horizontal and vertical components of the force that the screws exert on the beam?
13.2 Static Equilibrium and Centre of Mass

13. Up to what angle can this surface be tilted before this box of uniform density tips over?

14. Roxanne built a tower made of Lego blocks with the following shape.

There are 3 blocks to the left, and then all the other blocks shift to the right. What is the maximum number of blocks that Roxanne can put in this tower (total number of blocks) before the tower falls over to the right?

13.3 Equilibrium of an Accelerating Object

15. The box in this truck is 2 m high and 50 cm wide. The centre of mass of the box is in the middle of the box.

a) What is the maximum acceleration of the truck for which the box will not tip over?

b) What is the minimum value of the coefficient of friction that will ensure that the box will tip over before sliding?
16. The box in this truck is 2 m high and 1 m wide. The centre of mass of the box is in the middle of the box. What is the maximum acceleration of the truck for which the box will not tip over if the truck is going uphill on a 10° slope?

![Image](www.canstockphoto.com/delivery-cargo-truck-13682465.html)

17. In this curve, the bike is inclined 40° from the vertical. The radius of curvature of the curve is 100 m.

   a) What is the speed of the bike (in km/h)?

   b) What must be the minimum friction coefficient between the asphalt and the tire so that the bike take this turn at that speed without slipping?

![Image](motorcycle.com.vsassets.com/blog/wp-content/uploads/2013/10/Motorcycle-Cornering-Sparks-1014.jpg)

**Challenges**
(Questions more difficult than the exam questions.)

18. In the situation shown in the figure, there is no friction. The mass of the beam is 50 kg. What is the tension of the rope?

![Image](Eugene Hecht, Physique mécanique, groupe Modulo, 2006, P 235)
19. The following figure shows that the beam supports a load over its entire length. When the load is thus distributed, the force density \( f \) is given. For example, a force density of 50 N/m means that the total force is 50 m over a beam length of 1 m. However, the force is not evenly distributed here. It increases gradually so that there is no force near the pivot and that there is a force density of 900 N/m at the other end of the beam. What is the tension of the string?

ANSWERS

13.1 Static Equilibrium

1. Rope on the left = 784 N       Rope on the right = 588 N
2. \( T = 626.13 \text{ N} \)     pivot: 795.1 N at 38.0°
3. \( T = 653.3 \text{ N} \)     pivot: 792.8 N at 135.5°
4. Sam: 383.5 N     Joe: 596.5 N
5. Right hand: 140 N downwards     Left hand: 238 N upwards
6. 7.5 kg
7. \( T = 2377.7 \text{ N} \)     pivot: 1050.5 N at -9.9°
8. She cannot walk up to the end of the trunk
9. a) 62.3 kg     b) 1.045 m from her feet
10. 2.75 m
11. The force is 1470 N directed downwards; the torque is 3430 Nm clockwise
12. a) 1.388 m     b) \( V = 686 \text{ N upwards} \)     \( H = 0 \)
13.2 Static Equilibrium and Centre of Mass

13. 63.4°
14. 13

13.3 Equilibrium of an Accelerating Object

15. a) 2.45 m/s²   b) 0.25
16. 3.124 m/s²
17. a) 103.2 km/h   b) 0.839

Challenges

18. 443.1 N
19. 1990 N