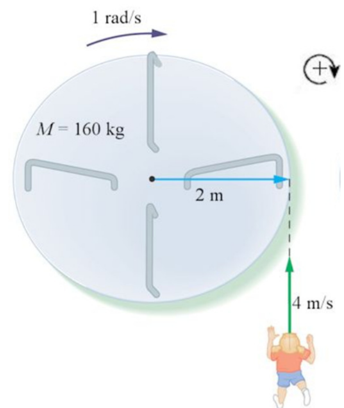
12 ROTATION

A 60 kg (fat) child runs at 4 m/s towards a circular plate that is initially spinning in the direction indicated in the diagram. What is the angular speed when the child is on the plate?



user.physics.unc.edu/~rowan/p24site/p24units/unit11/WCHAP11-7.html

www.lat34.com/2013/06/24/2013-x-games-real-street-semi-finals/

Discover the answer to this question in this chapter.

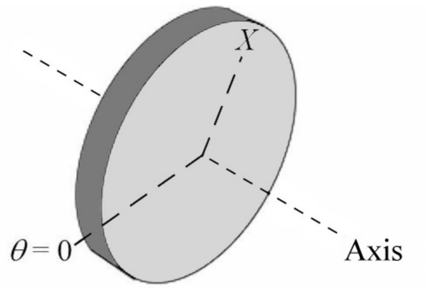
Rotation of objects will now be considered. From now on, an object can have two motions: a motion in a straight line (call *translation* or *linear motion*) and a rotating motion. For example, a ball rolling on an incline have both types of motions simultaneously.

In this chapter, rotation around a fixed axis (i.e. whose orientation does not change) will be considered most of the time. This, therefore, excludes the description of the motion of a spinning top, which turns on itself while the orientation of the axis of rotation changes.

12.1 ROTATIONAL KINEMATICS

Angular Position, Velocity and Acceleration

To describe the motion of a rotating object, the orientation of the object must be known. This orientation can be given with an angle.



Suppose that there is a mark on the object (X on the diagram). The orientation of the object can be described by specifying the angle between the direction this mark and a reference angle $\theta = 0$. This angle will be given in radians here. The positive direction for the angle is arbitrary and must be chosen each time.

The angular displacement corresponds to the change in orientation of the object.

Angular Displacement

$$\Delta\theta = \theta_2 - \theta_1$$

Angular velocity is used to determine if an object spins more or less quickly. If the object spins quickly, the angle changes quickly and if the object spins slowly, the angle changes slowly. This suggests that the angular velocity corresponds to the rate of change of the angle.

Average Angular Velocity

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous Angular Velocity

$$\omega = \frac{d\theta}{dt}$$

These speeds are in rad/s.

Angular speed is the absolute value of angular velocity.

In this chapter, only the rotation of rigid objects will be described. Rigid objects are solid objects that do not deform so that the angular velocity is the same for each atom of the object. A juice container is not a rigid object since it is possible to turn the container without turning the juice inside it. As the angular velocity of the juice is not the same as the angular velocity of the container then, it is not a rigid object. Note that at the beginning of this chapter, objects will also not change shape as they rotate.

The angular velocity is related to the period if the angular velocity is constant. Remember that the period (T) is the time it takes for the object to make one complete revolution. One revolution (2π rad) during the period of rotation (T) gives the following angular velocity.

Relationship between Angular Velocity and Period (if ω is Constant)

$$\omega = \frac{2\pi}{T}$$

However, the angular velocity can change. If a spinning wheel slows down, then its angular velocity decreases. To describe this change in angular velocity, the rate of change of angular velocity is used. This rate is the angular acceleration.

Average Angular Acceleration

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous Angular Acceleration

$$\alpha = \frac{d\omega}{dt}$$

These accelerations are in rad/s²

Formally, angular velocity and angular acceleration are vectors. With a fixed axis, it is not really necessary to consider them as vectors. Only the sign (depending on our chosen positive direction) is sufficient in this case.

Equations of Rotational Kinematics for Constant Angular Acceleration

If the angular acceleration is a constant, the definitions given above can be integrated to obtain the equations of rotational kinematics.

The results of these integrations are

$$\alpha = \frac{d\omega}{dt} \quad \rightarrow \quad \omega = \omega_0 + \alpha t$$

and

$$\omega = \frac{d\theta}{dt} \quad \rightarrow \quad \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

This was done rather quickly, but these are similar to the calculations made in the first chapter to get the equations of kinematics for linear motion. As in Chapter 1, these two equations can also be combined to form two other equations. The four following equations are thus obtained.

Rotational Kinematics Equations for Constant Angular Acceleration

$$\omega = \omega_o + \alpha t$$

$$\theta = \theta_0 + \omega_o t + \frac{1}{2}\alpha t^2$$

$$2\alpha\left(\theta - \theta_0\right) = \omega^2 - \omega_0^2$$

$$\theta = \theta_0 + \frac{1}{2}\left(\omega_0 + \omega\right)t$$

As in linear kinematics, one of these four equations will, most of the time, solve the problem directly.

Example 12.1.1

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The pulley in this diagram, initially at rest, begins to rotate in the positive direction with a constant angular acceleration of 3 rad/s².

a) What is the angular velocity 2 seconds later?

The velocity is

$$\begin{aligned}\omega &= \omega_0 + \alpha t\\ &= 0 + 3\tfrac{rad}{s^2}\cdot 2s\\ &= 6\tfrac{rad}{s}\end{aligned}$$

b) What is the number of revolutions made by the pulley in 10 seconds?

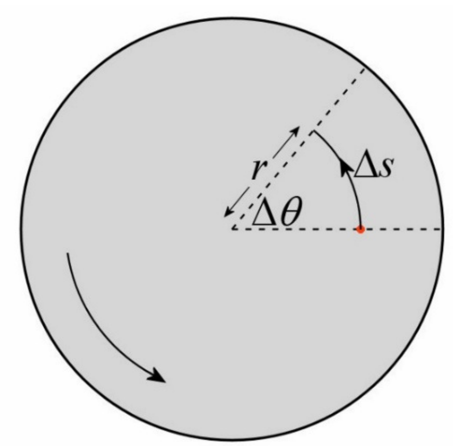
To find the number of revolutions made, the angular displacement must be found. If the initial position is the origin ($\theta_0 = 0$), the final angle is, therefore, the angular displacement. This angle is

$$\begin{aligned}\theta &= \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2\\ &= 0 + 0 + \frac{1}{2}\cdot 3\tfrac{rad}{s^2}\cdot\left(10s\right)^2\\ &= 150rad\end{aligned}$$

Therefore, the pulley has rotated 150 rad. As one revolution is made for each rotation of 2π rad, the number of revolutions is

$$N = \frac{150\,rad}{2\pi\,rad} = 23.87$$

The Distance Travelled, the Speed and the Acceleration at a Certain Place on the Rotating Object



At a certain distance from the axis on a rotating plate, a certain distance is travelled as the object rotates (Δs in the diagram). There is, of course, a link between the distance travelled (Δs) and the angular displacement of the object ($\Delta\theta$). Actually, this link comes from the definition of the angle in radians.

The definition of angle (in radians) is

$$\Delta\theta = \frac{\Delta s}{r}$$

Therefore,

Distance Travelled at a Distance *r* from the Axis if the Angular Displacement is $\Delta\theta$

$$\Delta s = r\Delta\theta$$

The speed of an object staying at distance r from the axis can now be found. If the previous equation is divided by the time, the average speed is obtained.

$$\frac{\Delta s}{\Delta t} = \frac{r\Delta\theta}{\Delta t}$$

If this time now tends to zero, the equation becomes

$$\frac{ds}{dt} = \frac{rd\theta}{dt}$$
$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

The last line comes from the fact that we are looking for the speed of an object staying at the same distance from the axis, which means that r is constant and can be taken outside the derivative. As the remaining derivatives are the speed and the angular speed, the equation becomes

Speed at a Distance *r* From the Axis

$$v = r\omega$$

Thus, the farther an object is from the axis of rotation, the faster it goes. This is what can be seen in this video.
http://www.youtube.com/watch?v=atzsv63a1FE
Forget Maurice's sexist comments and focus on the figure skaters. The skaters must skate faster the further away they are from the axis of rotation. This is because the speed (v) increases as the distance from the axis of rotation increases if the angular velocity (ω) is constant. In the clip, the angular speed is constant since each skater makes the same angle during the same time in order to keep the line of skaters straight.

Finally, the acceleration at a distance r from the axis can be found if the speed equation is derived.

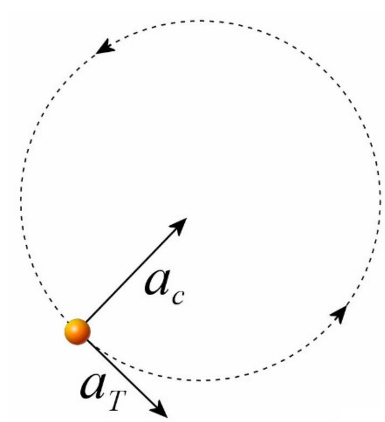
$$v = r\omega$$

$$\frac{dv}{dt} = \frac{d(r\omega)}{dt}$$

$$\frac{dv}{dt} = r\frac{d\omega}{dt}$$

$$a = r\alpha$$

As this speed is the tangential speed (the object always remains at the same distance r from the axis), the acceleration obtained is actually the tangential acceleration (a_T). In the diagram, this acceleration is shown as being in the same direction as the rotation (the speed would increase then). It could also have been directed in the opposite direction to the rotation (the speed would decrease then).



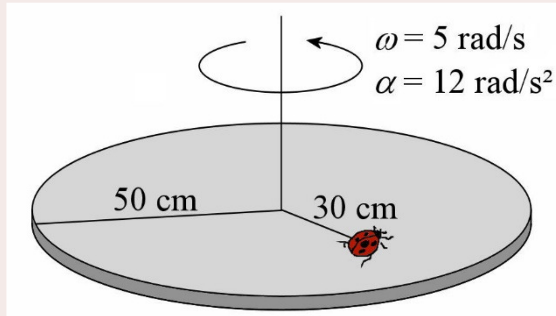
But be careful: a centripetal acceleration (a_c) is also present. Therefore, the acceleration is

Magnitude of the Acceleration at a Distance *r* from the Axis

Tangential Acceleration	$a_T = r\alpha$
Centripetal Acceleration	$a_c = \frac{v^2}{r} = r\omega^2$
Acceleration	$a = \sqrt{a_T^2 + a_c^2}$

Example 12.1.2

A bug is 30 cm from the axis of rotation of a rotating disk having a diameter of 1 m. The initial angular velocity is 5 rad/s, and it increases at the rate of 12 rad/s².



a) What is the speed of the bug at this time?

The speed is

$$\begin{aligned} v &= r\omega \\ &= 0.3m \cdot 5\tfrac{rad}{s} \\ &= 1.5\tfrac{m}{s} \end{aligned}$$

b) What is the magnitude of the acceleration at this time?

The magnitude of the acceleration is

$$a = \sqrt{a_T^2 + a_c^2}$$

The tangential and centripetal accelerations must be found first.

The tangential acceleration is

$$\begin{aligned} a_T &= r\alpha \\ &= 0.3m \cdot 12\tfrac{rad}{s^2} \\ &= 3.6\tfrac{m}{s^2} \end{aligned}$$

The centripetal acceleration is

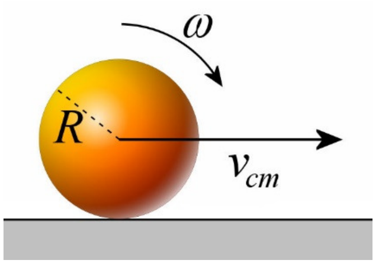
$$\begin{aligned} a_c &= r\omega^2 \\ &= 0.3m \cdot \left(5\tfrac{rad}{s}\right)^2 \\ &= 7.5\tfrac{m}{s^2} \end{aligned}$$

Therefore, the magnitude of the acceleration is

$$\begin{aligned} a &= \sqrt{a_T^2 + a_c^2} \\ &= \sqrt{\left(3.6\tfrac{m}{s^2}\right)^2 + \left(7.5\tfrac{m}{s^2}\right)^2} \\ &= 8.32\tfrac{m}{s^2} \end{aligned}$$

Rolling without Slipping Conditions

If a sphere or a cylinder rolls without slipping on the ground, the rotation must be done at a very precise rate. There is then a link between the speed of the centre of mass and the angular velocity because the object has to spin more quickly if the speed increases. This is common sense: If the speed of a car increases, the wheels must spin faster.



To find the link between the two, imagine that a wheel makes on revolution during time T. As it makes on revolution, it travels a distance equal to the circumference of the wheel. The speed is, therefore,

$$v_{cm} = \frac{2\pi R}{T}$$

As the wheel makes one revolution during the same time, the angular speed is

$$\omega = \frac{2\pi}{T}$$

Comparing these two equations, the link between speed and angular velocity is easily found.

Link Between Speed and Angular Velocity if There Is Rolling Without Slipping

$$v_{cm} = \omega R$$

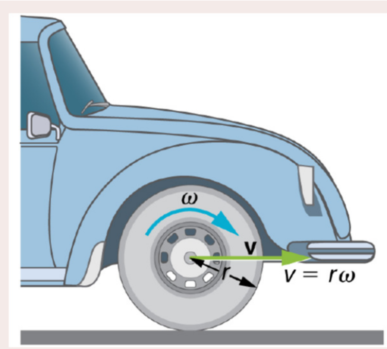
If this equation is derived with respect to time, the connection between acceleration and angular acceleration is found.

Link Between Acceleration and Angular Acceleration if There Is Rolling Without Slipping

$$a_{cm} = \alpha R$$

Example 12.1.3

A car travels at 90 km/h. How many revolutions per minute do the wheels make if they have a diameter of 60 cm?



College physics, openstax

The speed of the centre of mass of the wheel is the same as the speed of the car. Thus, the angular velocity of the wheel is

$$\omega = \frac{v_{cm}}{R} = \frac{25 \frac{m}{s}}{0.3m} = 83.33 \frac{rad}{s}$$

Therefore, the number of revolutions per second is

$$\frac{83.33 \frac{rad}{s}}{2\pi \frac{rad}{revolution}} = 13.26 \frac{revolution}{s}$$

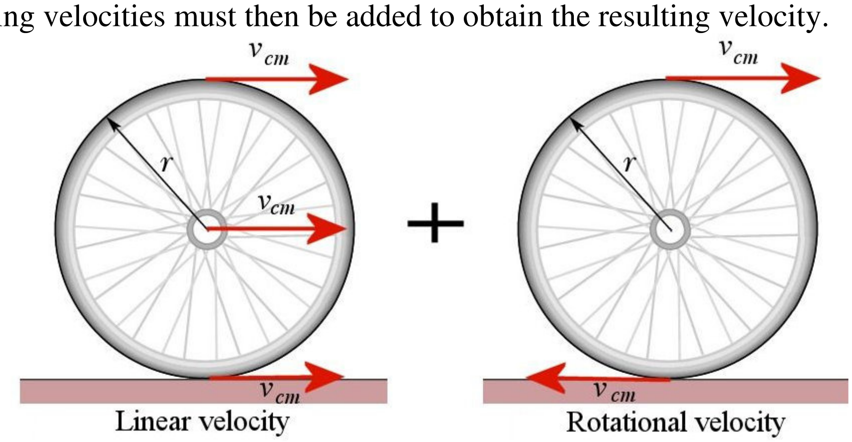
In one minute, the number of revolutions is

$$13.26 \frac{revolutions}{s} \cdot 60 \frac{s}{min} = 795.8 \frac{revolutions}{min}$$

The velocity at different points on the wheel can now be found. This velocity is the result of the addition of two velocities: the linear velocity, which is the same everywhere, and the rotational velocity, which increases with the distance from the axis. On the edge of the wheel, the rotation speed is equal to the speed of the centre of mass if there is no slipping, because

$$v = \omega r = \frac{v_{cm}}{R} R = v_{cm}$$

The following velocities must then be added to obtain the resulting velocity.



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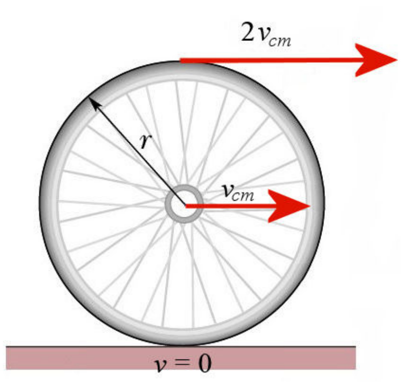
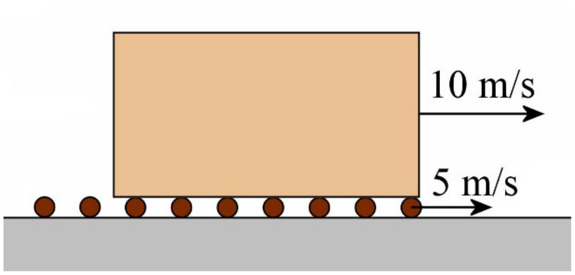


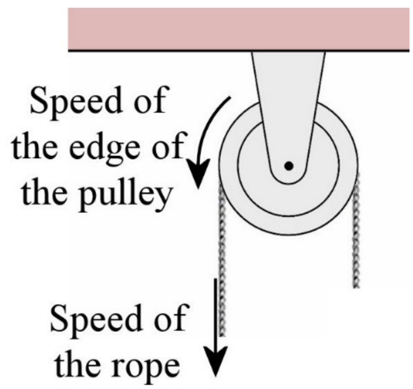
The result of this addition is shown in the diagram to the left. The top of the wheel goes at $2v_{cm}$, and the bottom of the wheel is at rest. It is quite normal for the lower part of the wheel to have no speed since it does not slip. If the ground has no speed, the bottom of the wheel must have the same speed if the wheel does not slip.

This means that if a car moves at 100 km/h, the top of the wheels moves at 200 km/h.



It also means that when a large block of stone rolling on logs is pushed, the block, in contact with the top of the logs, is moving twice as fast as the logs (see diagram).

Rope on a Pulley Conditions



When a rope passes through a pulley without slipping, the speed of the rope must be the same as the speed of the pulley at the place where the rope passes through the pulley.

This means that if the rope passes through the pulley at a distance *r* from the axis of rotation, the speed of the pulley at that distance from the axis must be equal to the speed of the rope. This implies that

$$v_{rope} = v_{pulley}$$
$$v_{rope} = \omega r$$

If this equation is derived with respect to time, the accelerations are obtained.

$$a_{rope} = \alpha r$$

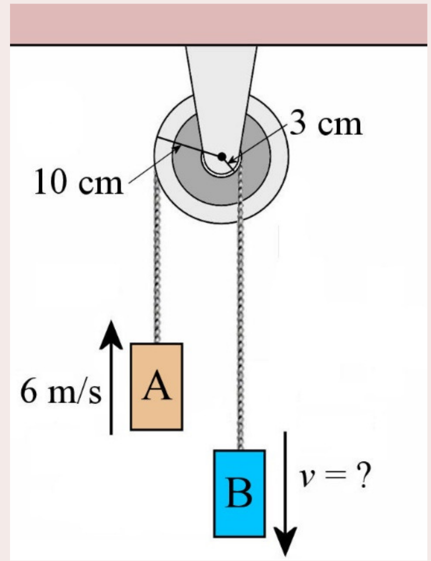
Therefore, the conditions are as follows.

Link Between Speeds and Accelerations for a Rope Passing Through a Pulley Without Slipping

$$v_{rope} = \omega r$$
$$a_{rope} = \alpha r$$

where *r* is the distance between the rope and the axis of rotation

Example 12.1.4



In the situation shown in the diagram, what is the speed of block B if the strings do not slip on the pulley?

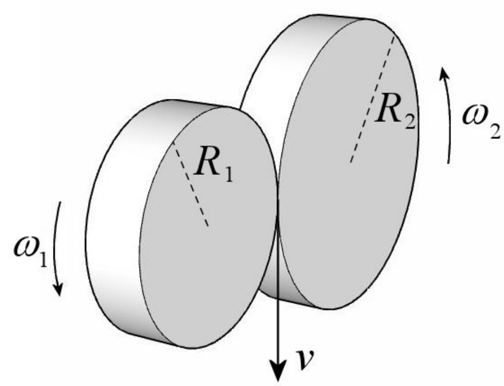
The string on the left has the same speed as block A. As the rope passes through the pulley 100 mm from the axis, the speed of the pulley 100 mm from the axis must be 6 m/s. Therefore, the angular speed of the pulley is

$$v_{rope\,1} = \omega r$$
$$6\tfrac{m}{s} = \omega \cdot 0.1m$$
$$\omega = 60\tfrac{rad}{s}$$

The rope on the right has the same speed as block B. As the rope passes through the pulley 30 mm from the axis, the speed of the pulley 30 mm from the axis must be equal to the speed of block B. This speed is

$$v_{rope\,2} = \omega r$$
$$v_B = 60\tfrac{rad}{s} \cdot 0.03m$$
$$v_B = 1.8\tfrac{m}{s}$$

Transmission of Rotational Motion



If two wheels are in contact and there is no slippage between the two, there is a link between the angular speeds of the wheels. As the wheels touch without slipping, the speed of the wheels must be the same at the point of contact. This means that

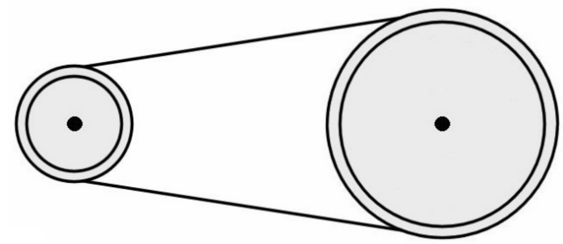
$$v_1 = v_2$$

As the speed is ωR, this becomes

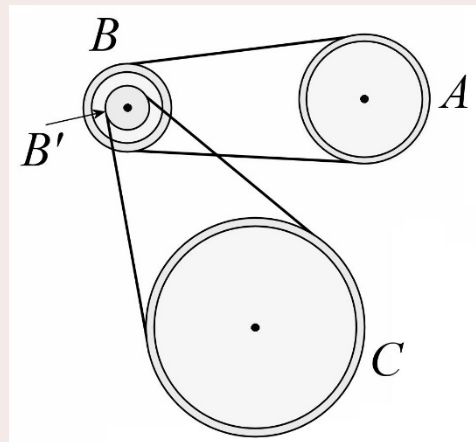
Rotational Motion Transmission

$$\omega_1 R_1 = \omega_2 R_2$$

This relationship remains valid even if the wheels are not in contact, but are instead connected by a belt or a chain. In this case, R represents the distance from the axis of rotation to the place where the strap or chain is in contact with the wheel.



Example 12.1.5



Pulley A, whose radius is 15 cm, spins with an angular velocity of 10 rad/s. Pulley B has a radius of 10 cm, pulley B', which is welded to pulley B, has a radius of 5 cm, and pulley C has a radius of 25 cm. What is the angular velocity of the pulley C?

The angular velocity of pulley B must be found first. Its angular velocity is

$$\omega_A R_A = \omega_B R_B$$
$$10\tfrac{rad}{s} \cdot 0.15m = \omega_B \cdot 0.1m$$
$$\omega_B = 15\tfrac{rad}{s}$$

Pulley B' spins with the same angular velocity as it is welded to pulley B. Then, the angular speed of pulley C is found with

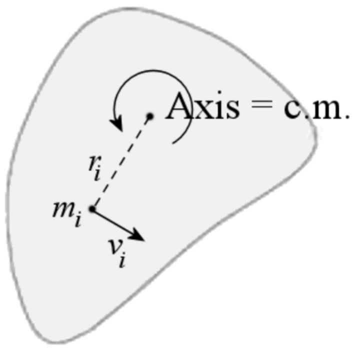
$$\omega_{B'} R_{B'} = \omega_C R_C$$
$$15\tfrac{rad}{s} \cdot 0.05m = \omega_C \cdot 0.25m$$
$$\omega_C = 3\tfrac{rad}{s}$$

12.2 ROTATIONAL KINETIC ENERGY AND MOMENT OF INERTIA

Object Spinning Around a Stationary Centre of Mass

The total kinetic energy is calculated by summing the kinetic energy of each little mass composing the object.

$$E_k = \sum \frac{1}{2} m_i v_i^2$$



As the speed at a distance r from the axis of rotation is $v = \omega r$, the energy is

$$E_k = \sum \frac{1}{2} m_i \left(\omega r_i\right)^2$$

As the angular velocity is the same for all the small masses, it is a constant that can be taken out of the sum. Then, the energy becomes

$$\begin{aligned} E_k &= \sum \frac{1}{2} m_i \left(\omega r_i\right)^2 \\ &= \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2 \end{aligned}$$

The quantity in parentheses is a quantity that comes up often in rotation. It is called the *moment of inertia* (name given by Euler in 1758).

Moment of Inertia of an Object

$$I = \sum m_i r_i^2$$

In this equation, r is the distance between the small mass and the axis of rotation (which is at the centre of mass here). The units for the moment of inertia are kg m².

Later, it will be shown that the moment of inertia indicates whether an object is hard to set into rotation. The higher the moment of inertia is, the harder it is to set the object into rotation. It is the rotational equivalent of mass, which measures the inertia for linear motion. However, in rotation, inertia depends not only on the mass but also on the shape of the object.

The term "moment" may intrigue you. It is not used here in the sense of "instant" but rather in its mathematical sense. A moment is a quantity multiplied by a distance to a certain power. If the power is two, as is the case here with r^2, it is an order 2 moment. Note that the centre of mass of an object is a moment of order 1. This terminology will be present everywhere in this chapter.

In the moment of inertia calculation, the distance between the small mass and the axis of rotation is used. Here, as the axis is at the centre of mass, the moment of inertia is calculated using the distances between each small mass and the centre of mass. To indicate that the moment of inertia is calculated here using the distances from the centre of mass, the moment of inertia is denoted I_{cm}.

Using the formula for the moment of inertia, the kinetic energy

$$E_k = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

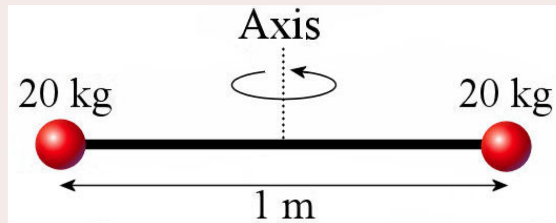
then becomes

Kinetic Energy of an Object Rotating Around a Stationary Axis Passing Through the Centre of Mass

$$E_k = \frac{1}{2} I_{cm} \omega^2$$

Example 12.2.1

The object shown in the diagram spins at the rate of 1 revolution per second. What is its kinetic energy? (The mass of the rod is neglected and the 20 kg masses are considered to be point masses.)



To calculate the kinetic energy, the moment of inertia of the object must be known first. As each 20 kg mass is 50 cm from the axis, we have

$$\begin{aligned} I_{cm} &= \sum m_i r_i^2 \\ &= m_1 r_1^2 + m_2 r_2^2 \\ &= 20kg \cdot (0.5m)^2 + 20kg \cdot (0.5m)^2 \\ &= 10kgm^2 \end{aligned}$$

Therefore, the kinetic energy of rotation is

$$\begin{aligned} E_k &= \frac{1}{2} I_{cm} \omega^2 \\ &= \frac{1}{2} \cdot 10kgm^2 \cdot \left(2\pi \tfrac{rad}{s}\right)^2 \\ &= 197.4J \end{aligned}$$

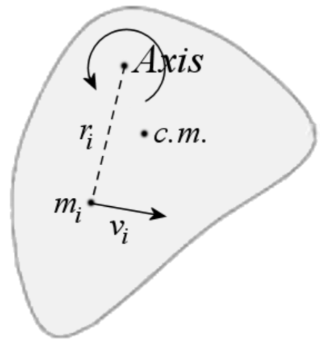
If the speed of each mass had been found, the kinetic energy could have been calculated by summing the $\frac{1}{2}mv^2$ of each mass. The answer would have been the same then.

Note that an object spinning in the air or in space must spin around its centre of mass.

Object Spinning Around a Stationary Axis Not at the Centre of Mass of the Object

Kinetic Energy Formula

The total kinetic energy is calculated by summing the kinetic energy of each little mass composing the object.



$$E_k = \sum \frac{1}{2} m_i v_i^2$$

As the speed at a distance r from the axis of rotation is $v = \omega r$, the energy is

$$E_k = \sum \frac{1}{2} m_i \left(\omega r_i\right)^2$$
$$= \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

In this formula, the famous sum of the mass times the distance squared is found. This sum is the moment of inertia

$$I = \sum m_i r_i^2$$

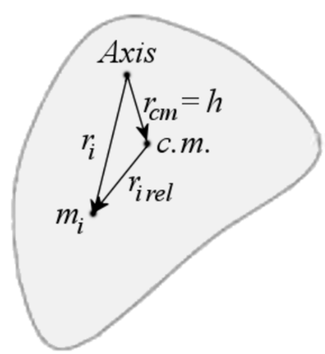
So the kinetic energy is

Kinetic Energy of an Object Rotating Around an Axis Not Passing Through the Centre of Mass

$$E_k = \frac{1}{2} I \omega^2$$

Moment of Inertia With an Axis Not Passing Through the Centre of Mass

The moment of inertia is this last calculation is different as the distances are not measured from the axis at the centre of mass, as was the case with I_{cm}. Instead, they are measured from an axis located somewhere else. This changes all the distances and thus completely changes the result of the sum. Therefore, the value of I of an object depends on the position of the axis of rotation.



Fortunately, it is possible to know the moment of inertia with any axis if I_{cm} is known. Since the position of a small mass is

$$\vec{r}_i = \vec{r}_{cm} + \vec{r}_{i\,rel}$$

(r_{rel} is the position of each mass measured relative to the centre of mass.)

The moment of inertia is then

$$I = \sum m_i \left(\vec{r}_{i\,rel} + \vec{r}_{cm}\right)^2$$
$$= \sum m_i \left(r_{i\,rel}^2 + 2\vec{r}_{i\,rel} \cdot r_{cm} + r_{cm}^2\right)$$
$$= \sum m_i r_{i\,rel}^2 + \sum m_i 2\vec{r}_{i\,rel} \cdot \vec{r}_{cm} + \sum m_i r_{cm}^2$$

$$
\begin{aligned}
&= \sum m_i r_{i\,rel}^2 + 2\left(\sum m_i \vec{r}_{i\,rel}\right)\cdot \vec{r}_{cm} + \left(\sum m_i\right) r_{cm}^2 \\
&= \sum m_i r_{i\,rel}^2 + 2\left(\sum m_i \vec{r}_{i\,rel}\right)\cdot \vec{r}_{cm} + m r_{cm}^2
\end{aligned}
$$

The second term vanishes since

$$\frac{1}{m}\sum m_i \vec{r}_{i\,rel} = \vec{r}_{cm\,rel}$$

is the position of the centre of mass relative to the centre of mass. It is obvious that the distance between the centre of mass and the centre of mass is zero! Therefore, $r_{cm\,rel} = 0$ and the term vanishes. The moment of inertia then becomes

$$I = \sum m_i r_{i\,rel}^2 + m r_{cm}^2$$

The first term is the moment of inertia when the axis is at the centre of mass (I_{cm}) since all the r_{rel} are the distances measured from the centre of mass. In the second term, r_{cm}^2 is the distance between the axis and the centre of mass, a distance called h. The moment of inertia is thus

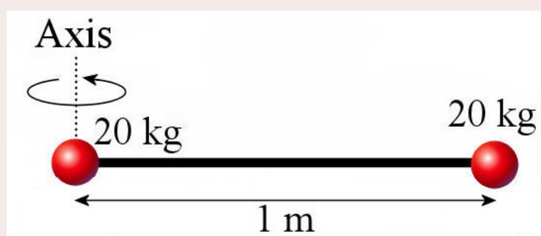
I From I_{cm} (Parallel Axes Theorem)

$$I = I_{cm} + mh^2$$

Note that these two axes (the one passing through the centre of mass and the one not passing through the centre of mass) must be parallel for this formula to be true.

Example 12.2.2

The object shown in the diagram spins at a rate of 1 revolution per second. What is its kinetic energy? (The mass of the rod is neglected and the 20 kg masses are considered to be point masses.)



To calculate the kinetic energy, the moment of inertia of the object must be known first. There are two ways to calculate this moment. The first is

$$
\begin{aligned}
I &= \sum m_i r_i^2 \\
&= m_1 r_1^2 + m_2 r_2^2 \\
&= 20kg \cdot (0m)^2 + 20kg \cdot (1m)^2 \\
&= 20kgm^2
\end{aligned}
$$

The second is

$$\begin{aligned} I &= I_{cm} + mh^2 \\ &= 10kgm^2 + 40kg \cdot (0.5m)^2 \\ &= 20kgm^2 \end{aligned}$$

The value of I_{cm} found in example 12.2.1 was used. As the centre of mass of this object is equidistant from the two balls, the distance between the centre of mass and the axis is h = 0.5 m.

Therefore, the kinetic energy is

$$\begin{aligned} E_k &= \frac{1}{2} I\omega^2 \\ &= \frac{1}{2} \cdot 20kgm^2 \cdot \left(2\pi \tfrac{rad}{s}\right)^2 \\ &= 394.8J \end{aligned}$$

Object Moving and Spinning Around its Centre of Mass

An object can have a linear motion while spinning around its centre of mass. This is the case for the wheels of a car or for a projectile turning on itself during its flight. Then the kinetic energy can be calculated with (formula at the end of the previous chapter)

$$E_k = \frac{1}{2} m v_{cm}^2 + \sum \frac{1}{2} m_i v_{i\,rel}^2$$

The velocity v_{rel} is the difference in speed between the centre of mass and the small mass.

$$\vec{v}_{rel} = \vec{v} - \vec{v}_{cm}$$

Since the speed of the centre of mass is subtracted, the effect of the linear motion is eliminated. This means that only the rotational speed must be considered in the calculation of v_{rel}. Therefore, $v_{rel} = \omega r$, and the kinetic energy is

$$\begin{aligned} E_k &= \frac{1}{2} m v_{cm}^2 + \sum \frac{1}{2} m_i v_{i\,rel}^2 \\ &= \frac{1}{2} m v_{cm}^2 + \sum \frac{1}{2} m_i \left(\omega r_{i\,rel}\right)^2 \\ &= \frac{1}{2} m v_{cm}^2 + \frac{1}{2} \left(\sum m_i r_{i\,rel}{}^2\right) \omega^2 \end{aligned}$$

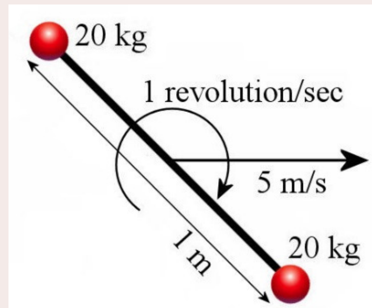
Since the term in parentheses is the moment of inertia calculated from the centre of mass, the energy becomes

Kinetic Energy of an Object Making Both Linear and Rotational Motion

$$E_k = \underbrace{\frac{1}{2}mv_{cm}^2}_{\text{Translation } E_k} + \underbrace{\frac{1}{2}I_{cm}\omega^2}_{\text{Rotation } E_k}$$

Therefore, the energy is a simple sum of the kinetic energies of translation and rotation.

Example 12.2.3



What is the kinetic energy of this object moving at 5 m/s and spinning at the rate of 1 revolution/s?

The kinetic energy is

$$E_k = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$

Since we already know that the moment of inertia of this object is 10 kgm² (example 12.2.1), the energy is

$$\begin{aligned} E_k &= \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 \\ &= \frac{1}{2}\cdot 40kg\cdot\left(5\tfrac{m}{s}\right)^2 + \frac{1}{2}\cdot 10kgm^2\cdot\left(2\pi\tfrac{rad}{s}\right)^2 \\ &= 500J + 197.4J \\ &= 697.4J \end{aligned}$$

12.3 MOMENT OF INERTIA OF EXTENDED OBJECTS

The moment of inertia of a system is

$$I = \sum m_i r_i^2$$

When there are only a few masses that make up the system, as in the previous examples where there were only two masses, it is not very long to calculate the moment of inertia. However, it can become quite long if the moment of inertia of a sphere, for example, is needed. Then, the sum must be done over all the atoms forming the sphere…

Fortunately, the calculation of the moment of inertia can also be done with calculus. The object is divided into small infinitesimal masses, and the sum is done with the following integral.

$$\lim_{m\to 0}\sum r_i^2 m_i = \int r^2\, dm$$

However, the calculation of the moment of inertia with this integral remains quite complex most of the time. For a two-dimensional object, two double integrals must be calculated while three triple integrals must be calculated for three-dimensional objects. As you have never calculated double or triple integrals of this kind, this type of calculation will not be done here. However, the calculation can be done quite easily for a one-dimensional object such as a rod.

To calculate the moment of inertia, the rod is divided into small pieces of infinitesimal length. The moment of inertia is then

$$I = \int x^2 \, dm$$

Each piece has a length *dx* and a mass *dm* and *x* is the distance between the small mass and the axis of rotation.

The linear density of the small piece is

$$\lambda = \frac{mass}{length} = \frac{dm}{dx}$$

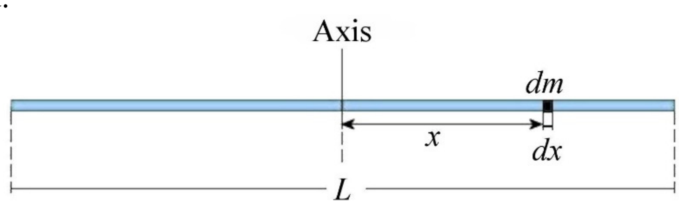
Therefore, the mass of the small piece is

$$dm = \lambda \, dx$$

Thus, the moment of inertia is given by

$$I = \int x^2 \lambda \, dx$$

This formula can now be used to calculate the moment of inertia of a uniform rod ($\lambda =$ constant) when the axis passes through the centre of mass. The following diagram represents this rod.



The moment of inertia is

$$I_{cm} = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \lambda \, dx$$

$$= \lambda \left[\frac{x^3}{3} \right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \lambda \left[\frac{L^3}{24} - \frac{-L^3}{24} \right]$$

$$= \lambda \frac{L^3}{12}$$

As the linear density of the rod is

$$\lambda = \frac{m}{L}$$

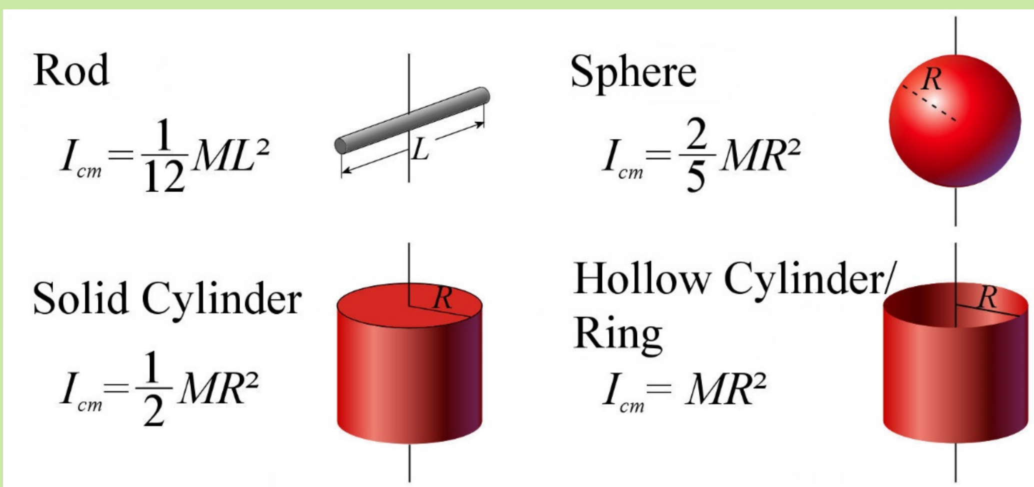
the moment of inertia is

$$I_{cm} = \frac{m}{L} \frac{L^3}{12}$$

$$= \frac{1}{12} mL^2$$

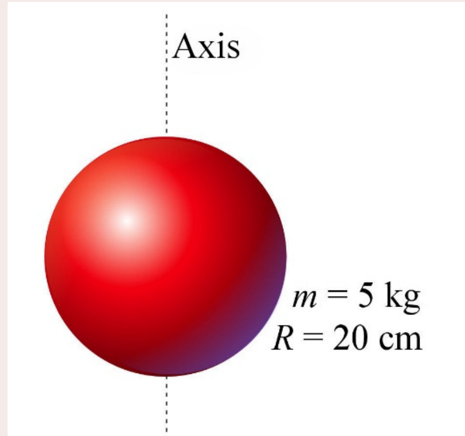
By doing similar calculations, but in three dimensions, the moment of inertia of multiple objects can be found. Here are the results for 4 common shapes.

Moment of Inertia of 4 Common Uniform Objects



Many other moments of inertia have been calculated. A more thorough list can be found on this site.
http://www.livephysics.com/tables-of-physical-data/mechanical/moment-of-inertia.html

Example 12.3.1

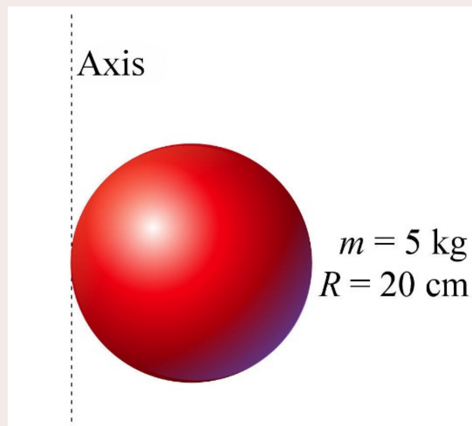


What is the moment of inertia of this sphere?

With a sphere and an axis of rotation passing through the centre of mass, moment of inertia is

$$\begin{aligned} I_{cm} &= \tfrac{2}{5} MR^2 \\ &= \tfrac{2}{5} \cdot 5kg \cdot (0.2m)^2 \\ &= 0.08kgm^2 \end{aligned}$$

Example 12.3.2



What is the moment of inertia of this sphere?

This time, the axis does not pass through the centre of mass of the sphere. The moment of inertia is therefore

$$I = I_{cm} + mh^2$$

The first term (I_{cm}) is the moment of inertia of a sphere when the axis passes through the centre of mass. This moment of inertia is

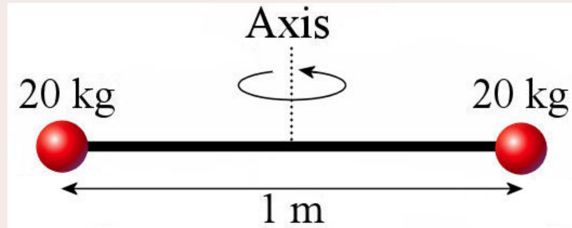
$$I_{cm} = \tfrac{2}{5} MR^2$$

In the second term, h is the distance from the axis of rotation to the centre of mass of the sphere. Here, this distance is 20 cm. The moment if inertia is thus

$$\begin{aligned} I &= \tfrac{2}{5} MR^2 + mh^2 \\ &= \tfrac{2}{5} \cdot 5kg \cdot (0.2m)^2 + 5kg \cdot (0.2m)^2 \\ &= 0.28kgm^2 \end{aligned}$$

Example 12.3.3

What is the moment of inertia of the object shown in the diagram? The mass of the rod is 1.2 kg, and radius of the spheres is 5 cm. (This time, we do not neglect the mass of the rod and do not consider that the two 20 kg spheres are point masses.)



This object consists of a rod and two spheres.

The moment of inertia of the rod (which rotates around its centre of mass) is

$$\begin{aligned} I_1 &= \frac{1}{12} mL^2 \\ &= \frac{1}{12} \cdot 1.2kg \cdot (0.9m)^2 \\ &= 0.081 kgm^2 \end{aligned}$$

The moment of inertia of the sphere on the right (which does not rotate around its centre of mass) is

$$\begin{aligned} I_2 &= \frac{2}{5} mR^2 + mh^2 \\ &= \frac{2}{5} \cdot 20kg \cdot (0.05m)^2 + 20kg \cdot (0.50m)^2 \\ &= 5.02 kgm^2 \end{aligned}$$

The moment of inertia of the sphere on the left (which does not rotate around its centre of mass) is identical to the moment of inertia of the sphere on the right.

$$I_3 = 5.02 kgm^2$$

Therefore, the total moment of inertia is

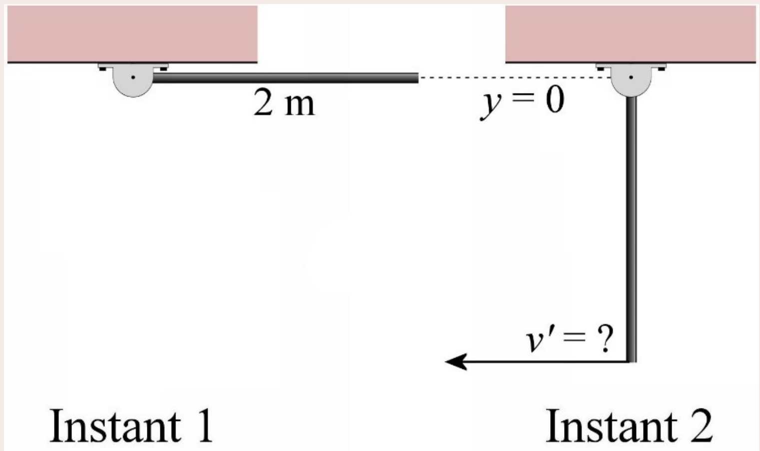
$$\begin{aligned} I &= I_1 + I_2 + I_3 \\ &= 0.081 kgm^2 + 5.02 kgm^2 + 5.02 kgm^2 \\ &= 10.121 kgm^2 \end{aligned}$$

12.4 MECHANICAL ENERGY CONSERVATION

The law of conservation of mechanical energy will now be applied to rotating objects. One of the critical steps now consists of choosing the correct kinetic energy formula that applies to the situation.

Example 12.4.1

A 2 m rod is held in a horizontal position. At one end of the 6 kg rod, there is a pivot. The rod is then released. What will the speed of the other end of the rod be when the rod is in a vertical position?



Mechanical Energy Formula

As there is only one object, the mechanical energy is

$$E_{mec} = E_k + mgy$$

With rotation, the kinetic energy is not simply $\frac{1}{2}mv^2$. This is an object rotating around a stationary axis not passing through the centre of mass. Therefore, the kinetic energy is

$$E_k = \frac{1}{2} I \omega^2$$

The moment of inertia is the moment of inertia of a rod with the axis at the end. This moment can be found with the parallel axes theorem.

$$\begin{aligned} I &= I_{cm} + mh^2 \\ &= \frac{1}{12} mL^2 + mh^2 \\ &= \frac{1}{12} \cdot 6kg \cdot (2m)^2 + 6kg \cdot (1m)^2 \\ &= 8kgm^2 \end{aligned}$$

h is equal to 1 m since the centre of mass is in the midpoint of the rod and the axis of rotation is at the end of the rod. The distance between the two is, therefore, equal to half the length of the rod.

Note that this calculation could also have been done in this way

$$\begin{aligned} I &= \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 \\ &= \frac{1}{12}mL^2 + \frac{1}{4}mL^2 \\ &= \left(\frac{1}{12} + \frac{1}{4}\right)mL^2 \\ &= \frac{1}{3}mL^2 \\ &= \frac{1}{3}\cdot 6kg\cdot(2m)^2 \\ &= 8kgm^2 \end{aligned}$$

The mechanical energy at instants 1 and 2 will now be calculated. The calculation is done with the origin $y = 0$ set at the height of the axis of rotation.

Mechanical Energy at Instant 1

At instant 1, the angular speed of the rod is zero and the centre of mass of the rod is at $y = 0$. Therefore, the energy is

$$\begin{aligned} E &= \frac{1}{2}I\omega^2 + mgy \\ &= 0 \end{aligned}$$

Mechanical Energy at Instant 1

At instant 2, the centre of mass of the rod is at $y = -1$ m. The energy is thus

$$\begin{aligned} E' &= \frac{1}{2}I\omega'^2 + mgy' \\ &= \frac{1}{2}I\omega'^2 + mg(-1m) \end{aligned}$$

Mechanical Energy Conservation

$$E = E'$$

$$0 = \frac{1}{2}I\omega'^2 + mg(-1m)$$

$$0 = \frac{1}{2}\cdot 8kgm^2\cdot\omega'^2 + 6kg\cdot 9.8\tfrac{N}{kg}\cdot(-1m)$$

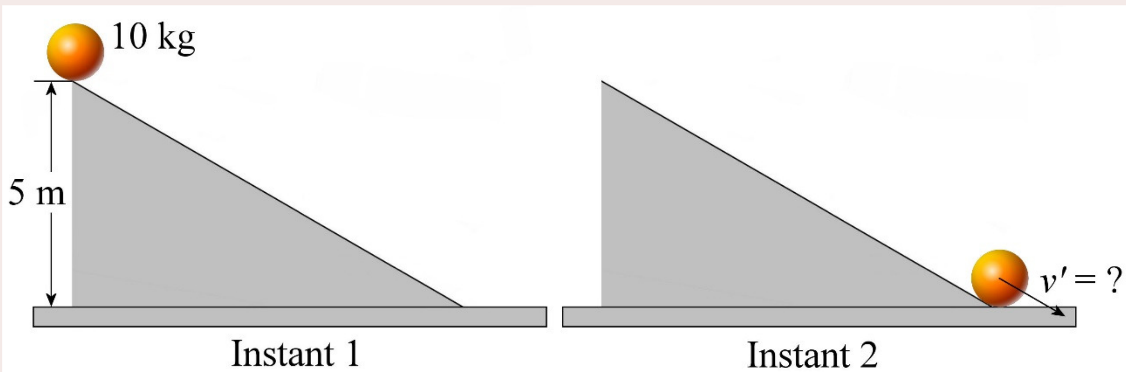
$$\omega' = 3.83\tfrac{rad}{s}$$

The speed of the end of the rod can finally be found from the angular speed.

$$\begin{aligned} v &= \omega r \\ &= 3.83 \tfrac{rad}{s} \cdot 2m \\ &= 7.66 \tfrac{m}{s} \end{aligned}$$

Example 12.4.2

A ball initially at rest rolls without slipping down a 5 m high incline (h = 5 m). The ball has a mass of 10 kg and a radius of 20 cm. What is the speed of the ball at the bottom of the slope?



Mechanical Energy Formula

As there is only one object, the mechanical energy is

$$E_{mec} = E_k + mgy$$

This object has a linear motion while spinning around its centre of mass. Therefore, the kinetic energy is

$$E_k = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

The mechanical energy is then

$$E_{mec} = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2 + mgy$$

Mechanical Energy at Instant 1

At instant 1, the speed and the angular speed are zero. The centre of mass of the ball is at $y = h + R$ = 5 m + 0.2 m = 5.2 m. (The origin y = 0 was set at the ground.) The energy is thus

$$\begin{aligned} E &= \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 + mgy \\ &= 0 + 0 + 10kg \cdot 9.8\tfrac{N}{kg} \cdot (5.2m) \\ &= 509.6J \end{aligned}$$

Mechanical Energy at Instant 2

At instant 2, the ball has a speed, and its centre of mass is at y = 0.2 m. The energy is thus

$$\begin{aligned} E' &= \frac{1}{2}mv_{cm}'^2 + \frac{1}{2}I_{cm}\omega'^2 + mgy' \\ &= \frac{1}{2}mv_{cm}'^2 + \frac{1}{2}I_{cm}\omega'^2 + 10kg \cdot 9,8\tfrac{N}{kg} \cdot (0.2m) \\ &= \frac{1}{2}mv_{cm}'^2 + \frac{1}{2}I_{cm}\omega'^2 + 19,6J \end{aligned}$$

To calculate this energy, the moment of inertia of the sphere must be found. This moment is

$$\begin{aligned} I_{cm} &= \frac{2}{5}mR^2 \\ &= \frac{2}{5} \cdot 10kg \cdot (0.2m)^2 \\ &= 0.16kgm^2 \end{aligned}$$

There is also a link between the angular velocity and the velocity of the centre of mass since the sphere rolls without slipping. This link is

$$v_{cm} = \omega R = \omega \cdot 0.2m$$

Therefore, the mechanical energy at the instant 2 is

$$\begin{aligned} E' &= \frac{1}{2}mv_{cm}'^2 + \frac{1}{2}I_{cm}\omega'^2 + 19.6J \\ &= \frac{1}{2} \cdot 10kg \cdot v_{cm}'^2 + \frac{1}{2} \cdot 0.16kgm^2 \cdot \left(\frac{v_{cm}'}{0.2m}\right)^2 + 19.6J \\ &= 5kg \cdot v_{cm}'^2 + 2kg \cdot v_{cm}'^2 + 19.6J \\ &= 7kg \cdot v_{cm}'^2 + 19.6J \end{aligned}$$

Mechanical Energy Conservation

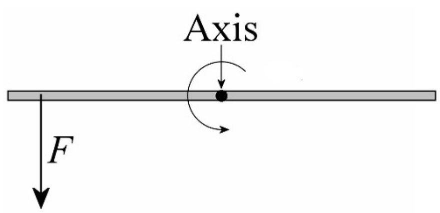
The law of conservation of mechanical energy then gives

$$E = E'$$
$$509.6J = 7kg \cdot v_{cm}'^2 + 19.6J$$
$$v_{cm}' = 8.367 \tfrac{m}{s}$$

12.5 TORQUE

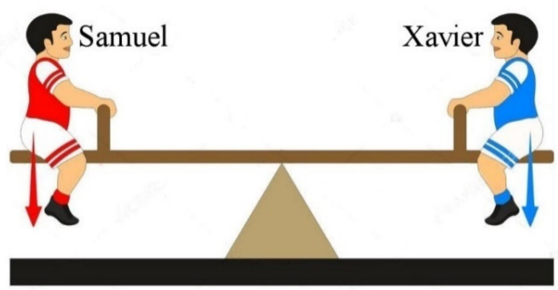
Torque Magnitude

We will now look at the rotational equilibrium of an object.



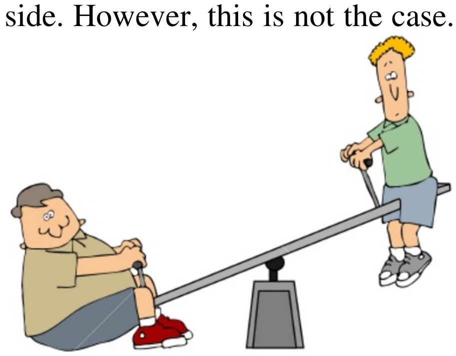
To achieve this, the effects of a force on the rotation of an object must be known. Suppose an object can spin around an axis of rotation. If a force is exerted on this object initially at rest, the object starts to turn. The force, therefore, causes angular acceleration.

To have an equilibrium of rotation, it is therefore necessary that the effect of all forces cancels each other out. For example, the forces exerted by these 2 twins on a swing cancel each other out. The force made by Samuel seeks to rotate the board counterclockwise while the force exerted by Xavier seeks to rotate the board clockwise. The effects of the two forces cancel each other out.



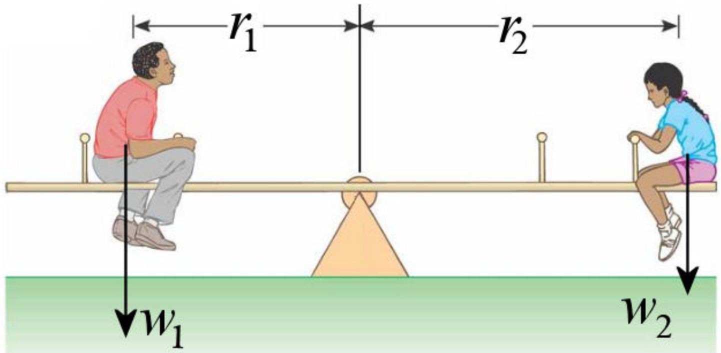
wdrfree.com/stock-vector/download/trigonometry-static-equilibrium-infographic-diagram-209863788

Xavier and Samuel exert identical forces on each side and one might therefore think that there is equilibrium if the forces are identical on each side. However, this is not the case. The magnitude of the force is not the only thing that counts for rotation, the point of application of the force is also very important. Let's illustrate this with an example. Suppose two people, one fat and one skinny, sit on a swing. If the two people are seated at the same distance from the axis of rotation, it is known that there can be no equilibrium. The person with the lowest mass finds himself perched up in the air.



francais.istockphoto.com/photo-9020464-balancoire-bascule-enfant-obese-cartoon-petits-garcons.php

However, it is possible to balance the swing even if the forces on each side are uneven. By changing the point of application of a force, the equilibrium can be restored. For a swing, this can be achieved by changing the sitting position of one of the people, as shown in this diagram.



www.chegg.com/homework-help/questions-and-answers/boy-mass-98kg-seesaw-41kg-girl-girl-distance-b-44m-center-seesaw-find-distance-seesaw-perf-q37449846

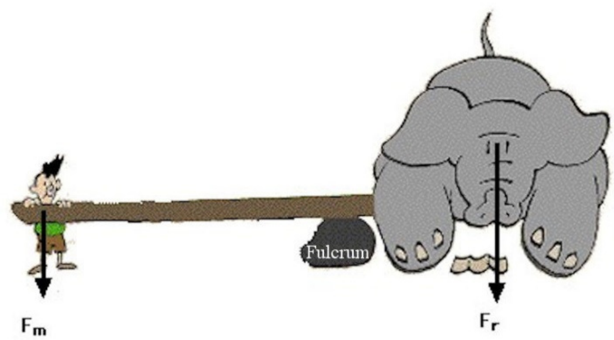
The fact that the force on the left is applied closer to the axis compensates for the force being larger. This demonstrates that the location of the point of application of the force is also important in rotation.

The equilibrium condition in this situation has been known for a long time. Archimedes already knew it in 250 BC. Using the variables shown in the last diagram, the condition is

$$w_1 r_1 = w_2 r_2$$

Thus, if the father has a weight twice as large as his daughter's weight, he must sit twice as close to the axis of rotation for the equilibrium to be reached.

With a small force, heavy objects could be lifted using such a lever. A child could lift an elephant if the distance between the axis and the point of application of the force is large enough on the side of the child. This is why Archimedes said that he could lift the Earth if he had a fulcrum and a long enough lever.



perso.b2b2c.ca/login/JP/mecanique/machsimp.html

Therefore, the force multiplied by the distance determines the effect of a force in rotational dynamics. This quantity is called the torque (or moment of force). It is denoted by the symbol τ.

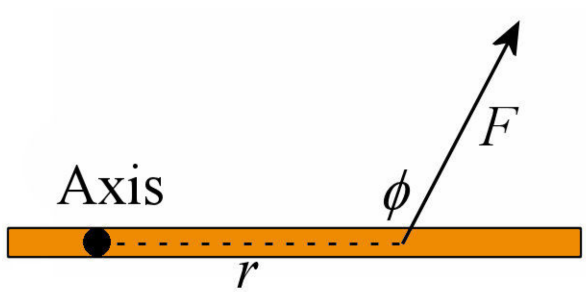
$$\tau = Fr$$

The SI units for torque are Nm. (Even if a Nm is equivalent a joule, the joule is never used for torque.)

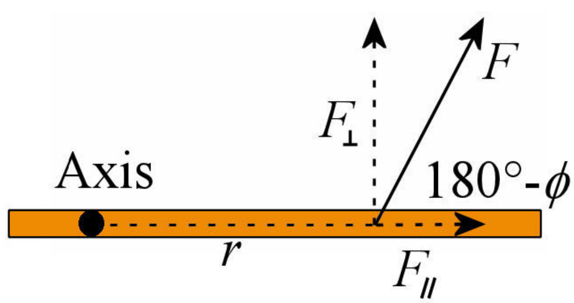
This first formula of the torque is not our most general formula since it is limited to forces perpendicular to a line going from the point of application of force to the axis of rotation. As soon as the third century BC, Archimedes had generalized this concept by examining the equilibrium of the inclined levers and levers with bend beams as shown in the image. Heron of Alexandria (1st century AD) and Jordanus de Nemore (13^{th} century) also obtained correct solutions for the equilibrium of such objects.

catalogue.museogalileo.it/object/FirstorderLeverWithBentBeam.html

The following diagram shows a situation when the force is not perpendicular to the line from the axis at the point of application of the force.



To determine how this force influence the rotation, it is resolved into two components. There is a component perpendicular to the line *r* and a component parallel to the line *r*.



Obviously, the parallel component cannot make the object spin; only the perpendicular component can make the object rotate. The magnitude of the torque exerted by this perpendicular component is

$$\tau = F_\perp r$$

The perpendicular component of the force *F* is

$$F_\perp = F \sin\left(180° - \phi\right)$$
$$F_\perp = F \sin\phi$$

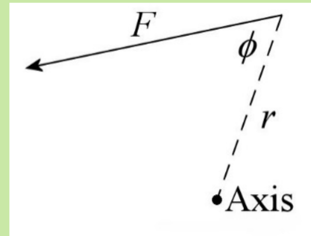
since sin(180°-ϕ) = sin ϕ. Thus, the general formula for the torque is

Torque (or Moment of Force)

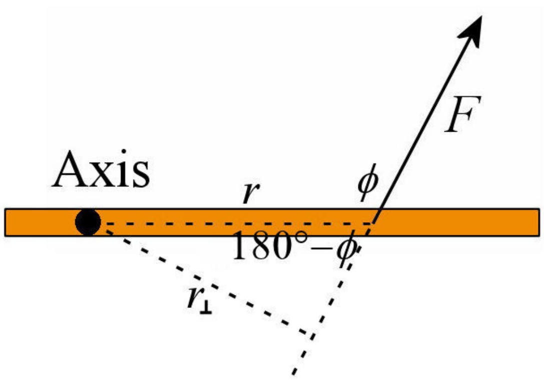
$$\tau = Fr\sin\phi$$

or

$$\tau = F_{\perp} r$$



There is another formula, equivalent to the last one, for the torque. To find it, a straight line extending the force vector is drawn. The shortest distance between this straight line and the axis of rotation is then measured ($r_{\perp}$ on the diagram). This shortest distance is



$$\frac{r_{\perp}}{r} = \sin\left(180° - \phi\right)$$

$$r_{\perp} = r\sin\left(180° - \phi\right)$$

$$r_{\perp} = r\sin\phi$$

This distance is called the *lever arm* or the *moment arm*. The moment of force

$$\tau = Fr\sin\phi$$

then becomes

Another Formula for Torque

$$\tau = Fr_{\perp}$$

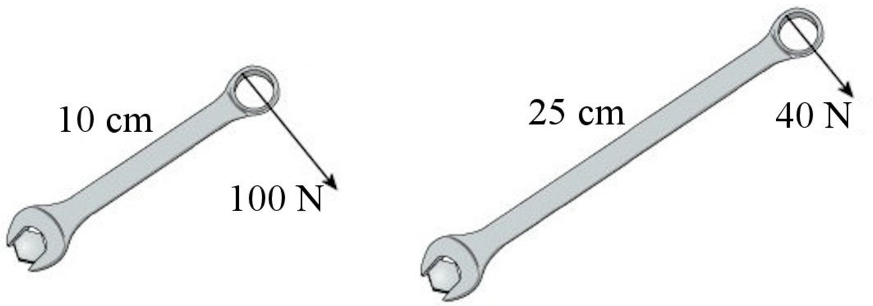
It is possible to live without this formula but it sometimes helps to quickly calculate the torque in specific situations, depending on the data that are initially given.

These three torque formulas clearly show that the distance is as important as the magnitude of the force. If you attempt to close a door by exerting a force on the hinges, the door does not close, because the force is exerted directly on the axis of rotation and the torque is zero since $r = 0$. If the force is exerted near the hinges (but not exactly on them), then the door can be closed, although the force exerted must be very large. This large force must compensate for the small distance between the point of application of the force and the axis of rotation. If the force is exerted on the other side of the door (on the side of the handle), the door can be closed with a weaker force because the distance is larger.

Tools

Torque is a crucial concept to master in order to understand the working of many tools. To illustrate this, suppose you want to unscrew a 2 cm diameter bolt which is very difficult to unscrew. Let's say it takes 10 Nm to unscrew it. You first try to unscrew the bolt with your hands. The force is then exerted on the edges of the bolt and the torque is quite small because the distance between the point of application of the force and the axis of rotation is tiny. Even with a force of 100 N, you would not be able to unscrew the bolt since the application of a 100 N force approximately 1 cm from the axis of rotation gives a torque of only 1 Nm. (100 N times 0.01 m). To succeed, you would need to exert a 1000 N force.

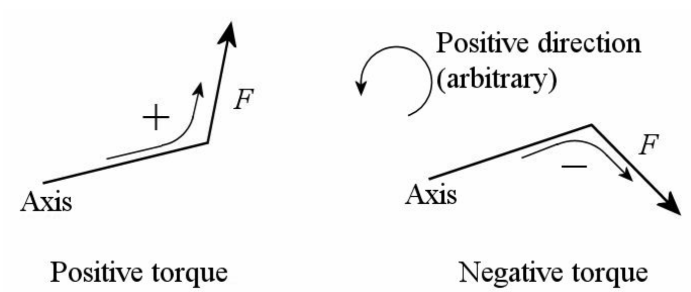
If you now take a wrench, the 10 Nm needed can be reached much more easily. If the wrench is 10 cm long, a force of 100 N is sufficient, and if the wrench is 25 cm long, a force of only 40 N is sufficient. This is much less than the 1000 N needed without tools!



www.physicsmastered.com/torque-magnitude-ranking-task/

Sign of the Torque

The sign of the torque must also be determined. A positive direction for the rotation must be chosen first. The following curve is then drawn: It starts from the axis, goes towards the point of application of the force and the turns in the direction of the force. If this curve then rotates in the same direction as the positive direction, the torque is positive. Conversely, if this curve rotates in the opposite direction to the positive direction, the torque is negative.



Net Torque

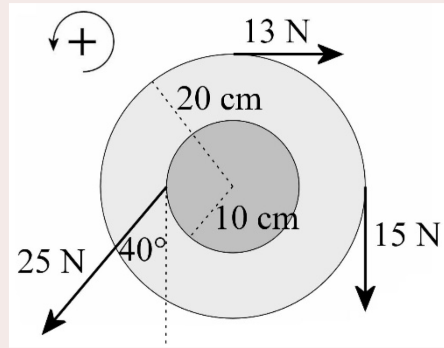
If several forces are acting on an object, the net torque or resulting torque is simply the sum of all the torques acting on the object.

Net Torque

$$\tau_{net} = \sum \tau$$

Example 12.5.1

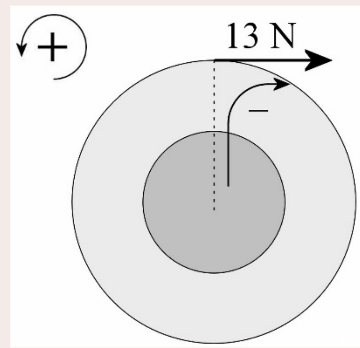
What is the net torque on this wheel?



The torque made by each force will be calculated separately.

The torque made by the 13 N force is

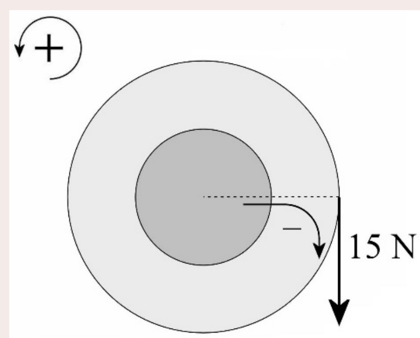
$$\begin{aligned}\tau_1 &= Fr\sin\phi \\ &= -13N \cdot 0.2m \cdot \sin 90° \\ &= -2.6Nm\end{aligned}$$



This torque is negative because it makes the object rotate in the negative direction since the curve that leaves from the axis, goes to the point of application of the force, and rotates in the direction of the force is in the opposite direction to the positive direction.

The torque made by the 15 N force is

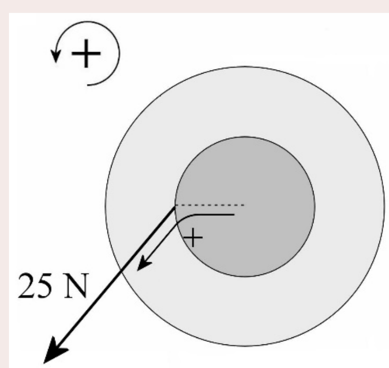
$$\begin{aligned}\tau_2 &= Fr\sin\phi \\ &= -15N \cdot 0.2m \cdot \sin 90° \\ &= -3Nm\end{aligned}$$



This torque is negative because it makes the object rotate in the negative direction since the curve that leaves from the axis, goes to the point of application of the force, and rotates in the direction of the force is in the opposite direction to the positive direction.

The torque made by the 25 N force is

$$\begin{aligned}\tau_3 &= Fr\sin\phi \\ &= 25N \cdot 0.1m \cdot \sin 130° \\ &= 1.915Nm\end{aligned}$$

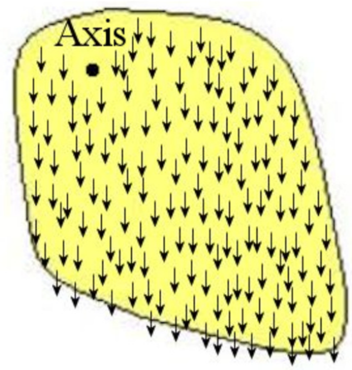


This torque is positive because it makes the object rotate in the positive direction since the curve that leaves from the axis, goes to the point of application of the force, and rotates in the direction of the force is in the same direction as the positive direction.

Therefore, the net torque is

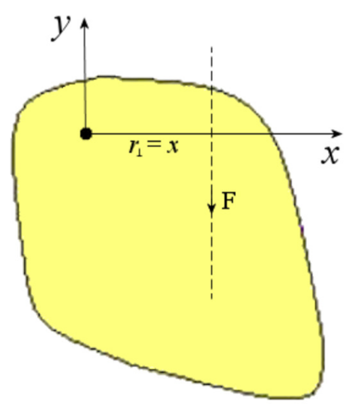
$$\begin{aligned}\tau_{net} &= -2.6Nm - 3Nm + 1.915Nm \\ &= -3.685Nm\end{aligned}$$

The Torque Made by the Gravitational Force



The gravitational force is exerted on each atom of the object. There are, therefore, many torques acting on an object.

Obviously, all these torques will not be added each time. Let's try to get a simpler result.



The torque exerted on one atom is

$$\begin{aligned}\tau &= m_i g r_{\perp i} \\ &= m_i g x_i\end{aligned}$$

The sum of all these torques is

$$\begin{aligned}\tau &= \sum m_i g x_i \\ &= \left(\sum m_i x_i\right) g\end{aligned}$$

But since

$$x_{cm} = \frac{\sum m_i x_i}{m}$$
$$\sum m_i x_i = m x_{cm}$$

the torque becomes

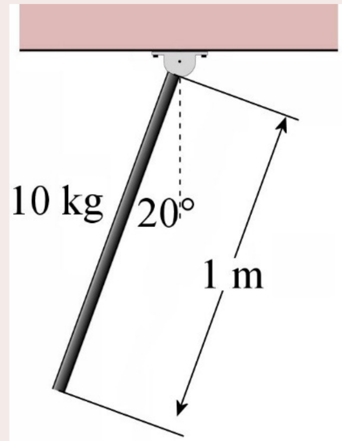
$$\tau = m x_{cm} g$$

This is the same as

$$\begin{aligned}\tau &= m r_{\perp cm} g \\ &= m g r_{cm} \sin\phi\end{aligned}$$

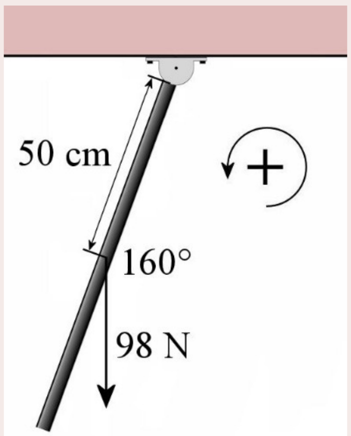
This equation means that the torque made by gravity is the same as if the full force of gravity was applied at the centre of mass. In fact, this is the proof that the point of application of the gravitational force must be set at the centre of mass.

Example 12.5.2



A 10 kg rod is fixed to a pivot as shown in the diagram. What is the torque made by the force of gravity on the rod?

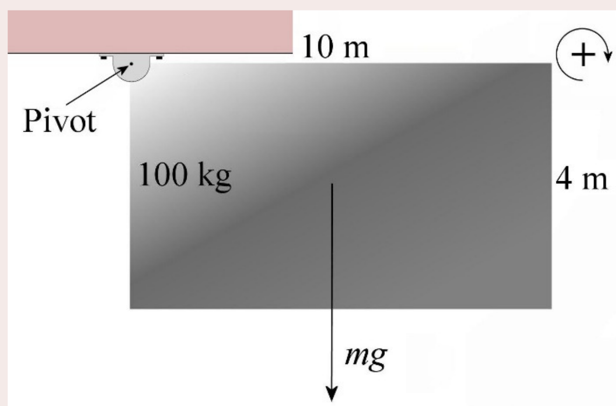
To calculate the torque made by gravity, consider that the force of gravity is exerted at the centre of mass, as in the diagram to the left.



Therefore, the torque is

$$\begin{aligned}\tau &= mgr_{cm}\sin\phi \\ &= 10kg \cdot 9.8\tfrac{N}{kg} \cdot 0.5m \cdot \sin 160° \\ &= 16.76Nm\end{aligned}$$

Example 12.5.3



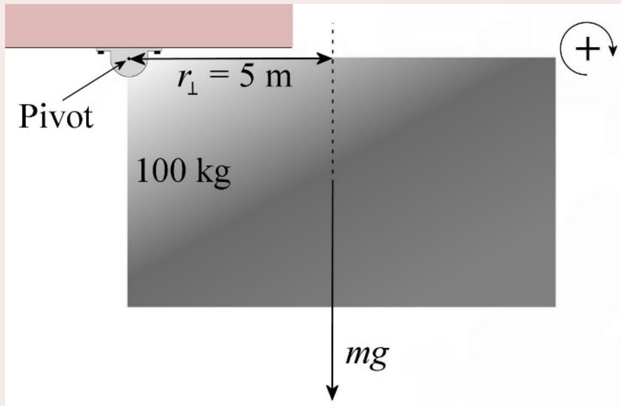
A 100 kg metallic plate is fixed to a pivot from one of its corners, as shown in the diagram. What is the torque made by the force of gravity acting on the plate?

Once again, the force of gravity is considered as being exerted at the centre of mass, as shown in the diagram.

To calculate the torque, the distance between the centre of mass and the axis must be calculated, as well as the angle between the force and the line connecting the axis and the centre of mass. When both distance and angle must be calculated, there is a good chance that the formula

$$\tau = Fr_{\perp}$$

allows calculating the torque more simply. In this situation, the lever arm ($r_{\perp}$) is 5 m according to the diagram below.



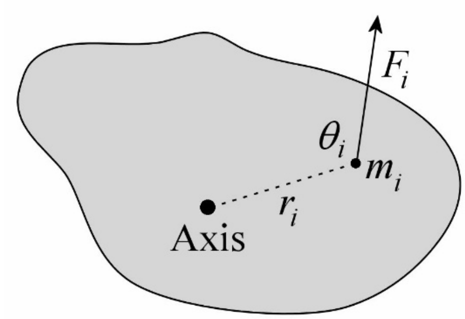
Therefore, the torque made by the force of gravitation is

$$\begin{aligned}\tau &= Fr_\perp \\ &= (mg)r_\perp \\ &= \left(100kg \cdot 9.8\tfrac{N}{kg}\right)\cdot 5m \\ &= 4900Nm\end{aligned}$$

12.6 ROTATIONAL DYNAMICS

Relationship Between τ and α

Suppose that a force is exerted on a small mass in an object (this force is the net force on the small mass). Then



$$\vec F_i = m_i \vec a$$

The tangential component of this equation is

$$F_\perp = ma_T$$

Then, we have

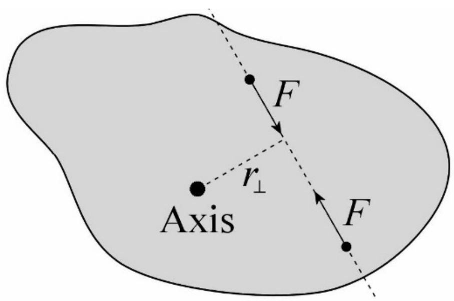
$$\begin{aligned}F_\perp &= ma_T \\ F_\perp r &= ma_T r \\ \tau &= mr^2 \frac{a_T}{r} \\ \tau &= mr^2\alpha\end{aligned}$$

These torques made on each small mass of the object are then added to obtain

$$\sum \tau = \sum m_i r_i^2 \alpha$$
$$\sum \tau = \left(\sum m_i r_i^2\right) \alpha$$
$$\sum \tau = I\alpha$$

This sum of torque includes torques made by internal forces and torques made by external forces.

$$\sum \tau_{int} + \sum \tau_{ext} = I\alpha$$



However, torques made by internal forces cancel each other out. Indeed, if two small masses attract each other, the two forces are of the same magnitude and are directed exactly towards each other. The line extending the forces is, therefore, the same for both forces, which means that the lever arm ($r_\perp$) is the same for both forces. Both forces thus exert the same torque, but with opposite signs. The two internal torques, therefore, cancel each other out and the equation becomes

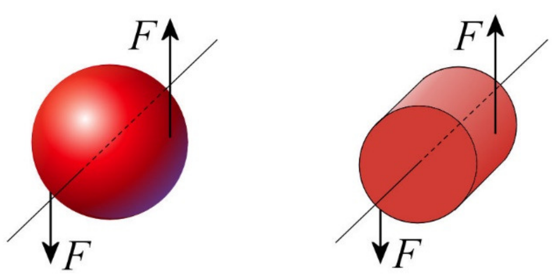
Newton's Second Law for Rotational Motion (for an Object That Keeps the Same Shape)

$$\sum \tau_{ext} = I\alpha$$

This formula was obtained by Euler in 1749.

(The formula applies only if the object does not change shape since at some point in the proof, $a_T = \alpha r$ was used. This formula was obtained with the derivative of $v = \omega r$ assuming that r is constant. This r would not be constant if the object changes shape.)

With this equation, it is obvious that an object with a high moment of inertia is harder to accelerate. To illustrate this idea, imagine that the same torque is exerted on two objects



having different moments of inertia. A sphere and a cylinder having the same mass (say 10 kg) and the same radius (say 20 cm) are considered. Two forces of the same magnitude (say 50 N) are exerted on each of them to make them spin. The torque on each object is the same because the forces are identical and they are exerted at the same distance from the axis of rotation. This torque is 20 Nm for each object. Even if the torques are the same, the effect is different because the moments of inertia are different. The moment of inertia of the sphere (0.16 kg m²) is smaller than the moment of inertia of the cylinder (0.2 kg m²). The angular acceleration of the sphere (125 rad/s²) is then larger than the angular acceleration of the cylinder (100 rad/s²) even if the torques and masses are identical.

The difference comes from the distribution of mass in the object. If the mass is closer to the axis of rotation, the object is easier to rotate. This is not surprising because the speed is smaller for a mass near the axe of a spinning rigid object. As it is easier to give a small speed to a mass, it is easier to give angular speed to an object having its mass closer to the axis. The mass of a sphere being more concentrated near the axis that the mass of a cylinder, the moment of inertia of the sphere is smaller and it is easier to start its rotation.

Here's another demonstration of the effect of the distribution of mass on the value of the moment of inertia. The two sticks in this clip have the same mass and length, but one is much easier to accelerate than the other because the mass is closer to the axis.
http://www.youtube.com/watch?v=m9weJfoW5J0

In the following video, two rules fall from a vertical position. The two rules have different masses and very different moments of inertia. As mass has no influence in free-fall, you might think that the two rules will hit the ground at the same time. However, one of the rules has a much larger moment of inertia and is much harder to rotate. It, therefore, falls with a smaller angular acceleration that the other rule.
http://www.youtube.com/watch?v=POHD6GRoZEI
It could be argued that mass added at the end of the rule also increases the torque exerted by gravity on the rule and that it should thus fall faster. It is possible that, in certain situations, the increase of the angular acceleration caused by the increase of the torque dominates the decrease of angular acceleration caused by the increase of the moment of inertia. However, for these falling rules, the increase in the moment of inertia dominates.

Moment of Inertia and Evolution

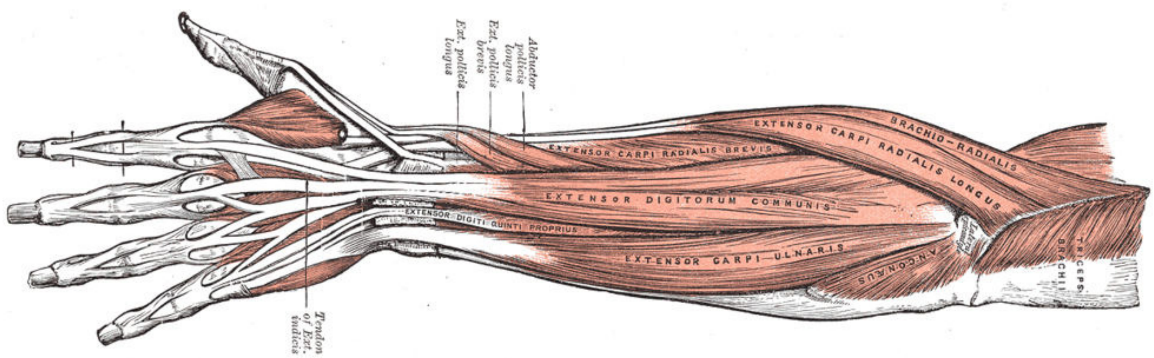
An animal greatly benefits if the moment of inertia of its limbs is small because the limb is then easier to rotate. If this occurs for feet or legs, then the animal can run faster, which is advantageous for catching preys or trying to escape a predator. To reduce the moment of inertia, the mass of the limb must be concentrated as close as possible to the axis of rotation. This can be achieved by placing the muscles near the joints. This image of a horse clearly shows that the rear leg muscles are concentrated near the joint (the hip) while the end of the leg is very slender. This concentration of mass greatly reduces the moment of inertia and

plus.google.com/112686995540721578944/posts allows much faster movements compared to a leg where the muscles are more evenly distributed.

Popeye is, therefore, a real freak of evolution since the mass of his arms is concentrated at the end of the arm rather than near the shoulder.

www.lacrampeauxdoigts.com/?p=1585

This concentration of the muscles near the joint is so advantageous that sometimes muscles are quite far from what they control. The muscles that control your fingers are mostly located in the forearm, near the elbow. They control the hand joints with long tendons going up to the fingers.

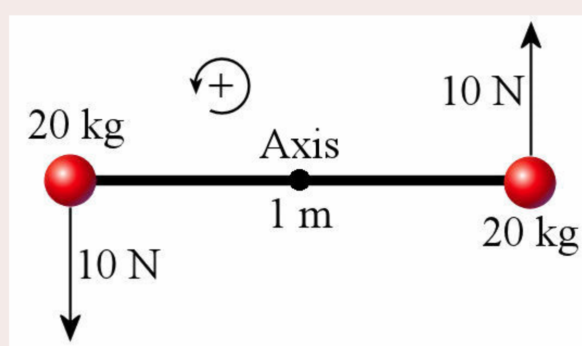


fr.wikipedia.org/wiki/Muscle_extenseur_des_doigts

Examples of Applications of Newton's Second Law for Rotational Motion

Example 12.6.1

The object shown in the diagram is initially at rest. What is the number of revolutions made in 10 seconds? Consider that the 20 kg masses are point masses and neglect the mass of the rod.



The number of revolutions can be found from the acceleration by using the laws of kinematics. This acceleration can be found from the moment of inertia and the net torque acting on the object.

The moment of inertia is

$$\begin{aligned} I &= \sum m_i r_i^2 \\ &= 20kg \cdot (0.5m)^2 + 20kg \cdot (0.5m)^2 \\ &= 10kgm^2 \end{aligned}$$

The net torque is

$$\begin{aligned} \tau_{net} &= \sum Fr\sin\phi \\ &= 10N \cdot 0.5m \cdot \sin 90° + 10N \cdot 0.5m \cdot \sin 90° \\ &= 10Nm \end{aligned}$$

Thus, the angular acceleration is

$$\begin{aligned} \alpha &= \frac{\tau_{net}}{I} \\ &= \frac{10Nm}{10kgm^2} \\ &= 1\tfrac{rad}{s^2} \end{aligned}$$

The angular position of the object after 10 seconds is, therefore,

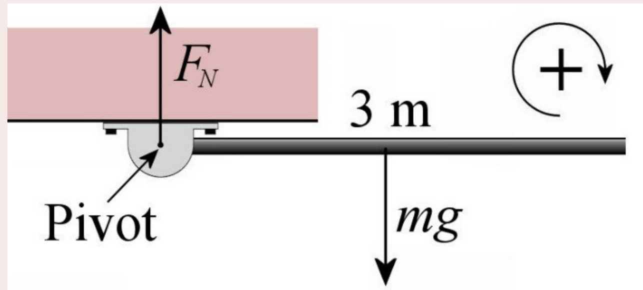
$$\begin{aligned} \theta &= \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \\ &= 0 + 0 + \frac{1}{2} \cdot 1\tfrac{rad}{s^2} \cdot (10s)^2 \\ &= 50rad \end{aligned}$$

Thus, the number of revolutions is

$$\begin{aligned} N &= \frac{\theta}{2\pi} \\ &= \frac{50rad}{2\pi} \\ &= 7.96 \end{aligned}$$

Example 12.6.2

What is the acceleration of the end of this 3 m long rod when it is released? There's no friction.



The acceleration is found with the angular acceleration using

$$a = \alpha r$$

This angular acceleration will be found with Newton's second law for rotational motion.

$$\tau_{net} = I\alpha$$

But to calculate this acceleration, the net torque and the moment of inertia are needed first.

Let's start with the net torque. There are two forces exerted on the rod: the weight and the normal force. Therefore, the net torque is

$$\begin{aligned}\sum \tau &= F_N \cdot 0m + mg \cdot \frac{L}{2} \cdot \sin 90° \\ &= mg\frac{L}{2}\end{aligned}$$

The distance is zero for the torque made by the normal force because the force is exerted directly on the axis of rotation.

Let's now find the moment of inertia. Here, the moment of inertia of a rod with the axis of rotation at the end must be known. This calculation has already been made in a previous example. The result was

$$\begin{aligned}I &= \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 \\ &= \frac{1}{3}mL^2\end{aligned}$$

Newton's second law for rotational motion then gives

$$\begin{aligned}\tau_{net} &= I\alpha \\ \cancel{m}g\frac{L}{2} &= \frac{1}{3}\cancel{m}L^2\alpha \\ 9{,}8\tfrac{N}{kg} \cdot \frac{3m}{2} &= \frac{1}{3} \cdot (3m)^2 \cdot \alpha\end{aligned}$$

$$\alpha = 4.9 \tfrac{rad}{s^2}$$

Thus, the acceleration of the end of the rod is

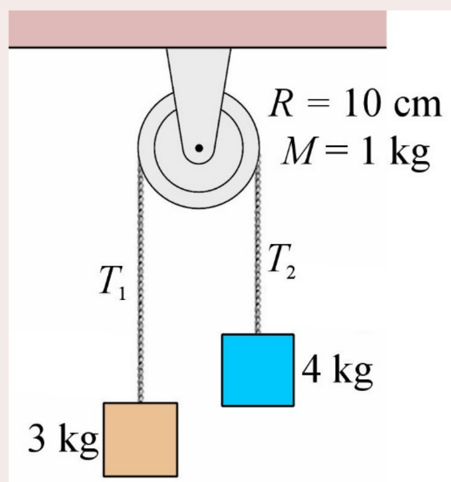
$$\begin{aligned} a &= \alpha r \\ &= 4.9 \tfrac{rad}{s^2} \cdot 3m \\ &= 14.7 \tfrac{m}{s^2} \end{aligned}$$

If an object is placed at the end of this rod, it will not be able to follow the motion of the rod when it starts to fall. The acceleration of the end of the rod is 14.7 m/s² while free-falling objects have an acceleration of 9.8 m/s². This is what is shown in this film. Coins are placed everywhere on a rule. At a certain distance from the axis, the acceleration of the rule is larger than *g* and the coins cannot follow the motion of the rule and they lose contact with the role.
http://www.youtube.com/watch?v=wQuwx7jYYyQ
It can be shown quite easily that the acceleration exceeds *g* at 2/3 of the length of the rod from the axis.

In some problems, there are objects having a linear motion and other objects having rotational motion. When this happens, the sum of forces must be done for objects having a linear motion and the sum of torques must be done for objects having a rotational motion.

Example 12.6.3

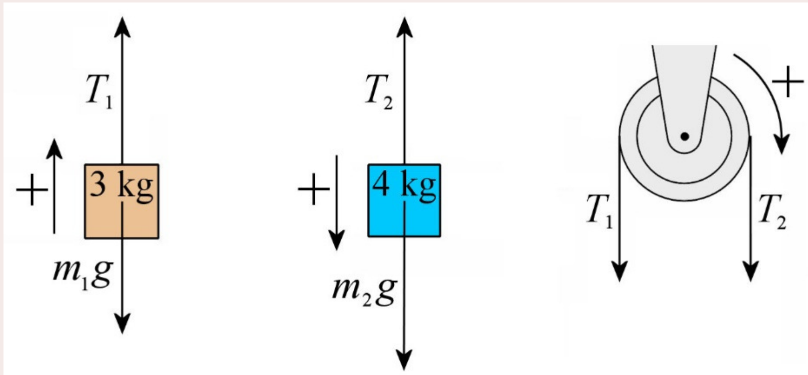
What are the accelerations of the blocks and the tension of the string in the situation shown in the diagram? There's no friction here (except between the rope and the pulley, so that the string does not slip on the pulley).



The sum of the forces on the blocks must be done since they have translation motion. The sum of the torques on the pulley must be done since it has a rotational motion. The acceleration and the tensions will then be found from those equations.

An important note here: the tension changes when the string passes through the pulley. In the previous chapters, it did not change, because the mass of the pulley was neglected. When the mass of the pulley is not neglected, part of the tension serves to accelerate the pulley and the tension changes. That is why T_1 and T_2 are used for the tension.

The forces exerted on the blocks and the pulleys are as follows.



Observe carefully how the positive directions were chosen. The *x*-axis was arbitrarily chosen to be directed upwards for the 3 kg block. If the 3 kg block moves in the positive direction, then the 4 kg block moves downwards. The positive *x*-axis must, therefore, be directed downwards for the 4 kg block. Also, if 3 kg block moves in the positive direction, then the pulley rotates in a clockwise direction. The positive direction for the pulley must then be directed clockwise.

The sum of forces on the 3 kg block is (with an *x*-axis pointing upwards)

$$\sum F_x = ma_x$$
$$-m_1 g + T_1 = m_1 a$$

The sum of forces on the 4 kg block is (with an *x*-axis pointing downwards)

$$\sum F_x = ma_x$$
$$m_2 g - T_2 = m_2 a$$

The sum of torques on the pulley is

$$\sum \tau = I\alpha$$
$$-T_1 R \sin 90° + T_2 R \sin 90° = I\alpha$$
$$-T_1 R + T_2 R = I\alpha$$

There are 3 equations and four unknowns (both tensions, the acceleration and the angular acceleration). These equations can be solved if a link between the two accelerations is found. As the string is tied to the blocks, it has the same acceleration as the blocks. If the string does not slip on the pulley, then

$$a_{string} = \alpha R$$
$$a = \alpha R$$

The torque equation then becomes

$$-T_1 R + T_2 R = I \frac{a}{R}$$

Using the formula for the moment of inertia of the pulley (a cylinder), this equation is

$$-T_1R + T_2R = \frac{1}{2}MR^2\frac{a}{R}$$
$$-T_1 + T_2 = \frac{1}{2}Ma$$

The three equations are now

$$-m_1g + T_1 = m_1a$$
$$m_2g - T_2 = m_2a$$
$$-T_1 + T_2 = \frac{1}{2}Ma$$

These could be resolved by solving for the tensions in the first two equations and substituting in the third equation. The three equations can also be added to obtain

$$(-m_1g + T_1) + (m_2g - T_2) + (-T_1 + T_2) = m_1a + m_2a + \frac{1}{2}Ma$$
$$-m_1g + m_2g = \left(m_1 + m_2 + \frac{1}{2}M\right)a$$

Using the numerical values, this is

$$-3kg\cdot 9.8\tfrac{N}{kg} + 4kg\cdot 9.8\tfrac{N}{kg} = \left(3kg + 4kg + \frac{1}{2}1kg\right)\cdot a$$
$$9.8N = 7.5kg\cdot a$$
$$a = 1.307\tfrac{m}{s^2}$$

The first tension is found with the equation of the forces on the 3 kg block.

$$-m_1g + T_1 = m_1a$$

$$-3kg\cdot 9.8\tfrac{N}{kg} + T_1 = 3kg\cdot 1.307\tfrac{m}{s^2}$$
$$T_1 = 33.32N$$

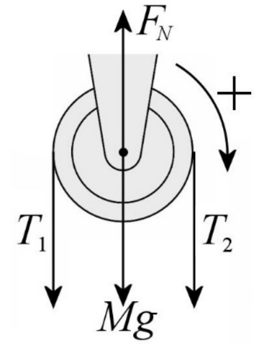
The second tension is found with the equation of the forces on the 4 kg block.

$$m_2g - T_2 = m_2a$$
$$4kg\cdot 9.8\tfrac{N}{kg} - T_2 = 4kg\cdot 1.307\tfrac{m}{s^2}$$
$$T_2 = 33.97N$$

Here are a few notes on this example.

If the mass of the pulley had been neglected, the acceleration would have been 1.4 m/s². With a 1 kg pulley, the acceleration is now 1.307 m/s². The acceleration is smaller since the mass of the pulley must be accelerated now. If there is more mass to accelerate with the same forces, the acceleration is smaller.

There is indeed a reduction of the rope tension when it passes through the pulley. As part of the force serves to accelerate the pulley, there is less tension in the rope on the other side of the pulley. The tension making a torque with the same signs as the acceleration will always be larger than the other tension.



The tensions are not the only two forces acting on the pulley. There's also the weight of the pulley and the normal force (at the contact between the pulley and the axis of rotation). The weight and the normal forces do not have the same magnitude because the normal force must cancel the weight and both tensions. These forces were not considered because the torque exerted by these forces vanishes since they are exerted on the axis of rotation. They exert no torque since the distance r (distance between the point of application of the force and the axis of rotation) is zero.

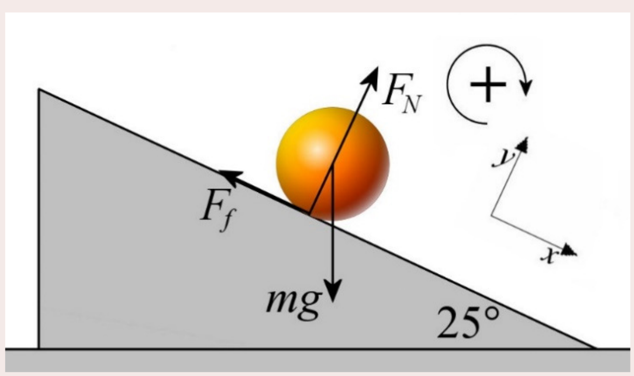
In some problems, an object can have both linear and rotation motions at the same time. When that happens, the sum of the forces <u>and</u> the sum of the torques exerted on the object must both be done.

Example 12.6.4

A ball rolls without slipping down a 25° incline.

a) What is the acceleration of the ball?

The forces exerted on the ball are: the weight, the normal force and the friction force.



As the ball makes both linear and rotational motions, the sum of the forces and the sum of the torque must be done. Furthermore, the centre of mass of the object must be taken as the axis of rotation since the axis is not fixed. Therefore, the equations are

$$\sum F_x = ma_x \quad \rightarrow \quad mg\cos(-65°) - F_f = ma$$
$$\sum F_y = ma_y \quad \rightarrow \quad mg\sin(-65°) + F_N = 0$$
$$\sum \tau = I\alpha \quad \rightarrow \quad \cancel{mg\cdot 0} + \cancel{F_N R\sin 0°} + F_f R\sin 90° = I\alpha$$

In this last equation, the distance is zero for the gravitational force because this force is exerted at the centre of mass and the axis of rotation is the centre of mass. The angle is zero for the normal force because the force is precisely directed towards the centre of mass and this makes the angle between the distance and the force vanish. (Alternatively, it can be argued that the lever arm is zero when the force is directed towards the axis of rotation.) The only remaining torque is the torque due to the force of friction. The latter is then the only force responsible for the rotation of the sphere. If the friction were to be removed, the sphere would simply slip without spinning (a situation that it has been treated in the chapter on forces).

Using the rolling without slipping conditions ($a_{cm} = \alpha R$) and the formula for the moment of inertia of a sphere, the sum of torques becomes

$$F_f R\sin 90° = I\alpha$$
$$F_f R = \left(\frac{2}{5}mR^2\right)\left(\frac{a}{R}\right)$$
$$F_f = \frac{2}{5}ma$$

This is the force of friction required for the sphere to roll without slipping.

Using the sum of the *x*-component of the forces, the acceleration can then be found.

$$mg\cos(-65°) - F_f = ma$$
$$\cancel{m}g\cos(65°) - \frac{2}{5}\cancel{m}a = \cancel{m}a$$
$$g\cos(65°) = a + \frac{2}{5}a$$
$$g\cos(65°) = \frac{7}{5}a$$
$$a = \frac{5}{7}g\cos(65°)$$
$$a = 2.958\tfrac{m}{s^2}$$

The acceleration is smaller than for an object sliding without any friction (which is $a = g\sin 65°$). It is quite normal for this acceleration to be smaller because there's a friction force now. This friction must be present so that the ball can roll without slipping.

b) What is the minimum friction coefficient required so that the ball rolls without slipping?

The friction force exerted on the ball is

$$\begin{aligned} F_f &= \frac{2}{5} ma \\ &= \frac{2}{5} m \frac{5}{7} g \cos(65°) \\ &= \frac{2}{7} mg \cos(65°) \end{aligned}$$

As this is static friction, this force must be less than the static friction maximum. This means that

$$\frac{2}{7} mg \cos(65°) \leq \mu_s F_N$$

According to the sum of the *y*-component of the forces, the normal force is

$$\begin{aligned} mg \sin(-65°) + F_N &= 0 \\ F_N &= -mg \sin(-65°) \\ F_N &= mg \sin(65°) \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{2}{7} \cancel{mg} \cos(65°) &\leq \mu_s \cancel{mg} \sin(65°) \\ \mu_s &\geq \frac{2}{7} \frac{\cos(65°)}{\sin(65°)} \\ \mu_s &\geq \frac{2}{7 \tan 65°} \end{aligned}$$

The minimum friction coefficient is then

$$\begin{aligned} \mu_{s\,\min} &= \frac{2}{7 \tan 65°} \\ &= 0.1332 \end{aligned}$$

If the coefficient of friction were to be less than this value, there would not be enough friction to make the ball roll without slipping. The ball would rotate as the friction force exerts a torque, but not quickly enough to meet the condition of rolling without slipping.

The acceleration of an object rolling down an incline is, therefore, influenced by the moment of inertia of the object. In all cases, the only thing that changes the value of the acceleration, it is the value of the constant in front of mR^2 in the formula for the moment of inertia. To have a more general result, the acceleration can be calculated using the moment of inertia $I = kmR^2$. (k is a number. It is 2/5 for a sphere, 1/2 for a solid cylinder and 1 for a hollow cylinder.) When this is done, the acceleration obtained is

$$a = \frac{g \sin\theta}{1+k}$$

where θ is the angle of the plane. Therefore, the accelerations are

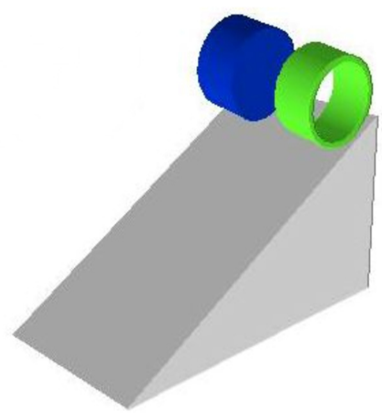
$$a_{sphere} = \frac{5}{7} g \sin\theta$$

$$a_{solid\ cylinder} = \frac{2}{3} g \sin\theta$$

$$a_{hollow\ cylinder} = \frac{1}{2} g \sin\theta$$

Thus, in a race between a solid cylinder and a hollow cylinder, the solid cylinder always wins because its acceleration is larger, regardless of the masses and the radius of the cylinders. Be amazed by a demonstration of this.
http://www.youtube.com/watch?v=gO2CRb8FHLA

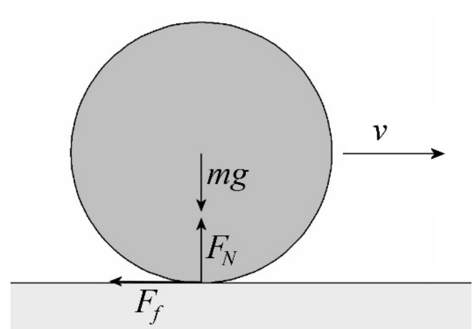
The mass of the hollow cylinder is concentrated very far from the axis of rotation, which makes it harder to turn. This gives it a higher moment of inertia and this is why it loses the race. In a race between a sphere and a solid cylinder, the sphere would win because it is easier to rotate since the mass of the sphere is a little more concentrated near the axis of rotation.



webspace.utexas.edu/cokerwr/www/index.html/inertiarace.htm

Why Does a Rolling Object Slow Down?

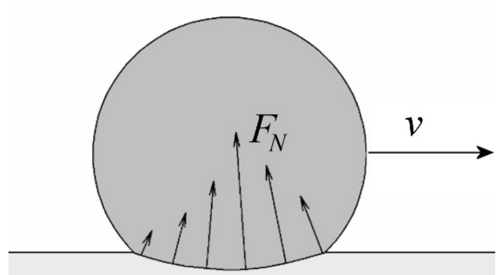
It's possible to think that a rolling object slows down because there is friction. Indeed, the forces on the object are as follows.



The friction force being in the opposite direction to the velocity, it actually slows down the sphere.

However, there is a problem with rotation. According to what can be seen in the diagram, the weight and normal force do not exert any torque on the sphere. Only the friction force is exerting a torque but it is not in the right direction. Indeed, the torque made by the friction force is in the same direction as the angular speed, which means that the friction force is making the angular speed increase. So, with these forces, the sphere is slowing down while turning faster! Obviously, there is something fishy.

In reality, the situation is a bit subtler than that. There is a deformation of the object and of the surface at the point of contact, so that the situation looks more like this (the deformation is exaggerated in this diagram).



www.lhup.edu/~dsimanek/scenario/rolling.htm

In this case, there are normal forces distributed over the entire surface of contact but the normal forces are slightly larger on the side where the ball is moving. These normal forces on this side of the ball are exerting a torque in a direction opposite to the rotation of the ball, thereby decreasing the angular speed of the ball (in addition to cancel the torque exerted by the friction force and the normal forces on the other side of the ball).

Motion in Space

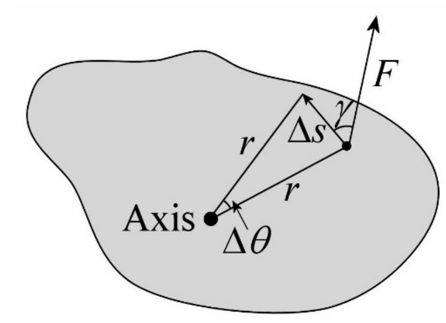
When a force is exerted on an object in space, this force will obviously accelerate the object. However, there is a good chance that the force will also generate a torque on the object. As soon as the direction of the force is not aligned with the center of mass of the object, there is a torque acting on the object and this torque will give an angular acceleration to the object. It is therefore very difficult to exert a force on an object in space without it generating rotation.

That's why the next scene, from the movie *Gravity*, is not really possible.
https://www.youtube.com/watch?v=y4isfJNtK98
It seems pretty well done if you only think about the acceleration generated by the force. However, the effects of the torque are not well represented. Clearly, the direction of the force is not aligned with the center of mass of the system formed by the astronaut and the fire extinguisher, and each burst of gas should generate a rotation. The rotation would then constantly change the direction of the force and it would be practically impossible to move in the desired direction.

To be able to steer a ship in space, the places where the forces are applied must have been well chosen to ensure that the forces will not cause unwanted rotations of the ship.

12.7 WORK AND POWER

Work Formulas



The work done by a constant torque on a spinning object is (note that Δs is an arc of a circle)

$$W = F\Delta s\cos\gamma$$
$$W = F\Delta s\sin\left(90° + \gamma\right)$$

However, as the angle between the force and the distance is $90°+\gamma$, the torque is

$$\tau = Fr\sin\left(90° + \gamma\right)$$

Then, the work is

$$\begin{aligned} W &= F\Delta s\sin\left(90° + \gamma\right) \\ &= Fr\sin\left(90° + \gamma\right)\frac{\Delta s}{r} \\ &= \tau\frac{\Delta s}{r} \end{aligned}$$

As the length of the arc of a circle divided by the radius is the angular displacement

$$\Delta\theta = \frac{\Delta s}{r}$$

the work is

Work Done by a Constant Torque

$$W = \tau\Delta\theta$$

If the torque is not constant, the angular displacement must be divided into small angles, and the work done is calculated for this infinitesimal rotation. With an infinitesimal displacement, the angle is so small that the force does not have time to change and it can be considered as being constant. Then, the work is

$$dW = \tau d\theta$$

If all these infinitesimal works are then summed, the work is

Work Done by a Variable Force

$$W = \int \tau\, d\theta$$

If the force is conservative, Then $W = -\Delta U$, and

Potential Energy if the Force Acting on a Rotating Object is Conservative

$$U = -\int \tau \, d\theta$$

Power Formulas

Once again, power is defined as being the work divided by the time required to do this work. For average power, this gives

Average Power

$$\overline{P} = \frac{W}{\Delta t}$$

For instantaneous power, the calculation is made by taking a very small time, an infinitesimal time.

$$P = \frac{dW}{dt} = \frac{\tau \, d\theta}{dt}$$

Therefore,

Instantaneous Power

$$P = \tau \omega$$

Work-Energy Theorem for Rotation

As for a motion in a straight line, the change in rotational kinetic energy can be calculated with the work. Here is the proof.

The work on a small part of an object is

$$\begin{aligned} W &= \frac{1}{2} m v'^2 - \frac{1}{2} m v^2 \\ &= \frac{1}{2} m \omega'^2 r'^2 - \frac{1}{2} m \omega^2 r^2 \end{aligned}$$

Note that different r were used. Maybe the object changed shape.

If the work on each small part is added to calculate the net work, the result is

$$\begin{aligned} W_{net} &= \sum \frac{1}{2} m\omega'^2 r'^2 - \sum \frac{1}{2} m\omega^2 r^2 \\ &= \frac{1}{2}\left(\sum mr'^2\right)\omega'^2 - \frac{1}{2}\left(\sum mr^2\right)\omega^2 \end{aligned}$$

Since the terms in brackets are the moments of inertia before and after the change, the net work is

$$W_{net} = \frac{1}{2} I'\omega'^2 - \frac{1}{2} I\omega^2$$

As the terms on the right are the rotational kinetic energies, this means that

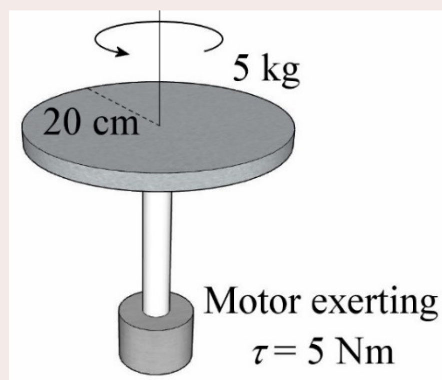
Work-Energy Theorem for Rotational Motion

$$W_{net} = \Delta E_k$$

The work-energy theorem, therefore, also works for rotational motion. Also note that this is the first time that we have a formula that can be applied to an object that changes shape since the *r*'s could be different at instant 2. If the distances between the masses and the axis can change, then the object can change shape. On the other hand, the object is still rigid since ω is still the same everywhere in the object at instant 1 and instant 2.

Example 12.7.1

A motor exerting a torque of 5 Nm spins a circular plate initially at rest. The plate has a mass of 5 kg and a radius of 20 cm.



a) What is the average power of the motor during the first 10 seconds?

To find the average power, the work done by the motor must first be calculated. The torque is known, but the angular displacement is not. This angle is found with the angular acceleration which is found with

$$\tau_{net} = I\alpha$$

To find it, the moment of inertia of the plate must be found. This moment is

$$\begin{aligned} I &= \frac{1}{2} mR^2 \\ &= \frac{1}{2} \cdot 5kg \cdot (0.2m)^2 \end{aligned}$$

$$= 0.1 kg\ m^2$$

The angular acceleration is, therefore,

$$\begin{aligned}\alpha &= \frac{\tau}{I} \\ &= \frac{5Nm}{0.1kg\ m^2} \\ &= 50 \tfrac{rad}{s^2}\end{aligned}$$

The angular displacement in 10 seconds is thus

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\Delta\theta = 0 + \frac{1}{2}\cdot 50 \tfrac{rad}{s^2}\cdot (10s)^2$$
$$\Delta\theta = 2500 rad$$

Therefore, the work is

$$\begin{aligned}W &= \tau\Delta\theta \\ &= 5Nm \cdot 2500 rad \\ &= 12,500 J\end{aligned}$$

and the average power is

$$\begin{aligned}\overline{P} &= \frac{W}{\Delta t} \\ &= \frac{12,500J}{10s} \\ &= 1250W = 1.68hp\end{aligned}$$

b) What is the instantaneous power of the motor 10 seconds after the start of the motion?

After 10 seconds, the angular speed is

$$\begin{aligned}\omega &= \omega_0 + \alpha t \\ &= 0 + 50 \tfrac{rad}{s^2}\cdot 10s \\ &= 500 \tfrac{rad}{s}\end{aligned}$$

The instantaneous power is thus

$$\begin{aligned} P &= \tau \omega \\ &= 5Nm \cdot 500 \tfrac{rad}{s} \\ &= 2500W = 3.35hp \end{aligned}$$

(Note that the change in kinetic energy in 10 seconds

$$\begin{aligned} \Delta E_k &= \frac{1}{2} I\omega^2 - \frac{1}{2} I\omega_0^2 \\ &= \frac{1}{2} \cdot 0.1kg\, m^2 \cdot \left(500 \tfrac{rad}{s}\right)^2 - 0 \\ &= 12{,}500J \end{aligned}$$

is equal to the work as stated by the work-energy theorem.)

12.8 ANGULAR MOMENTUM

Let's start with a definition of the angular momentum. Later, it will be shown why this quantity is useful.

Angular Momentum Definition

General Formula

The angular momentum of a mass in relation to an axis is given by the following vector product.

$$\vec{L} = \vec{r} \times \vec{p}$$

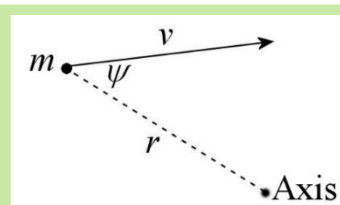
where p is its momentum. According to this definition, this is like the moment of momentum. The angular momentum is actually a vector, but here the magnitude of this vector is the only thing needed (with the correct sign depending on the direction of rotation). According to what is known about a vector product, the magnitude of the angular momentum

$$\left|\vec{L}\right| = \left|\vec{r} \times m\vec{v}\right|$$

is

Angular Momentum (Moment Cinétique in French)

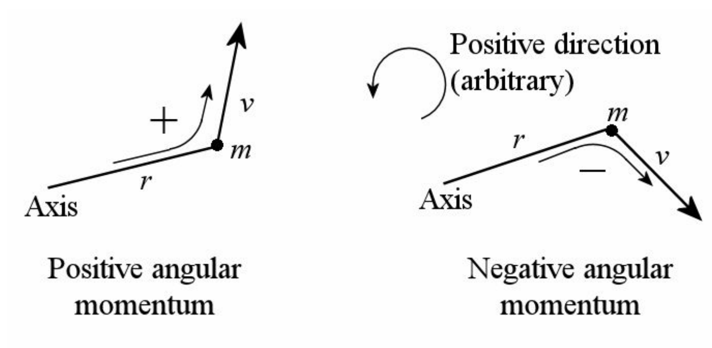
$$L = mvr \sin \psi$$



It may seem odd to have a velocity in any direction like this since the speed must be perpendicular to the radius if the mass is part of an object rotating around an axis. In fact, this definition of angular momentum is very general and allows it to be applied to something other than a rigid rotating object. For example, it can be used to describe the motion of one object orbiting around another.

Sign of Angular Momentum

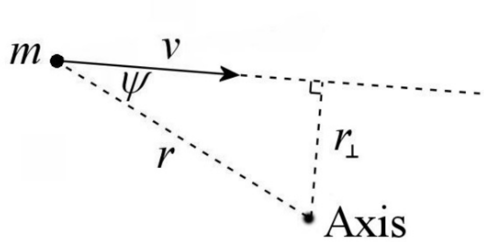
The sign of the angular momentum is found in a way similar to the sign of the torque. We start by choosing what the positive direction will be for rotation. Then start from the axis, go towards the mass and turn in the direction of the velocity. If the rotation obtained is in the same direction as the positive direction chosen, the angular momentum is positive. If the rotation obtained is in the opposite direction of the positive direction chosen, the angular momentum is negative.



Angular Momentum of a Mass Moving in a Straight Line

One might think that the angular momentum of an object moving in a straight line at constant speed is not constant since r and ψ are constantly changing. However, the changes in these two variables cancel each other so that L remains constant.

It is quite easy to understand why by examining the diagram on the left. Then, it can be seen that



$$\frac{r_\perp}{r} = \sin\psi$$

where $r_\perp$ is the smallest distance between the trajectory (in a straight line) of the particle and the axis of rotation. Then, the angular momentum can be written as

$$\begin{aligned} L &= mvr\sin\psi \\ &= mvr_\perp \end{aligned}$$

As the value of $r_\perp$ is constant, the angular momentum is constant.

The angular momentum can, therefore, be calculated with the following formula.

Angular Momentum of a Mass Travelling in a Straight Line

$$L = mvr_\perp$$

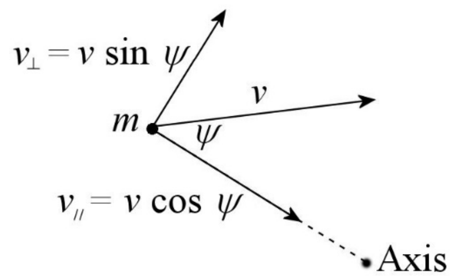
Precisely, $r_\perp$ is the smallest distance between the rotation axis and the trajectory of the centre of mass of the object moving in a straight line.

Angular Momentum of a Rigid Rotating Object

To have the angular momentum of a rigid object rotating around an axis, the angular momenta of all the particles of the object are added.

$$L = \sum mvr\sin\psi$$

Since $v \sin\psi$ is the component of the velocity perpendicular to r, we have



$$L = \sum mv_\perp r$$

As $v_\perp = \omega r$, the angular momentum is

$$\begin{aligned} L &= \sum m\omega rr \\ &= \sum mr^2\omega \end{aligned}$$

As the object is rigid, ω is constant and the equation becomes

$$L = \left(\sum mr^2\right)\omega$$

Since this sum is the moment of inertia, the angular momentum is

Angular Momentum of a Rigid Rotating Object

$$L = I\omega$$

Note that the object could change shape. *Rigid* only means that ω is the same everywhere in the object. This does not prevent it from changing shape (stretching for example).

Newton's 2nd Law (Again!)

The derivative of the angular momentum is

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

As

$$\vec{F}_{net} = \frac{d\vec{p}}{dt} \qquad \text{et} \qquad \vec{v} = \frac{d\vec{r}}{dt}$$

the derivative is

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}_{net} + \vec{v} \times \vec{p}$$

However, $\vec{v} \times \vec{p} = 0$ since the vectors v and p are parallels. Thus

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}_{net}$$

Here, only the magnitude of this vector is of interest. As the magnitude of the vector product on the right is

$$\begin{aligned} \left|\vec{r} \times \vec{F}_{net}\right| &= rF_{net} \sin\phi \\ &= \tau_{net} \end{aligned}$$

the result is

Newton's Second Law for Rotational Motion

$$\tau_{net} = \frac{dL}{dt}$$

This equation is more general than the other version of Newton's 2nd law for rotation, which was $\tau_{net} = I\alpha$. Let's see how $\tau_{net} = I\alpha$ is obtained from the new version of this law for a rotating object.

$$\begin{aligned} \tau_{net} &= \frac{dL}{dt} \\ \tau_{net} &= \frac{d(I\omega)}{dt} \\ \tau_{net} &= I\frac{d\omega}{dt} \\ \tau_{net} &= I\alpha \end{aligned}$$

We notice that to obtain the correct result, I must be a constant so that it can be put out of the derivative. Thus, the first version is valid only if the moment of inertia is a constant (this is what we assumed when we proved $\tau_{net} = I\alpha$).

This new version is more general because it can be applied if the moment of inertia is not constant. It is only when the moment of inertia I is constant (so when the object does not change shape) that the net torque is equal to $I\alpha$.

Angular Momentum Conservation

Condition for Which There is Conservation

The sum of the torques acting on a system is

$$\sum_{system} \tau_{net} = \sum_{system} \frac{dL}{dt}$$

$$\sum_{system} \tau_{net} = \frac{d\left(\sum_{system} L\right)}{dt}$$

$$\sum_{system} \tau_{net} = \frac{dL_{tot}}{dt}$$

The sum of all the net torques acting on the system is simply the sum of all the torques acting on the system (since the sum of a sum is a sum).

$$\sum_{system} \tau = \frac{dL_{tot}}{dt}$$

The torques acting on a system are made by internal or external forces. As the sum of the internal torque must vanish, the equation becomes

$$\sum_{system} \tau_{\text{ext}} + \cancel{\sum_{system} \tau_{\text{int}}} = \frac{dL_{tot}}{dt}$$

$$\sum_{system} \tau_{\text{ext}} = \frac{dL_{tot}}{dt}$$

If the sum of external torques is zero, then

$$0 = \frac{dL_{tot}}{dt}$$

$$L_{tot} = \text{constant}$$

This brings us to

Law of Conservation of Angular Momentum

$$L_{tot} = \text{constant} \qquad \text{if} \qquad \sum_{system} \tau_{ext} = 0$$

This is a new conservation principle that can be used to solve problems. It is mostly this aspect of the kinetic moment that will interest us here.

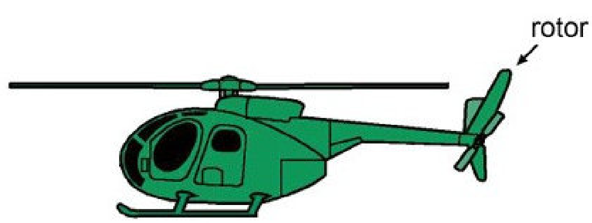
Experimental Demonstrations

To illustrate this principle, consider the example of a person (me) sitting on a swivel bench (without friction) holding in his hands a wheel whose axis is vertical.

If the person starts spinning the wheel in one direction, the person starts spinning in the opposite direction.
http://www.youtube.com/watch?v=VefD0BLTGYA
Initially, the angular momentum is zero since nothing spins. If the person does not touch the ground and there is no friction, there is no external torque and the angular momentum must always remain zero. Thus, if the person spins the wheel in one direction (chosen as the positive direction), then the person has to spin in the opposite direction (negative direction) so that the sum of the angular momentum of the wheel (positive) and of the person (negative) still vanishes. Since the moment of inertia of the person is larger than the moment of inertia of the wheel, the angular velocity of the person is smaller than the wheel's angular velocity.

This is also what is happening on this rotating platform.
http://www.youtube.com/watch?v=w6QaxdppJaE
Initially, everything is at rest and the angular momentum is zero. When the person starts to walk in one direction, the plate begins to rotate in the opposite direction. The angular momentum of the plate has the same magnitude as the angular momentum of the person but with an opposite sign so that the total angular momentum is still zero.

This law is used here in this Simpsons episode. When Skinner begins to run in one direction on a container, the container starts rotating in the opposite direction.
http://www.youtube.com/watch?v=Cnros82SUf0



seniorphysics.com/physics/eei.html

This law also explains the crucial role played by the tail rotor on a helicopter. Initially, when the main rotor does not spin, the angular momentum is zero. When the main rotor starts, the angular momentum must remain zero because the torque made on the main rotor comes from an internal torque. The bottom of the helicopter should, therefore, start to spin in a direction opposite to the main rotor, although with much less angular velocity because the moment of inertia of the bottom is much larger than that of the rotor. When the helicopter is touching the ground before takeoff, there is an external force (friction force exerted by the ground) that prevents the bottom from spinning. Then, the main rotor can

spin while the bottom of the helicopter remains at rest because angular momentum does have to be conserved when there are external torques. However, when the helicopter leaves the ground, the external torque made by the friction force with the ground disappears and the bottom of the chopper should begin to spin in a direction opposite to the rotation of the main rotor. The tail rotor then prevents this rotation. (It also allows the driver to control the heading of the helicopter.) If the tail rotor breaks, the helicopter must get to the ground as fast as possible before the rotation of the cockpit becomes too large. This is what happens in these short video clips. In each case, the problems began when the tail rotor broke.
http://www.youtube.com/watch?v=hnK9bGCvYtU
http://www.youtube.com/watch?v=vMaF4X9AtAI
http://www.youtube.com/watch?v=q3idQKi5EqM (what a pilot!)

Returning to the swivel bench, the person can also start with a wheel spinning in one direction and then reverse the wheel.
http://www.youtube.com/watch?v=0rVcJXWROvQ
In this case, the wheel has initially some angular momentum (the initial direction of rotation is chosen as the positive direction). When the wheel is inverted, it spins in the opposite direction and its angular momentum becomes negative. However, as there is no external torque (the person does not touch the ground), the angular momentum must be conserved. Therefore, the person must start to spin in the positive direction (the same direction as the initial direction of rotation of the wheel). The sum of the angular momentum of the person (positive) and of the angular momentum of the wheel (which is now negative) must match the initial angular momentum of the wheel (which was positive).
Main Uses of the Law of Conservation of Angular Momentum

The conservation of angular momentum has two main utilities.

1. Collisions involving rotational motion.
2. When the moment of inertia of an object changes.

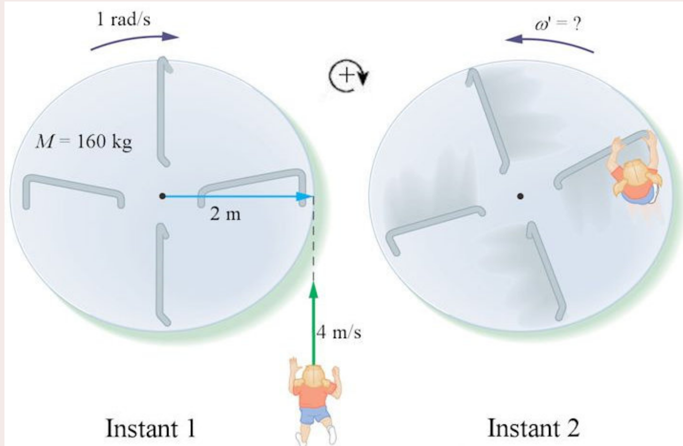
Conservation of Angular Momentum in Collisions

In rotation, angular momentum plays the same role as momentum in linear motion. In a collision, the impact between the two objects generates only internal forces. As the sum of the torques created by internal forces vanishes, the angular momentum is conserved if the sum of external torques is zero (or if they are neglected).

The law of angular momentum must be applied to collisions where there are both rotational and linear motions. A child running in a straight line who then jumps on a rotating platform is an example of this kind of collision. Initially, there is a linear motion whereas there is a rotational motion at the end.

Example 12.8.1

A 60 kg (fat) child runs at 4 m/s towards a circular plate that is initially spinning in the direction indicated in the diagram. What is the angular speed when the child is on the plate?



user.physics.unc.edu/~rowan/p24site/p24units/unit11/WCHAP11-7.html

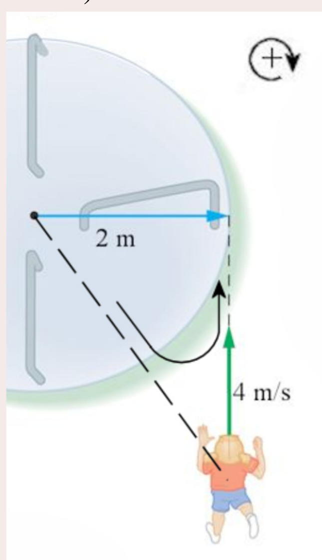
As this is a collision involving a rotational motion, the law of conservation of angular momentum will be used.

Angular Momentum at Instant 1

The angular momentum at instant 1 is

$$\begin{aligned} L &= L_{plate} + L_{child} \\ &= I\omega + mvr_{\perp} \end{aligned}$$

Here, the formula $I\omega$ is used for the object that rotates (the plate) and the formula $mvr_{\perp}$ is used for the object moving in a straight line (the child). As the moment of inertia of the plate is $\frac{1}{2}MR^2$, the angular momentum is



$$\begin{aligned} L &= \frac{1}{2}MR^2\omega + mvr_{\perp} \\ &= \frac{1}{2}\cdot 160kg \cdot (2m)^2 \cdot 1\tfrac{rad}{s} + -60kg \cdot 4\tfrac{m}{s} \cdot 2m \end{aligned}$$

The angular momentum of the plate is positive because it spins in the positive direction. The child's angular momentum is negative because the arrow that starts from the axis, goes to the child and then rotates in the direction of velocity is in the opposite rotation direction chosen as the positive direction.

Thus, the angular momentum at instant 1 is

$$\begin{aligned} L &= L_{plate} + L_{child} \\ &= 320\tfrac{kgm^2}{s} - 480\tfrac{kgm^2}{s} \\ &= -160\tfrac{kgm^2}{s} \end{aligned}$$

Angular Momentum at Instant 2

At instant 2, the angular momentum is

$$\begin{aligned} L' &= I'\omega' \\ &= I'_{plate}\omega' + I'_{child}\omega' \\ &= \frac{1}{2}Mr^2\omega' + mr^2\omega' \\ &= \left(\frac{1}{2}\cdot 160kg\cdot(2m)^2 + 60kg\cdot(2m)^2\right)\cdot\omega' \\ &= 560kgm^2\cdot\omega' \end{aligned}$$

Angular Momentum Conservation

The law of conservation of angular momentum then gives

$$\begin{aligned} L &= L' \\ -160\tfrac{kgm^2}{s} &= 560kg\,m^2\cdot\omega' \\ \omega' &= -0.2857\tfrac{rad}{s} \end{aligned}$$

As it is negative, the plate spins counterclockwise (since our positive is in a clockwise direction).

Moment of Inertia Changes

Basic Formula

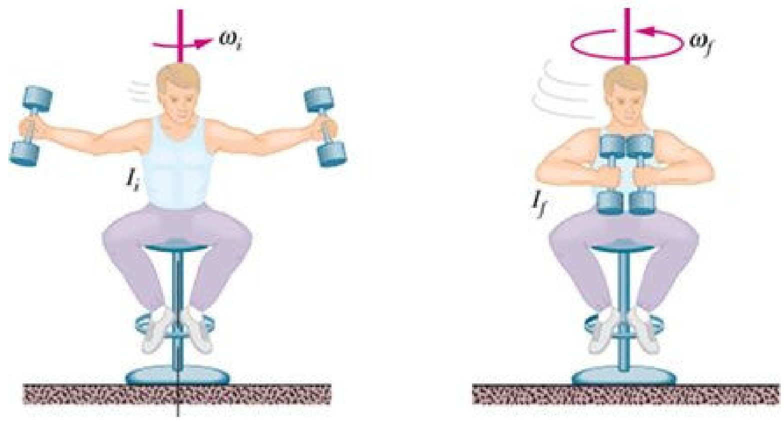
This second application allows finding the angular velocity of a spinning object if it changes its shape. If the shape modification is done with internal forces, the angular momentum is conserved and

$$\begin{aligned} L &= L' \\ I\omega &= I'\omega' \end{aligned}$$

When the shape changes, the moment of inertia changes and the angular velocity must change so that the product $I\omega$ remains constant.

Experimental Demonstrations

Imagine a person sitting on a swivel bench holding masses in his hands. At first, his arms are extended horizontally. If he brings the masses closer to the axis of rotation while spinning, the moment of inertia decreases and the angular speed increases.



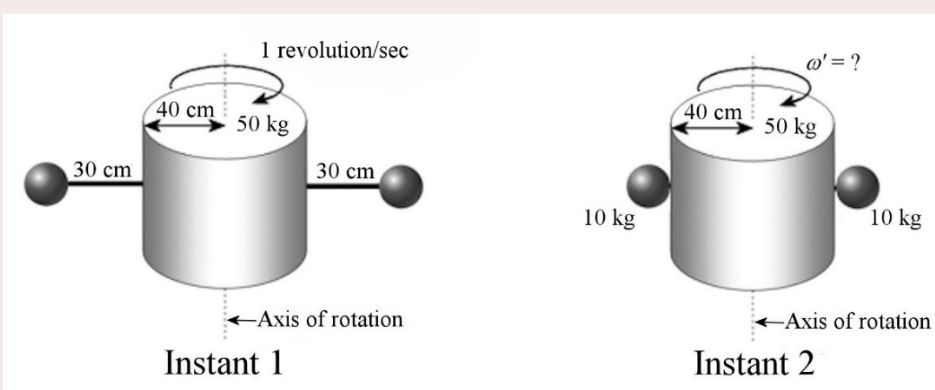
www.rakeshkapoor.us/ClassNotes/classnotes.php?notes=TorqueAndAngularMomentum&index=1

http://www.youtube.com/watch?v=wlKUjsSJRvg

This is what happens when a figure skater spins faster and faster. No forces are exerted by the feet of the skater to turn more rapidly. The angular speed simply increases because the figure skater concentrates its mass closer to the axis of rotation to decrease its moment of inertia.
http://www.youtube.com/watch?v=AQLtcEAG9v0
Note how she tries to increase her moment of inertia at the start when she pushes on the ice to start spinning. Beginning with a high moment of inertia, she may decrease it further and obtain a larger angular velocity.
A demonstration can also be done on a rotating platform like those found in some parks. Put several people on the edge of the platform and then give the platform a small angular speed. Then, tell all the people to move towards the centre. The moment of inertia then decreases and the platform spins faster (it may not be easy to move towards the centre because an important centripetal force will be required as the angular speed increases). Since I do not have any friend, I did it all by myself…
http://www.youtube.com/watch?v=6S5fyCFg30g

Example 12.8.2

What will the angular speed of this object be if its shape changes as shown in the diagram? (Consider that the 10 kg masses are point masses.)



As there is a change of moment of inertia, the principle of angular momentum conservation will be used.

Angular Momentum at Instant 1

At instant 1, the angular momentum is

$$\begin{aligned} L &= I\omega \\ &= \left(I_{cylinder} + I_{spheres}\right)\omega \\ &= \left(\frac{1}{2}m_c r_c^2 + 2m_s r^2\right)\omega \\ &= \left(\frac{1}{2}\cdot 50kg\cdot\left(0.4m\right)^2 + 2\cdot 10kg\cdot\left(0.7m\right)^2\right)\cdot 2\pi\tfrac{rad}{s} \\ &= \left(4kgm^2 + 9.8kgm^2\right)\cdot 2\pi\tfrac{rad}{s} \\ &= 86.708\tfrac{kgm^2}{s} \end{aligned}$$

Angular Momentum at Instant 2

At instant 2, the angular momentum is

$$\begin{aligned} L' &= I'\omega' \\ &= \left(I_{cylinder} + I_{spheres}\right)\omega' \\ &= \left(\frac{1}{2}m_c r_c^2 + 2m_s r^2\right)\omega' \\ &= \left(\frac{1}{2}\cdot 50kg\cdot\left(0.4m\right)^2 + 2\cdot 10kg\cdot\left(0.4m\right)^2\right)\cdot\omega' \\ &= \left(4kgm^2 + 3.2kgm^2\right)\cdot\omega' \\ &= 7.2kgm^2\cdot\omega' \end{aligned}$$

Angular Momentum Conservation

The law of conservation of angular momentum then gives

$$L = L'$$
$$86.708\tfrac{kgm^2}{s} = 7.2kgm^2\cdot\omega'$$
$$\omega = 12.04\tfrac{rad}{s}$$

The angular speed thus increased from 1 revolution per second to 1.92 revolutions per second. When the masses get nearer the axis of rotation, the angular velocity increased as expected.

Angular Velocity of an Object in Circular Motion

When an object makes a circular motion and the sum of the external torques vanishes, the angular momentum is conserved. This is the case with this coin.
https://www.youtube.com/watch?v=0khvqYjKoK4
Actually, there is a small external torque because of friction and the angular momentum slowly decreases here. Despite this, the effect of the conservation of angular momentum can be seen. As the coin descends into the hole, it gets closer to the axis of rotation and this reduces the moment of inertia. However, as the angular $I\omega$ must remain constant (or almost constant), the angular velocity of the coin must increase. This increase in the angular velocity (equivalent to a decreasing period) is quite evident in the clip.

SUMMARY OF EQUATIONS

Angular Displacement

$$\Delta\theta = \theta_2 - \theta_1$$

Average Angular Velocity

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous Angular Velocity

$$\omega = \frac{d\theta}{dt}$$

Relationship between Angular Velocity and Period (if ω is Constant)

$$\omega = \frac{2\pi}{T}$$

Average Angular Acceleration

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous Angular Acceleration

$$\alpha = \frac{d\omega}{dt}$$

Rotational Kinematics Equations for Constant Angular Acceleration

$$\omega = \omega_o + \alpha t$$

$$\theta = \theta_0 + \omega_o t + \frac{1}{2}\alpha t^2$$

$$2\alpha\left(\theta - \theta_0\right) = \omega^2 - \omega_0^2$$

$$\theta = \theta_0 + \frac{1}{2}\left(\omega_0 + \omega\right)t$$

Distance Travelled at a Distance *r* from the Axis if the Angular Displacement is $\Delta\theta$

$$\Delta s = r\Delta\theta$$

Speed at a Distance *r* from the Axis

$$v = \omega r$$

Magnitude of the Acceleration at a Distance *r* from the Axis

Tangential Acceleration $a_T = r\alpha$

Centripetal Acceleration $a_c = \frac{v^2}{r} = r\omega^2$

Acceleration $a = \sqrt{a_T^2 + a_c^2}$

Link between Speed and Angular Velocity if there is Rolling without Slipping

$$v_{cm} = \omega R$$

Link between Acceleration and Angular Acceleration if there is Rolling without Slipping

$$a_{cm} = \alpha R$$

Link between Speeds and Accelerations for a Rope Passing through a Pulley without Slipping

$$v_{rope} = \omega r$$

$$a_{rope} = \alpha r$$

where *r* is the distance between the rope and the axis of rotation

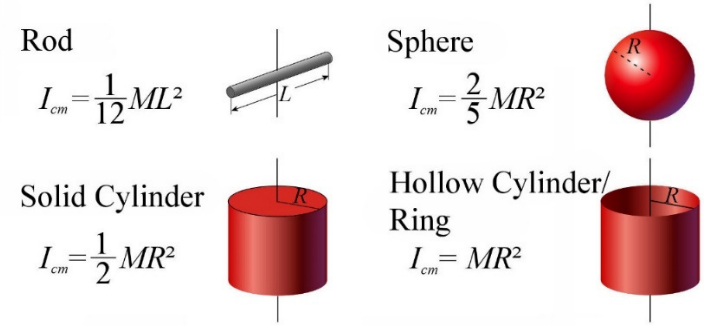
Rotational Motion Transmission

$$\omega_1 R_1 = \omega_2 R_2$$

Moment of Inertia of an Object

$$I = \sum m_i r_i^2$$

Moment of Inertia of 4 Common Uniform Objects



***I* from I_{cm} (parallel axis theorem)**

$$I = I_{cm} + mh^2$$

Kinetic Energy of an Object Rotating around a Stationary Axis Passing through the Centre of Mass

$$E_k = \frac{1}{2} I_{cm} \omega^2$$

Kinetic Energy of an Object Rotating around an Axis not Passing through the Centre of Mass

$$E_k = \frac{1}{2} I \omega^2$$

Kinetic Energy of an Object Making both Linear and Rotational Motion

$$E_k = \underbrace{\frac{1}{2} m v_{cm}^2}_{\text{Translation } E_k} + \underbrace{\frac{1}{2} I_{cm} \omega^2}_{\text{Rotation } E_k}$$

Torque (or Moment of Force)

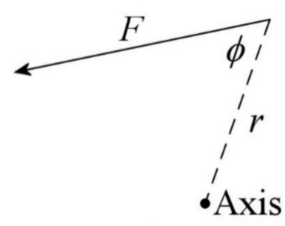
$$\tau = Fr\sin\phi$$

or

$$\tau = F_\perp r$$

or

$$\tau = F r_\perp$$



Net Torque

$$\tau_{net} = \sum \tau$$

Newton's Second Law for Rotational Motion (for an Object That Keeps the Same Shape)

$$\sum \tau_{ext} = I\alpha$$

Work Done by a Constant Torque

$$W = \tau \Delta\theta$$

Work Done by a Variable Force

$$W = \int \tau \, d\theta$$

Potential Energy if the Force Acting on a Rotating Object is Conservative

$$U = -\int \tau \, d\theta$$

Average Power

$$\overline{P} = \frac{W}{\Delta t}$$

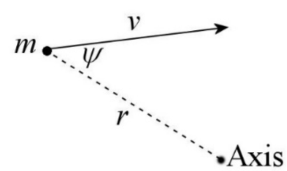
Instantaneous Power

$$P = \tau \omega$$

Work-Energy Theorem for Rotational Motion

$$W_{net} = \Delta E_k$$

Angular Momentum



$$L = mvr\sin\psi$$

Angular Momentum of a Mass Travelling in a Straight Line

$$L = mvr_{\perp}$$

Angular Momentum of a Rigid Rotating Object

$$L = I\omega$$

Newton's Second Law for Rotational Motion

$$\tau_{net} = \frac{dL}{dt}$$

Law of Conservation of Angular Momentum

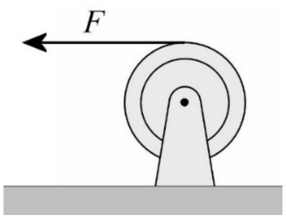
$$L_{tot} = \text{constant} \qquad \text{if} \qquad \sum_{system} \tau_{ext} = 0$$

EXERCISES

12.1 Rotational Kinematics

1. What is the average angular velocity of the Earth due to its rotation on itself?

2. The angular acceleration of this wheel initially at rest is 5 rad/s².



   a) What is the angular velocity of the wheel after 10 seconds?
   b) How many revolutions does the wheel make in 10 seconds?
   c) How much time does the wheel take to make 50 revolutions?

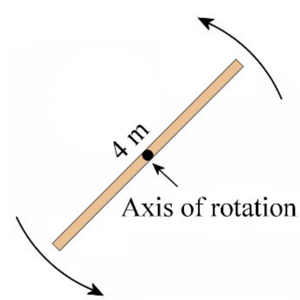
3. The angular speed of a wheel decreases from 120 RPM (revolutions per minute) to 80 RPM in 20 seconds. How many revolutions did the wheel make during this time?

4. The angular position of a rotating disk is given by the following formula.

$$\theta = 10\tfrac{rad}{s}t - 0,5\tfrac{rad}{s^2}t^2$$

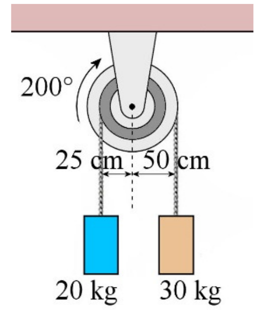
   a) How many revolutions does the disk make in 5 seconds?
   b) What is the average angular velocity of the disk during these 5 seconds?
   c) What is the angular velocity of the disk at t = 4 s?
   d) What is the angular acceleration of the disk at t = 2 s?
   e) When does the angular velocity of the disk vanish?

5. This rod spins about an axis as shown in the diagram. The initial angular speed is 60 RPM while it is only 12 RPM after 0.2 seconds. The angular acceleration is constant.

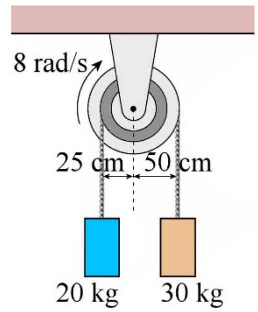


   a) What is the speed of the ends of the rod at t = 0 s?
   b) What is the angular acceleration of the rod?
   c) What is the centripetal acceleration of the ends of the rod at t = 0 s?
   d) What is the tangential acceleration of the ends of the rod at t = 0 s?
   e) What is the magnitude of the acceleration of the ends of the rod at t = 0 s?

6. The pulley shown in this diagram has rotated 200 °. What was the displacement of each block?



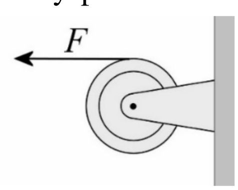
7. The pulley shown in this diagram spins with an angular velocity of 8 rad/s in a clockwise direction. What is the velocity of each block?



8. A car having a speed of 40 m/s slows down and stops over a distance of 160 m with a constant acceleration. The wheels have a diameter of 80 cm.

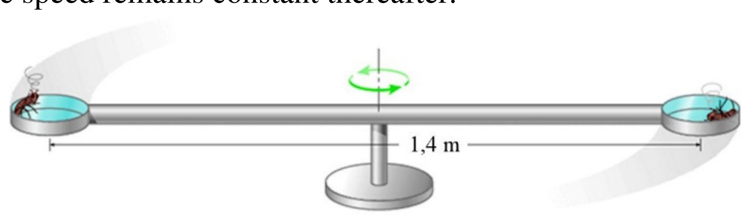
   a) What is the initial angular velocity of the wheels?
   b) What is the angular acceleration of the wheels during braking?

9. A pulley is initially at rest. A rope wound around the pulley is then pulled, which gives a constant angular acceleration to the pulley. At some point during this motion, the pulley makes 60 revolutions whilst its angular velocity passes from 90 RPM to 300 RPM.



   a) What is the angular acceleration of the pulley?
   b) How many revolutions did the pulley make when its speed increased from 0 RPM at 90 RPM?

10. In a biology experiment to stun ants, Marilou places 2 of these insects in the apparatus shown in the diagram. Initially, the apparatus does not spin. When Marilou starts the device, the rod has an angular acceleration of 8 rad/s² for 4 seconds. The speed remains constant thereafter.



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a) How many revolutions do the apparatus make during the first 5 seconds?
b) How long does it take for the apparatus to make 100 revolutions?

11. In the situation shown in the diagram, the largest toothed wheel has a radius of 25 cm and the smallest toothed wheel has a radius of 15 cm. The large-toothed wheel has an angular velocity of 1200 RPM.

a) What is the angular speed of the small toothed wheel?
b) What is the speed of the chain linking the two toothed wheels?

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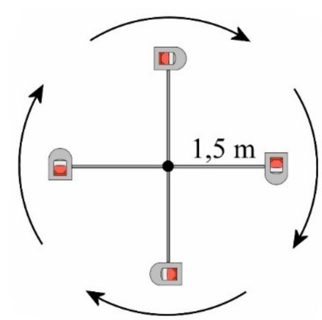
12.2 Rotational Kinetic Energy and Moment of Inertia

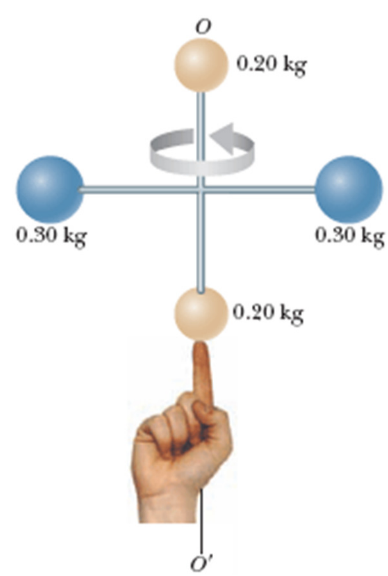
12. The small cars at the end of these metal rods of this ride each has a mass of 120 kg.

a) What is the moment of inertia of the ride when it rotates around an axis located at the centre if the masses of the rods are neglected and if the cars are considered as being point masses?
b) What is the kinetic energy of this ride if it rotates with an angular speed of 12 RPM?



13. In her clown classes, Natasha must spin an object as shown in the diagram. The rods have a negligible mass and a length of 30 cm (from one ball to another).

a) What is the moment of inertia of this object when she spins it around the axis shown in the diagram if the balls are considered as being point masses?
b) What is the kinetic energy of this object when she spins it around the axis shown in the diagram with an angular velocity of 2 rad/s?



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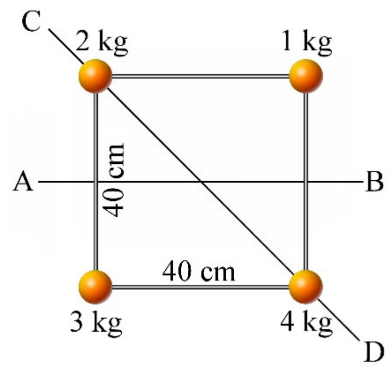
14. What is the moment of inertia of this object if the axis of rotation is…

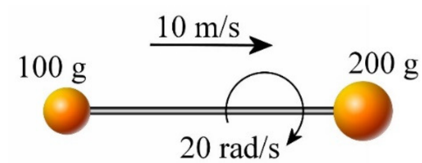
a) the line AB?
b) the line CD?

(The rods have negligible masses and the balls are considered as point masses.)



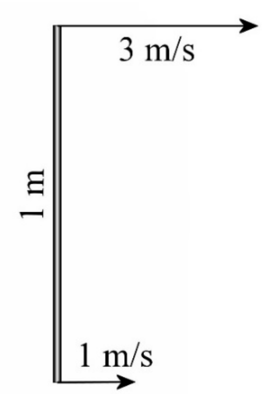
15. What is the kinetic energy of this object composed of two masses connected by a 60 cm long rod if its linear speed is 10 m/s and if it spins around its centre of mass with an angular velocity of 20 rad/s?

(The rod has a negligible mass and the balls are considered as point masses.)



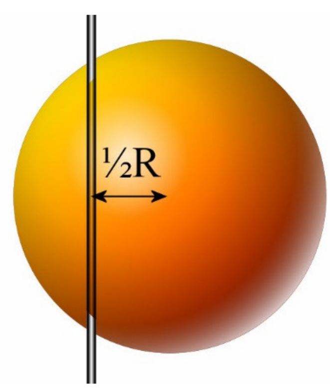
16. The diagram indicates the speed of the two ends of a rod at a specific time. The rod is 1 m long and has a mass of 3 kg.

a) What is the rotational kinetic energy of rotation of the rod?
b) What is the linear kinetic energy of rotation of the rod?
c) What is the total kinetic energy of rotation of the rod?



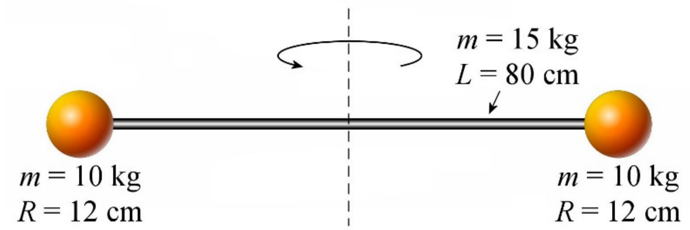
12.3 Moment of Inertia of Extended Object

17. What is the moment of inertia of this solid sphere with a mass of 2 kg and a radius of 6 cm if the axis of rotation is the rod that passes through the sphere?



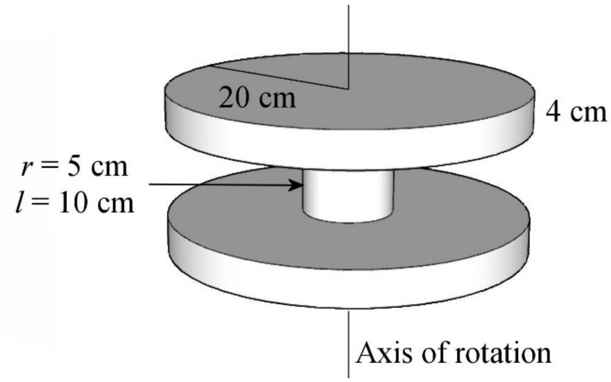
18. What is the moment of inertia of this object if the axis of rotation is the one shown in the diagram?

(The rod does not have a negligible mass and the balls are not considered as point masses.)

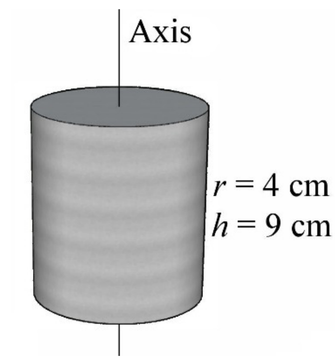


19. What is the moment of inertia of this object made of a material with a density of 5000 kg/m³ if the axis of rotation is as shown in the diagram?

(The large disk at the bottom is identical to the large disk at the top.)

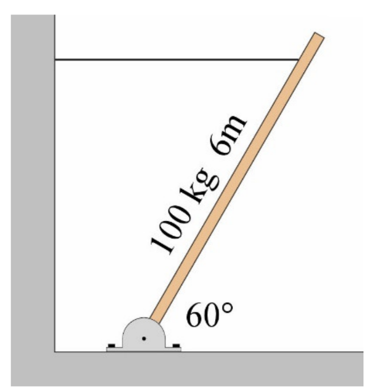


20. Here is a tin can filled with air. The walls (side, lid and bottom) are thin and made of a material having a surface density of 1 g/cm². What is the moment of inertia of this tin can if the axis of rotation is as shown in the diagram?

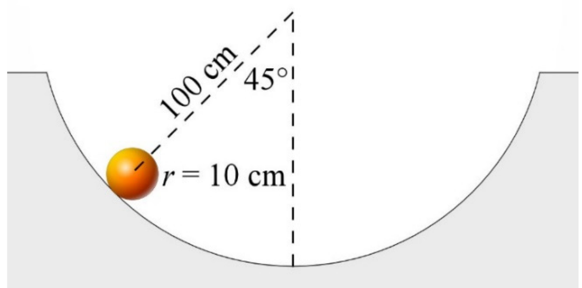


12.4 Mechanical Energy Conservation

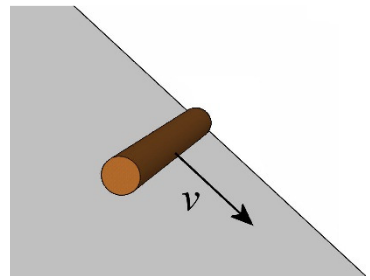
21. This beam held by a cable has a mass of 100 kg and a length of 6 m. With what speed will the end of the beam hit the ground if the cable breaks?



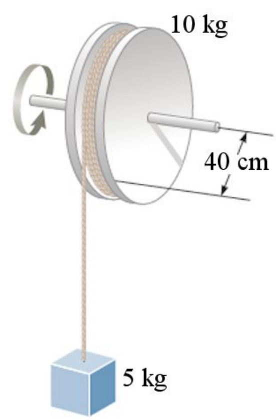
22. An 800 g ball having a radius of 10 cm is held at rest at the position shown in the diagram. The ball is then released (without pushing it) and it rolls in the bowl without slipping. What is the speed of the ball when it is at the lowest point of the bowl?



23. A 4000 kg log having a 50 cm diameter and a 10 m length falls from a truck and then rolls along a 40° slope. Throughout the descent, the log rolls without slipping. At the top of the slope, the log had a speed of 5 m/s. What is the speed of the log at the bottom of the slope if the slope is 400 m long?



24. A 5 kg mass is initially at rest in the situation shown in the diagram. The mass then descends 8 m. What is the speed of the 5 kg mass when it is 8 m lower? (Consider that the pulley is a disk.)



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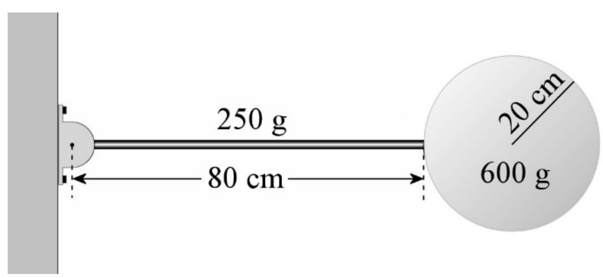
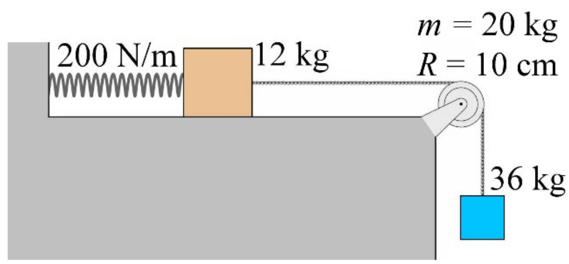
25. The object shown in the diagram is composed of a rod and a disk. This object, initially at rest in the position shown in the diagram, is then released. What will the angular speed of this object be when the rod is vertical?



26. In the situation shown in the diagram, the spring is neither stretched nor compressed initially. What will the speed of the blocks be after a displacement of 1 m if they were initially at rest?

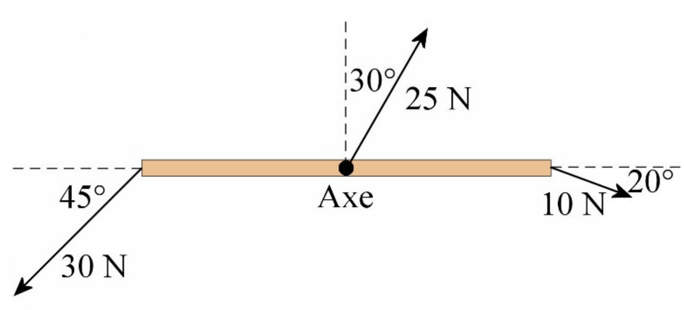
(Consider that the pulley is a disk and that there is no friction between the 12 kg block and the surface.)



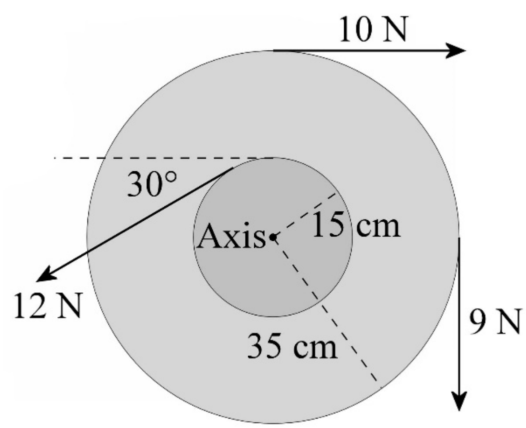
12.5 Torque

27. What is the net torque on this rod?

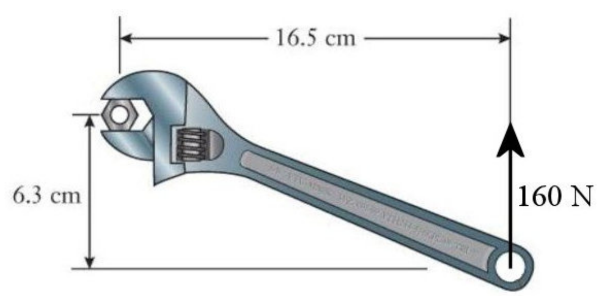
(The axis of rotation is at the centre of the rod and the rod is 4 m long.)



28. What is the net torque exerted on this wheel?



29. What is the torque made by the 160 N force exerted on the wrench?



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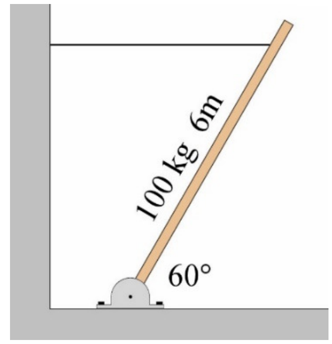
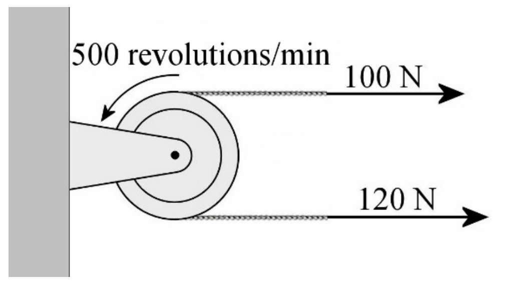
12.6 Rotational Dynamics

30. This 100 kg beam retained by a cable has a length of 6 m. What is the angular acceleration of the rod immediately after the rope has broken?

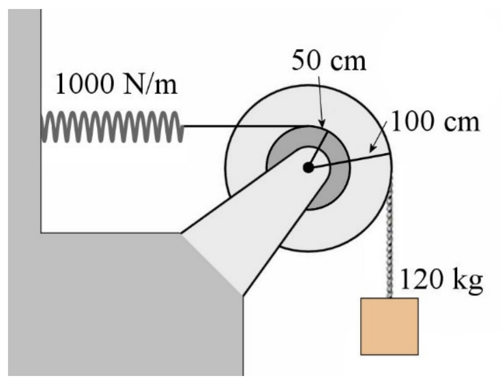


31. The pulley shown in the diagram has a moment of inertia of 0.5 kgm² and a 300 mm radius.

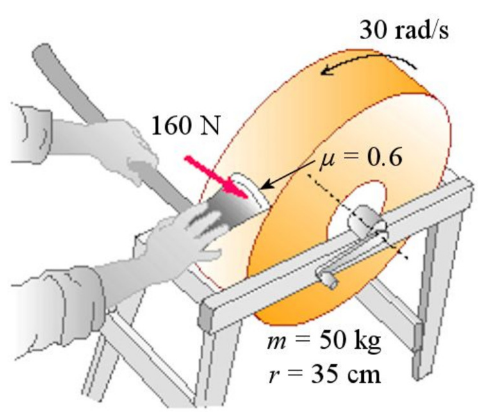
    a) What is the angular acceleration of the pulley?
    b) What will its angular speed be in 10 seconds (in RPM)?



32. There is an equilibrium in the situation shown in the diagram. What is the stretching of the spring?



33. With a crank, Gertrude gave an angular velocity of 30 rad/s to a grinding wheel. She then sharpens her axe by pushing it on the grinding wheel with a 160 N force. As there is friction between the wheel and the axe, the wheel slows down. How long can she sharpen her axe before the angular speed reaches 10 rad/s?



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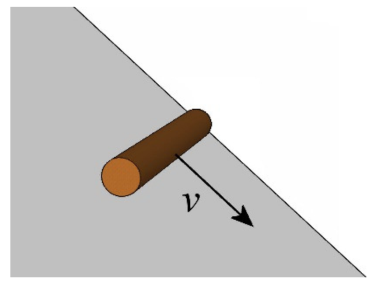
34. A torque acts on a wheel initially at rest to give it an angular velocity of 30 rad/s in 10 s. The torque then disappears and the wheel stops in 120 seconds because of the torque due to friction (that is assumed to be constant) opposed to the rotation of the wheel. This torque due to friction opposing the rotation was also present when the torque was exerted to accelerate the wheel. The moment of inertia of the wheel is of 0.5 kgm².

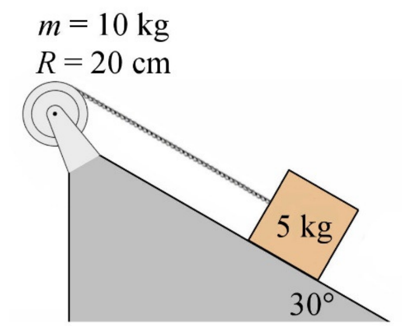
a) What is the torque made by the friction?
b) What was the torque exerted to give the wheel a 30 rad/s angular speed?

35. A 4000 kg log having a 50 cm diameter and a 10 m length falls from a truck and then rolls along a 40° slope. Throughout the descent, the log rolls without slipping.

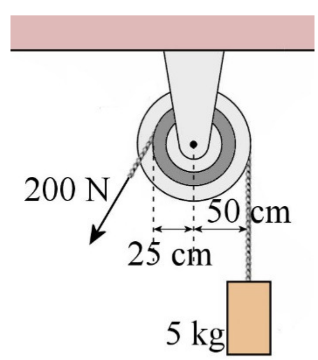
a) What is the acceleration of the log?
b) What is the minimum value of the coefficient of friction required for the log to roll without slipping?



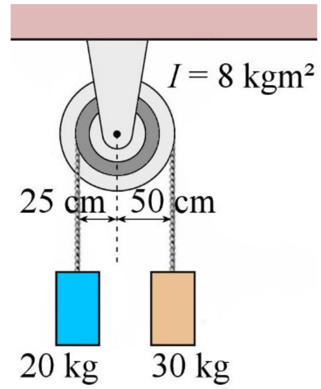
36. A rope wound around a cylinder is tied to a 5 kg block, as shown in the diagram. What is the acceleration of the block if there is no friction between the block and the slope?



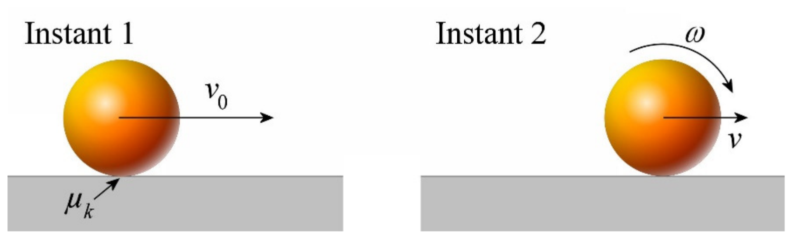
37. The pulley shown in this diagram has a moment of inertia of 8 kgm². What is the acceleration of the 5 kg mass?



38. The pulley shown in this diagram has a moment of inertia of 8 kgm². What is the acceleration of each block?



39. A ball was just hit so that it moves at v_0 = 3.5 m/s without rolling (which means that it is slipping on the surface). However, the friction force between the ball and the ground exerts a torque. This torque will increase the angular speed of the ball so that the ball will roll without slipping at the end. What will the speed of the ball be when it rolls without slipping?



12.7 Work and Power

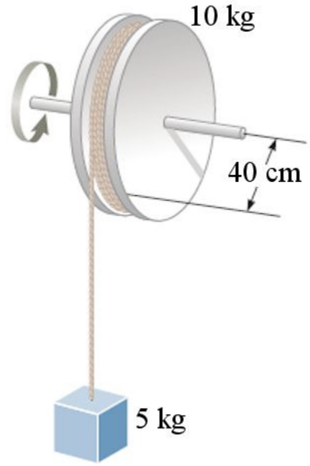
40. A 0.45 Nm constant torque makes a sphere initially at rest spins. The sphere has a mass of 10 kg and a radius of 20 cm.

a) What is the angular displacement of the sphere in 10 seconds?
b) What is the work done by this torque in 10 seconds?
c) What is the angular velocity of the sphere after 10 seconds (using the work-energy theorem)?

41. A motor makes a sphere initially at rest spin so that the angular velocity is 2400 RPM after 30 seconds. The sphere has a radius of 10 cm and a mass of 20 kg. What is the average power of the motor?

42. The power of the engine of a Kia Optima is 274 horsepower when the engine runs at 6000 RPM. What is the torque exerted by the engine then?

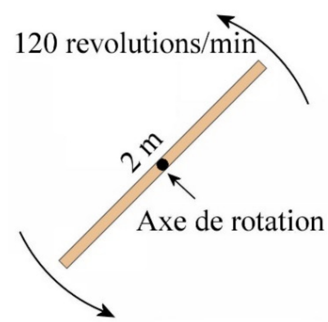
43. A motor makes this pulley spin with a constant angular acceleration in order to lift this block. Initially, the block is at rest on the ground. 3 seconds after the motor has started lifting the block, the block is 4.5 m from the ground and has a speed of 3 m/s. Consider that the pulley is a disk.



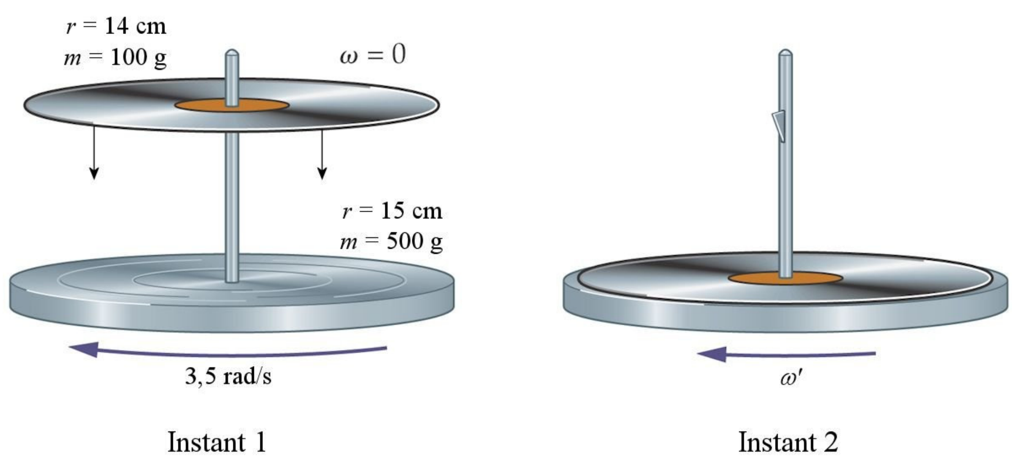
a) What is the work done by the motor (which corresponds to the external work)?
b) What is the average power of the motor during the first 3 seconds?
c) What is the instantaneous power of the motor 3 seconds after the motor has started lifting the block?

12.8 Angular momentum

44. What is the angular momentum of this 300 g rod?

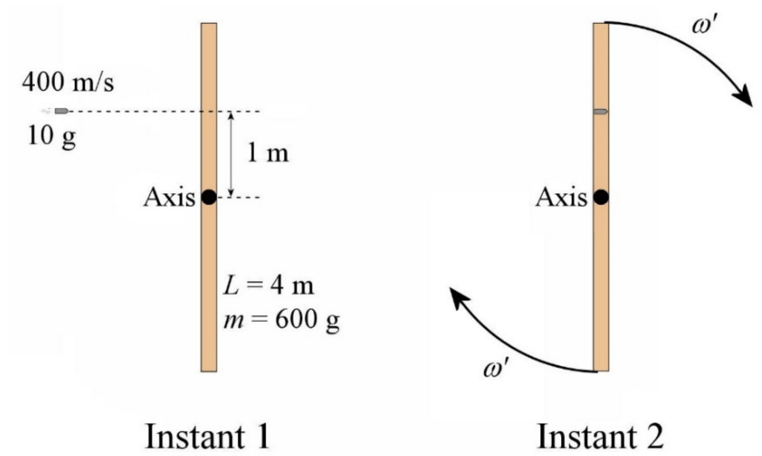


45. A disk falls on a rotating platform such as illustrated in the diagram. What is the angular speed at instant 2?



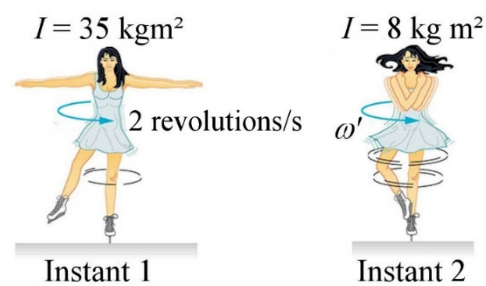
user.physics.unc.edu/~rowan/p24site/p24units/unit11/WCHAP11-7.html

46. A bullet hits a rod initially at rest that can rotate about an axis passing through its centre of mass as shown in the diagram. The bullet then gets lodged in the rod. What is the angular velocity of the rod after the collision?

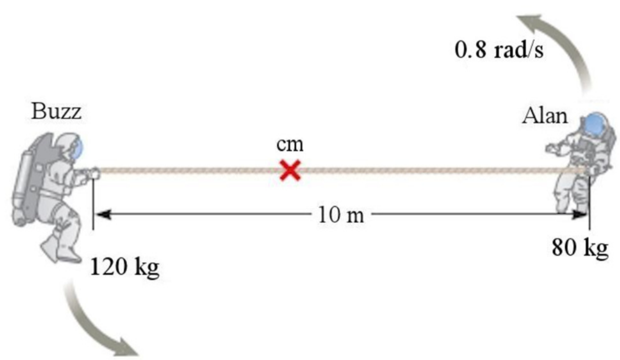


47. What is the rotational speed of this figure skater at the instant 2 (in revolutions per seconds)?

physics.stackexchange.com/questions/81791/conservation-of-angular-momentum-experiment



48. Buzz and Alan are two astronauts floating in space as shown in the diagram. Initially, they are 10 m one from each other and they revolve around the centre of mass of the two astronauts with an angular velocity of 0.8 rad/s. Buzz then pulls on the rope so that the distance between Buzz and Alan is now only 4 m.



www.chegg.com/homework-help/questions-and-answers/astronauts-having-mass-m-connected-rope-length-d-having-negligible-mass-isolated-space-orb-q3272102

a) What is the moment of inertia about the centre of mass before Buzz pulls on the rope? (The rope has negligible mass and the astronauts are considered as point masses.)
b) What is the moment of inertia about the centre of mass after Buzz has pulled on the rope? (The rope has negligible mass and the astronauts are considered as point masses.)

c) What is the angular speed of the astronauts when they are 4 m from each other?
d) What is the speed of each astronaut when they are 4 m from each other?
e) What is the tension of the rope when the two astronauts are 4 m from each other?
f) By how much does the kinetic energy of the system increase when the distance between the astronauts changes from 10 m to 4 m?
g) What is the work done by Buzz to change the distance between the astronauts from 10 m to 4 m?

49. Marjorie is on the edge of a rotating platform which rotates with an angular speed of 1.3 rad/s. What will the angular speed of the plate be if she moves to the centre of the plate? (Considering Marjorie as a point mass.)

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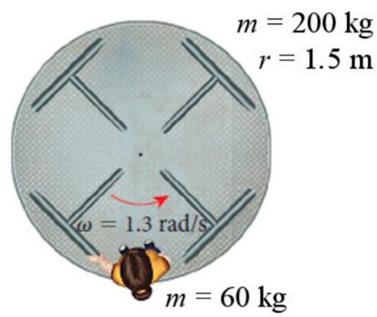
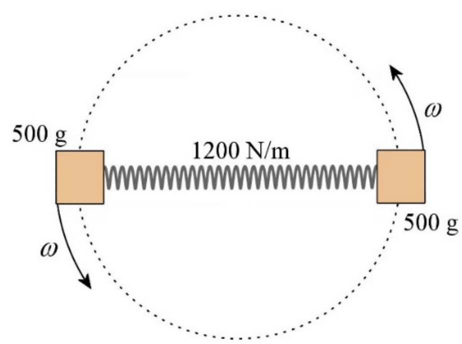


Challenges

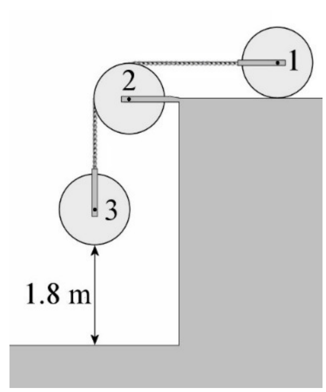
(Questions more difficult than the exam questions.)

50. This system composed of two 500 g masses, and a spring rotates around its centre of mass. The mechanical energy of this system is 126 J, and the spring constant is 1200 N/m. The length of spring when it is neither stretched nor compressed is 10 cm. (There is no gravitation and consider the two masses as point masses.)



a) What is the stretching of the spring?
b) What is the angular speed?

51. Three identical solid cylinders are fixed together as illustrated on the diagram. When cylinder 3 moves downwards, cylinder 1 rolls without slipping on the surface and the rope makes cylinder 2 spin (the rope does not slip on the cylinder). What will the speed of cylinder 3 be immediately before hitting the ground if the system is initially at rest? In this problem, neglect the mass of the brackets that connect cylinders 1 and 3 to the rope.



ANSWERS

12.1 Rotational Kinematics

1. 7.272×10^{-5} rad/s
2. a) 50 rad/s b) 39.79 revolutions c) 11.21 s
3. 33.33 revolutions
4. a) 5.968 revolutions b) 7.5 rad/s c) 6 rad/s d) -1 rad/s² e) 10 s
5. a) 12.566 m/s b) -25.13 rad/s² c) 78.96 m/s² d) 50.27 m/s² e) 93.6 m/s²
6. 20 kg block: 0,8727 m upwards 30 kg block: 1,745 m downwards
7. 20 kg block: 2 m/s upwards 30 kg block: 4 m/s downwards
8. a) 100 rad/s b) -12.5 rad/s²
9. a) 1.191 rad/s² b) 5.934 revolutions
10. a) 15.28 revolutions b) 21.635 s
11. a) 2000 RPM b) 31.418 m/s

12.2 Rotational Kinetic Energy and Moment of Inertia

12. a) 1080 kgm² b) 852.73 J
13. a) 0.0135 kgm² b) 0.027 J
14. a) 0.4 kgm² b) 0.32 kgm²
15. 19.8 J
16. a) 0.5 J b) 6 J c) 6.5 J

12.3 Moment of Inertia of Extended Object

17. 0.00468 kgm²
18. 6.3232 kgm²
19. 1.0102 kgm²
20. 4.423×10^{-4} kgm²

12.4 Mechanical Energy Conservation

21. 12.36 m/s
22. 2.025 m/s
23. 58.18 m/s
24. 8.854 m/s
25. 4.541 rad/s
26. 2.952 m/s

12.5 Torque

27. 35.586 Nm counterclockwise
28. 4.85 Nm clockwise
29. 26.4 Nm counterclockwise

12.6 Rotational Dynamics

30. 1.225 rad/s²
31. a) 12 rad/s² counterclockwise
    b) 1646 RPM counterclockwise
32. 2.352 m
33. 1.823 s
34. a) 0.125 Nm opposed to the rotation b) 1.625 Nm same direction as the rotation
35. a) 4.2 m/s² b) 0.2797
36. 2.45 m/s²
37. 1.378 m/s²
38. 20 kg block: 1.463 m/s² upwards 30 kg block: 2.925 m/s² downwards
39. 2.5 m/s

12.7 Work and Power

40. a) 140.625 rad b) 63.28 J c) 28.125 rad/s
41. 84.22 W
42. 325.32 Nm
43. a) 265.5 J b) 88.5 W c) 177 W

12.8 Angular Momentum

44. 1.2566 kgm²/s
45. 2.98 rad/s
46. 4.938 rad/s
47. 8.75 revolutions/s
48. a) 4800 kgm² b) 768 kgm² c) 5 rad/s d) Buzz: 8 m/s Alan: 12 m/s
    e) 4800 N f) 8064 J g) 8064 J
49. 2.08 rad/s

Challenges

50. a) 30 cm b) 60 rad/s
51. 3.429 m/s