Chapter 11 Solutions

1. The *x*-coordinate of the centre of mass is

$$\begin{aligned} x_{cm} &= \frac{1}{m}\sum xm \\ &= \frac{1}{m}\left(x_1m_1 + x_2m_2 + x_3m_3\right) \\ &= \frac{1}{10kg}\left(0.5\cdot 3kg + 3.5m\cdot 3kg + 2m\cdot 4kg\right) \\ &= \frac{20kgm}{10kg} \\ &= 2m \end{aligned}$$

The *y*-coordinate of the centre of mass is

$$\begin{aligned} y_{cm} &= \frac{1}{m}\sum ym \\ &= \frac{1}{m}\left(y_1m_1 + y_2m_2 + y_3m_3\right) \\ &= \frac{1}{10kg}\left(2.5m\cdot 3kg + 2.5m\cdot 3kg + 0.5m\cdot 4kg\right) \\ &= \frac{17kgm}{10kg} \\ &= 1.7m \end{aligned}$$

The centre of mass is therefore at the position (2 m, 1.7 m).

2. We have the following three masses:

1) A 3 kg mass at (0,0).
2) A 2 kg mass at (10 m, 0 m).
3) A 1 kg mass at (5 m, 8.66 m).

(This last position is y_3 = 10m sin (60°).)

The *x*-coordinate of the centre of mass is

$$\begin{aligned} x_{cm} &= \frac{1}{m}\sum xm \\ &= \frac{1}{m}\left(x_1m_1 + x_2m_2 + x_3m_3\right) \\ &= \frac{1}{6kg}\left(0\cdot 3kg + 10m\cdot 2kg + 5m\cdot 1kg\right) \\ &= \frac{25kgm}{6kg} \\ &= 4.17m \end{aligned}$$

The *y*-coordinate of the centre of mass is

$$\begin{aligned} y_{cm} &= \frac{1}{m}\sum ym \\ &= \frac{1}{m}\left(y_1m_1 + y_2m_2 + y_3m_3\right) \\ &= \frac{1}{6kg}\left(0m\cdot 3kg + 0m\cdot 2kg + 8.66m\cdot 1kg\right) \\ &= \frac{8.66kgm}{6kg} \\ &= 1.44m \end{aligned}$$

The centre of mass is therefore at the position (4.17 m, 1.44 m).

3. The *x*-coordinate of the centre of mass is

$$x_{cm} = \frac{1}{m}\sum xm$$

$$x_{cm} = \frac{1}{m}\left(x_1m_1 + x_2m_2 + x_3m_3 + x_4m_4\right)$$

$$4\text{m} = \frac{1}{7\text{kg} + m_4}\left(1\text{m}\cdot 1\text{kg} + 3\text{m}\cdot 4\text{kg} + 6\text{m}\cdot 2\text{kg} + 10\text{m}\cdot m_4\right)$$

$$4\text{m} = \frac{1}{7\text{kg} + m_4}\left(25\text{kgm} + 10\text{m}\cdot m_4\right)$$

If we solve for m_4, we have

$$4\text{m} = \frac{1}{7kg + m_4}\left(25\text{kgm} + 10\text{m}\cdot m_4\right)$$
$$4\text{m}\cdot\left(7kg + m_4\right) = 25\text{kgm} + 10\text{m}\cdot m_4$$
$$28\text{kgm} + 4\text{m}\cdot m_4 = 25\text{kgm} + 10\text{m}\cdot m_4$$
$$3\text{kgm} = 6\text{m}\cdot m_4$$
$$m_4 = 0.5\text{kg}$$

4. The rod begins at $x = 0$ and ends at $x = 3$. Therefore

$$\begin{aligned}
x_{cm} &= \frac{1}{m}\int_0^{3\text{m}} \lambda x\,dx \\
&= \frac{1}{m}\int_0^{3\text{m}} \left(1\tfrac{\text{kg}}{\text{m}^3}\cdot x^2 + 1\tfrac{\text{kg}}{\text{m}}\right) x\,dx \\
&= \frac{1}{m}\int_0^{3\text{m}} \left(1\tfrac{\text{kg}}{\text{m}^3}\cdot x^3 + 1\tfrac{\text{kg}}{\text{m}}\cdot x\right) dx \\
&= \frac{1}{m}\left[\tfrac{1}{4}\tfrac{\text{kg}}{\text{m}^3}\cdot x^4 + \tfrac{1}{2}\tfrac{\text{kg}}{\text{m}}\cdot x^2\right]_0^{3\text{m}} \\
&= \frac{1}{m}\left[\tfrac{1}{4}\tfrac{\text{kg}}{\text{m}^3}\cdot(3\text{m})^4 + \tfrac{1}{2}\tfrac{\text{kg}}{\text{m}}\cdot(3\text{m})^2\right] \\
&= \frac{1}{m}\cdot 24.75\text{kgm}
\end{aligned}$$

The mass of the rod must be found. The mass is simply the sum of all the infinitesimal masses *dm*. It is therefore

$$\begin{aligned}
m &= \int_0^{3\text{m}} \lambda\,dx \\
&= \int_0^{3\text{m}} \left(1\tfrac{\text{kg}}{\text{m}^3}\cdot x^2 + 1\tfrac{\text{kg}}{\text{m}}\right) dx \\
&= \left[\tfrac{1}{3}\tfrac{\text{kg}}{\text{m}^3}\cdot x^3 + 1\tfrac{\text{kg}}{\text{m}}\cdot x\right]_0^{3\text{m}} \\
&= \left[\tfrac{1}{3}\tfrac{\text{kg}}{\text{m}^3}\cdot(3\text{m})^3 + 1\tfrac{\text{kg}}{\text{m}}\cdot 3\text{m}\right] \\
&= 12kg
\end{aligned}$$

The x-coordinate of the centre of mass is then

$$\begin{aligned} x_{cm} &= \frac{24.75\text{kgm}}{m} \\ &= \frac{24.75\text{kgm}}{12\text{kg}} \\ &= 2.0625m \end{aligned}$$

5. Each rod is replaced by a point mass located at the centre of the rod. So we have the following masses (using an *x*-axis whose origin is at the left end of the beam):

1) A 30 kg mass at $x = 1$ m.
2) A 20 kg mass at $x = 3$ m.
3) A 10 kg mass at $x = 5$ m.

The *x*-coordinate of the centre of mass is then

$$\begin{aligned} x_{cm} &= \frac{1}{m}\sum xm \\ &= \frac{1}{m}\left(x_1m_1 + x_2m_2 + x_3m_3\right) \\ &= \frac{1}{60kg}\left(1m\cdot 30kg + 3m\cdot 20kg + 5m\cdot 10kg\right) \\ &= \frac{140kgm}{60kg} \\ &= 2.333m \end{aligned}$$

6. Each rod is replaced by a point mass located at the centre of the rod. The mass of each rod is proportional to the length of the rod. The lengths of the rods are:

Rod 1 (horizontal rod): $m_1 = \lambda \cdot 30$ cm

Rod 2 (Tilted rod to the left): $m_2 = \lambda \cdot \sqrt{500}$ cm

Rod 2 (Tilted rod to the right): $m_3 = \lambda \cdot \sqrt{800}$ cm

So we have the following masses.

1) The mass m_1 at (15 cm, 0 cm).
2) The mass m_2 at (5 cm, 10 cm).
3) The mass m_3 at (20 cm, 10 cm).

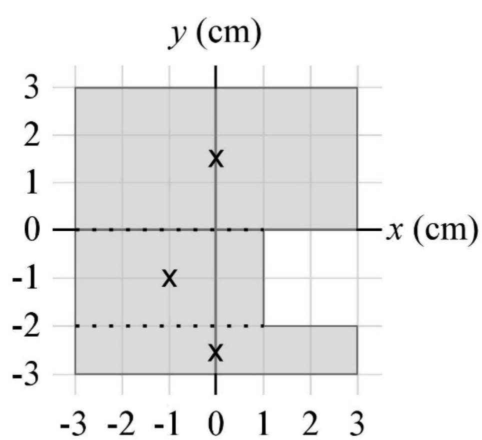
The *x*-coordinate of the centre of mass is then

$$\begin{aligned}
x_{cm} &= \frac{1}{m}\sum xm \\
&= \frac{1}{m}\left(x_1m_1 + x_2m_2 + x_3m_3\right) \\
&= \frac{15cm\cdot\left(\lambda\cdot 30cm\right) + 5cm\cdot\left(\lambda\cdot\sqrt{500}cm\right) + 20cm\cdot\left(\lambda\cdot\sqrt{800}cm\right)}{\lambda\cdot 30cm + \lambda\cdot\sqrt{500}cm + \lambda\cdot\sqrt{800}cm} \\
&= \frac{15cm\cdot 30cm + 5cm\cdot\sqrt{500}cm + 20cm\cdot\sqrt{800}cm}{30cm + \sqrt{500}cm + \sqrt{800}cm} \\
&= 13.98cm
\end{aligned}$$

and the y-coordinate of the centre of mass is

$$\begin{aligned}
y_{cm} &= \frac{1}{m}\sum ym \\
&= \frac{1}{m}\left(y_1m_1 + y_2m_2 + y_3m_3\right) \\
&= \frac{0cm\cdot\left(\lambda\cdot 30cm\right) + 10cm\cdot\left(\lambda\cdot\sqrt{500}cm\right) + 10cm\cdot\left(\lambda\cdot\sqrt{800}cm\right)}{\lambda\cdot 30cm + \lambda\cdot\sqrt{500}cm + \lambda\cdot\sqrt{800}cm} \\
&= \frac{10cm\cdot\sqrt{500}cm + 10cm\cdot\sqrt{800}cm}{30cm + \sqrt{500}cm + \sqrt{800}cm} \\
&= 6.28cm
\end{aligned}$$

7. The plate is separated into three rectangular plates (note that there are several ways to do it). Each of these plates is then replaced by a point mass located at the centre of the plate (the x's on the diagram).



The mass of these plates is proportional to the surface of the plate. The area of each plate is

Plate 1 (top plate): $m_1 = \sigma \cdot 18$ cm².

Plate 2 (middle plate): $m_2 = \sigma \cdot 8$ cm².

Plate 3 (bottom plate): $m_3 = \sigma \cdot 6$ cm².

We have the following masses:

1) The mass m_1 at (0 cm, 1,5 cm).
2) The mass m_2 at (-1 cm, -1 cm).
3) The mass m_3 at (0 cm, -2.5cm).

The x-coordinate of the centre of mass is then

$$\begin{aligned}
x_{cm} &= \frac{1}{m}\sum xm \\
&= \frac{1}{m}\left(x_1m_1 + x_2m_2 + x_3m_3\right) \\
&= \frac{0cm \cdot \left(\sigma \cdot 18cm^2\right) + -1cm \cdot \left(\sigma \cdot 8cm^2\right) + 0cm \cdot \left(\sigma \cdot 6cm^2\right)}{\sigma \cdot 18cm^2 + \sigma \cdot 8cm^2 + \sigma \cdot 6cm^2} \\
&= \frac{-1cm \cdot 8cm^2}{18cm^2 + 8cm^2 + 6cm^2} \\
&= -0.25cm
\end{aligned}$$

and the y-coordinate of the centre of mass is

$$\begin{aligned}
y_{cm} &= \frac{1}{m}\sum ym \\
&= \frac{1}{m}\left(y_1m_1 + y_2m_2 + y_3m_3\right) \\
&= \frac{1,5cm \cdot \left(\sigma \cdot 18cm^2\right) + -1cm \cdot \left(\sigma \cdot 8cm^2\right) + -2,5cm \cdot \left(\sigma \cdot 6cm^2\right)}{\sigma \cdot 18cm^2 + \sigma \cdot 8cm^2 + \sigma \cdot 6cm^2} \\
&= \frac{1,5cm \cdot 18cm^2 + -1cm \cdot 8cm^2 + -2,5cm \cdot 6cm^2}{18cm^2 + 8cm^2 + 6cm^2} \\
&= 0.125cm
\end{aligned}$$

8. The plate is separated into two plates. There is a square plate and a triangular plate. Each of these plates is then replaced by a point mass located at the centre of the plate.

The height of the triangular plate is 0.2 m tan(60°) = $\frac{\sqrt{3}}{5}$ m = 0.34641 m.

The mass of these plates is proportional to the area of the plate. The areas of the plates are

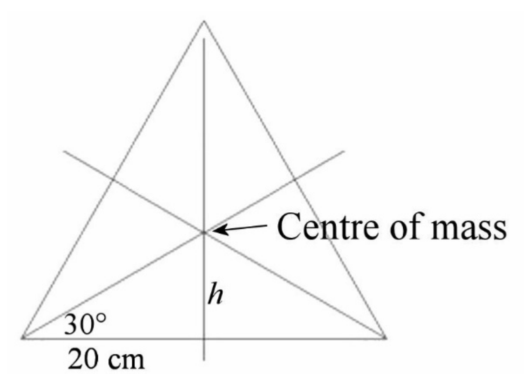
Plate 1 (square): $m_1 = \sigma \cdot area = 60\frac{kg}{m^2} \cdot (0.4m \times 0.4m) = 9.6kg$

Plate 1 (triangle): $m_2 = \sigma \cdot area = 60\frac{kg}{m^2} \cdot \frac{1}{2}\left(0.4m \times \frac{\sqrt{3}}{5} m\right) = 4.157kg$

An axis system whose origin is in the lower-left corner of the square plate is used.

There is no need to make a long calculation to find the *x*-coordinate of the position of the centre of mass because there is a vertical axis of symmetry at *x* = 20 cm. The *x*-coordinate of the position of the centre of mass must, therefore, be at *x* = 20 cm. However, more calculation must be made to find the *y*-coordinate of the position of the centre of mass.

The triangle is replaced by a point mass located at the intersection of the axes of symmetry of the triangle.



The height of the centre of mass from the base of the triangle is

$$h = 20cm \cdot \tan(30°) = 11.547cm$$

(Which is one third of the height of the triangle.)

Once the plates are replaced by point masses, we have the following masses.

1) A 9.6 kg mass at (20 cm, 20 cm).
2) A 4.157 kg mass at (20 cm, 51.547 cm).

The *y*-coordinate of the centre of mass is then

$$\begin{aligned} y_{cm} &= \frac{1}{m}\sum ym \\ &= \frac{1}{m}(y_1 m_1 + y_2 m_2) \\ &= \frac{20cm \cdot 9.6kg + 51.547cm \cdot 4.157kg}{9.6kg + 4.157kg} \\ &= 29.53cm \end{aligned}$$

The centre of mass is therefore at (20 cm, 29.53 cm) with the axes used.

9. An axis system whose origin is at the centre of the rectangular plate is used here.

Let's imagine that we have a rectangular plate without a hole formed of two plates.

1) A rectangular plate with a hole.
2) A circular plate which would plug the hole.

The following formula gives the *x*-coordinate of the position of the centre of mass of this plate without a hole.

$$x_{cm} = \frac{1}{m}\sum x_i m_i$$
$$x_{cm} = \frac{m_1 \cdot x_{1cm} + m_2 \cdot x_{2cm}}{m_{\text{plate without a hole}}}$$

Obviously, the position of the centre of mass of the plate without a hole is at the centre of the plate, so at $x_{cm} = 0$. The equation then becomes

$$0 = \frac{m_1 \cdot x_{1cm} + m_2 \cdot x_{2cm}}{m_{\text{plaque sans trou}}}$$
$$0 = m_1 \cdot x_{1cm} + m_2 \cdot x_{2cm}$$

We can then solve for the position of the centre of mass of the plate with a hole.

$$0 = m_1 \cdot x_{1cm} + m_2 \cdot x_{2cm}$$
$$x_{1cm} = \frac{-m_2 \cdot x_{2cm}}{m_1}$$

The centre of mass of the plate that plugs the hole is at the centre of this plate, so at

$$x_{2cm} = 2cm$$
$$y_{2cm} = 1cm$$

The masses are found by multiplying the density by the area of the plate. Therefore

$$m_2 = \sigma \cdot \pi (2cm)^2$$
$$m_1 = \sigma \cdot \left[14cm \cdot 10cm - \pi (2cm)^2\right]$$

We then have

$$\begin{aligned} x_{1cm} &= \frac{-m_2 \cdot x_{2cm}}{m_1} \\ &= \frac{-\sigma \cdot \pi (2cm)^2 \cdot 2cm}{\sigma \cdot \left[14cm \cdot 10cm - \pi (2cm)^2\right]} \\ &= \frac{-\pi (2cm)^2 \cdot 2cm}{14cm \cdot 10cm - \pi (2cm)^2} \\ &= -0.19722cm \end{aligned}$$

Using the same procedure for the *y*-coordinate, we have

$$\begin{aligned} y_{1cm} &= \frac{-m_2 \cdot y_{2cm}}{m_1} \\ &= \frac{-\sigma \cdot \pi (2cm)^2 \cdot 1cm}{\sigma \cdot \left[14cm \cdot 10cm - \pi (2cm)^2\right]} \\ &= \frac{-\pi (2cm)^2 \cdot 1cm}{14cm \cdot 10cm - \pi (2cm)^2} \\ &= -0.0986cm \end{aligned}$$

10. a)

Initially, the two astronauts are at rest and the total momentum of the system formed by the two astronauts and the rope is zero.

$$p_{x\,tot} = 0$$

If Buzz has a speed of 3 m/s, then the momentum of the system becomes, if the mass of the rope is neglected,

$$\begin{aligned} p'_{x\,tot} &= m_{Buzz} \cdot v'_{Buzz} + m_{rope} \cdot v'_{rope} + m_{Alan} \cdot v'_{Alan} \\ &= 120kg \cdot 3\tfrac{m}{s} + 0kg \cdot v'_{rope} + 80kg \cdot v'_{Alan} \\ &= 360\tfrac{kgm}{s} + 80kg \cdot v'_{Alan} \end{aligned}$$

Since the forces between the astronauts and the rope are all internal forces, the momentum is conserved. Thus

$$p_{x\,tot} = p'_{x\,tot}$$
$$0\tfrac{kgm}{s} = 360\tfrac{kgm}{s} + 80kg \cdot v'_{Alan}$$
$$v'_{Alan} = -4.5\tfrac{m}{s}$$

b) When there are only internal forces, the centre of mass remains at the same place. Using an *x*-axis whose origin is at the initial position of Buzz, the *x*-coordinate of the centre of mass is

$$\begin{aligned} x_{cm} &= \frac{1}{m}\sum xm \\ &= \frac{1}{m}\left(x_1m_1 + x_2m_2\right) \\ &= \frac{1}{200kg}\cdot\left(0m\cdot 120kg + 10m\cdot 80kg\right) \\ &= 4m \end{aligned}$$

A second after his departure, Buzz is now at $x = 3$ m, but the centre of mass always stays at the same place. Therefore

$$x_{cm} = \frac{1}{m}\sum xm$$
$$x_{cm} = \frac{1}{m}\left(x_1m_1 + x_2m_2\right)$$
$$4m = \frac{1}{200kg}\cdot\left(3m\cdot 120kg + x_2\cdot 80kg\right)$$
$$x_2 = 5{,}5m$$

Buzz being at $x = 3$ m and Allan being at $x = 5.5$ m, the distance between the two is 2.5 m.

c) As the centre of mass always remaining at the same place here, the two astronauts will meet at the centre of mass. Buzz will, therefore, travel 4 m at a speed of 3 m/s. He will, therefore, arrive at the centre of mass 1.333 s after the start of the motion.

(You can also do it with Alan. The latter will travel 6 m at a speed of 4.5 m/s, which will also take him 1.333 s.)

11. a)

We will consider that the system consists of the train (with the canon) and the cannonball. The train will be noted with index 1 and the ball with index 2. The position of the centre of mass of the system is

$$x_{cm} = \frac{1}{m}\left(x_{1\,cm}m_1 + x_2 m_2\right)$$

We don't know exactly where the centre of mass of the train is ($x_{1\ cm}$ in the equation), but this does not matter.

When the ball moves towards the other end of the car, the centre of mass stays at the same position. The train will move a distance d towards the left and the ball will be 20 m - d from its starting point (don't forget that the right wall of the train will move towards the left so that the ball will hit the wall before travelling 20 m). The x-coordinate of the centre of mass is then

$$x'_{cm} = \frac{1}{m}\left(\left(x_{1\,cm} - d\right)m_1 + \left(x_2 + 20m - d\right)m_2\right)$$

Since the centre of mass has remained at the same position, we have

$$\begin{gathered}
x_{cm} = x'_{cm} \\
\frac{1}{m}\left(x_{1\,cm}m_1 + x_2 m_2\right) = \frac{1}{m}\left(\left(x_{1\,cm} - d\right)m_1 + \left(x_2 + 20m - d\right)m_2\right) \\
x_{1\,cm}m_1 + x_2 m_2 = \left(x_{1\,cm} - d\right)m_1 + \left(x_2 + 20m - d\right)m_2 \\
x_{1\,cm}m_1 + x_2 m_2 = x_{1\,cm}m_1 - d\cdot m_1 + x_2 m_2 + 20\mathrm{m}\cdot m_2 - d\cdot m_2 \\
0 = -d\cdot m_1 + 20\mathrm{m}\cdot m_2 - d\cdot m_2 \\
d\cdot m_1 + d\cdot m_2 = 20\mathrm{m}\cdot m_2 \\
d = \frac{20\mathrm{m}\cdot m_2}{m_1 + m_2} \\
d = \frac{20\mathrm{m}\cdot 10kg}{12{,}000kg + 10kg} \\
d = 0.01665m \\
d = 1.665cm
\end{gathered}$$

b) Initially, the total momentum is zero

$$p_{x\,tot} = 0$$

After the departure of the cannonball, the momentum is

$$\begin{gathered}
p'_{x\,tot} = m_1\cdot v'_1 + m_2\cdot v'_2 \\
p'_{x\,tot} = 12{,}000kg\cdot v' + 10kg\cdot 500\tfrac{m}{s} \\
p'_{x\,tot} = 12{,}000kg\cdot v' + 5000\tfrac{kgm}{s}
\end{gathered}$$

Since the forces are internal forces, the momentum is conserved.

$$p_{x\,tot} = p'_{x\,tot}$$
$$0\tfrac{kgm}{s} = 12,000kg \cdot v'_1 + 5000\tfrac{kgm}{s}$$
$$v'_1 = -0.4167\tfrac{m}{s}$$

c) As internal forces do not change the speed of the centre of mass and the centre of mass was at rest initially, then the speed of the centre of mass remains zero.

12. We will consider that the system is composed of the rowboat and Sebastian. The boat will be noted with index 1 and Sebastian with index 2. The *x*-coordinate of the centre of mass is

$$x_{cm} = \frac{1}{m}\left(x_{1\,cm}m_1 + x_2 m_2\right)$$

We don't know exactly where the centre of mass of the train is (x_{1cm} in the equation), but this does not matter.

When Sebastian moves towards the other end of the boat, the centre of mass stays at the same position. The boat moves 40 cm to the right, and Sebastian is now 3 m - 0.40 m = 2.60 m from its starting point. The *x*-coordinate of the centre of mass is now

$$x'_{cm} = \frac{1}{m}\left(\left(x_{1\,cm} + 0.4\text{m}\right)m_1 + \left(x_2 - 2.6\text{m}\right)m_2\right)$$

Since the centre of mass remains at the same position, we have

$$x_{cm} = x'_{cm}$$
$$\frac{1}{m}\left(x_{1\,cm}m_1 + x_2 m_2\right) = \frac{1}{m}\left(\left(x_{1\,cm} + 0.4\text{m}\right)m_1 + \left(x_2 - 2.6\text{m}\right)m_2\right)$$
$$x_{1\,cm}m_1 + x_2 m_2 = \left(x_{1\,cm} + 0.4\text{m}\right)m_1 + \left(x_2 - 2.6\text{m}\right)m_2$$
$$x_{1\,cm}m_1 + x_2 m_2 = x_{1\,cm}m_1 + 0.4\text{m}\cdot m_1 + x_2 m_2 - 2.6\text{m}\cdot m_2$$
$$0 = 0.4\text{m}\cdot m_1 - 2.6\text{m}\cdot m_2$$
$$0 = 0.4\text{m}\cdot m_1 - 2.6\text{m}\cdot 60\text{kg}$$
$$m_1 = 390kg$$

13. a) The x-component of the velocity of the centre of mass is

$$\begin{aligned} v_{cm\,x} &= \frac{1}{m}\sum m_i v_{xi} \\ &= \frac{1}{2500kg}\cdot\left(1200kg\cdot 25\tfrac{m}{s} + 1300kg\cdot 0\tfrac{m}{s}\right) \\ &= 12\tfrac{m}{s} \end{aligned}$$

The y-component of the velocity of the centre of mass is

$$\begin{aligned} v_{cm\,y} &= \frac{1}{m}\sum m_i v_{yi} \\ &= \frac{1}{2500kg}\cdot\left(1200kg\cdot 0\tfrac{m}{s} + 1300kg\cdot 15\tfrac{m}{s}\right) \\ &= 7.8\tfrac{m}{s} \end{aligned}$$

The velocity of the centre of mass is therefore

$$\vec{v}_{cm} = \left(12\vec{i} + 7.8\vec{j}\right)\tfrac{m}{s}$$

b) As there are only internal forces in a collision and as the internal forces cannot change the velocity of the centre of mass, the velocity of the centre of mass remains the same after the collision.

$$\vec{v}_{cm} = \left(12\vec{i} + 7.8\vec{j}\right)\tfrac{m}{s}$$

c) As there are only internal forces in a collision and as the internal forces cannot change the velocity of the centre of mass, the velocity of the centre of mass remains the same after the collision.

$$\vec{v}_{cm} = \left(12\vec{i} + 7.8\vec{j}\right)\tfrac{m}{s}$$

14. We will use an axis system where $x = 0$ and $y = 0$ is located at the point of departure of the rocket, on the ground.

Despite the explosion, the centre of mass of the rocket will continue its movement exactly as if the explosion did not occur. The centre of mass will, therefore, continue to move upwards and then falls exactly at the point of departure of the rocket. Let's find the position of the centre of mass by finding where the rocket would be 6 seconds after the engine has stopped if the explosion did not occur.

As the rocket would have made a purely vertical motion, it would have always remained at $x = 0$ m. Then $x_{cm} = 0$.

For the height, we have a free-falling object which began its motion at $y_0 = 60$ m with a velocity of $v_{y0} = 25$ m/s. After 6 seconds, the position of the centre of mass is therefore

$$\begin{aligned} y_{cm} &= y_0 + v_{0y}t - \frac{1}{2}gt^2 \\ &= 60m + 25\tfrac{m}{s}\cdot 6s - \frac{1}{2}\cdot 9.8\tfrac{m}{s^2}\cdot(6s)^2 \\ &= 33.6m \end{aligned}$$

Since the x-coordinate of the position of the centre of mass is 0, the position of the 1.2 kg piece is

$$\begin{aligned} x_{cm} &= \frac{x_1m_1 + x_2m_2}{m_1+m_2} \\ 0m &= \frac{(-90m)\cdot 0.5kg + x_2\cdot 1.2kg}{0.5kg + 1.2kg} \\ x_2 &= 37.5m \end{aligned}$$

Since the y-coordinate of the position of the centre of mass is 33.6 m, the position of the 1.2 kg piece is

$$\begin{aligned} y_{cm} &= \frac{y_1m_1 + y_2m_2}{m_1+m_2} \\ 33.6m &= \frac{(0m)\cdot 0.5kg + y_2\cdot 1.2kg}{0.5kg + 1.2kg} \\ y_2 &= 47.6m \end{aligned}$$

The second piece is therefore 37.5 m to the east of the starting point and is at an altitude of 47.6 m.

15. a) We have

$$\begin{aligned} v_{cmx} &= \frac{1}{m}\sum m_i v_{xi} \\ &= \frac{1}{1.2kg}\cdot\left(0.2kg\cdot 20\tfrac{m}{s} + 1kg\cdot 5\tfrac{m}{s}\right) \\ &= 7.5\tfrac{m}{s} \end{aligned}$$

b) The kinetic energy of the centre of mass is

$$\begin{aligned} E_{k\,cm} &= \frac{1}{2}mv_{cm}^2 \\ &= \frac{1}{2}\cdot 1.2kg\cdot\left(7.5\tfrac{m}{s}\right)^2 \\ &= 33.75J \end{aligned}$$

c) The relative velocity to the centre of mass of the 200 g ball is

$$20\tfrac{m}{s} - 7.5\tfrac{m}{s} = 12.5\tfrac{m}{s}$$

The relative velocity to the centre of mass of the 1000 g ball is

$$5\tfrac{m}{s} - 7.5\tfrac{m}{s} = -2.5\tfrac{m}{s}$$

Therefore, the kinetic energy relative to the centre of mass is

$$\begin{aligned} E_{k\,rel} &= \frac{1}{2}\cdot 0.2kg\cdot\left(12.5\tfrac{m}{s}\right)^2 + \frac{1}{2}\cdot 1kg\cdot\left(-2.5\tfrac{m}{s}\right)^2 \\ &= 18.75J \end{aligned}$$

d) As the velocity of the centre of mass does not change in a collision and as the two objects stick together, the two objects have the same velocity as the centre of mass. Therefore

$$v_x' = 7.5\tfrac{m}{s}$$

e) As the collision does not change the speed of the centre of mass, the kinetic energy of the centre of mass remains the same

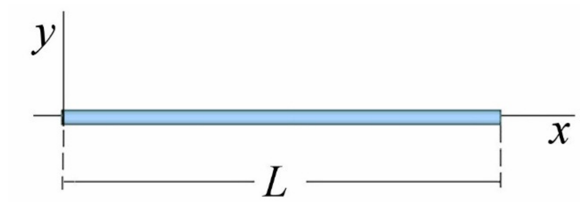
$$E_{k\,cm} = 33.75J$$

f) As the two objects have the same velocity as the centre of mass, the relative velocities become zero and the kinetic energy relative to the centre of mass becomes zero. (This is true for all completely inelastic collisions.)

16. The x-component of the position of the centre of mass is given by

$$x_{cm} = \frac{1}{m}\int \lambda x\,dx$$

With the system of coordinates shown in the figure, the rod goes from $x = 0$ to $x = L$. Therefore, the integral is



$$\begin{aligned} x_{cm} &= \frac{1}{m}\int_0^L \lambda x\,dx \\ &= \frac{1}{m}\int_0^L \left(kx^n\right) x\,dx \\ &= \frac{k}{m}\int_0^L x^{n+1}\,dx \\ &= \frac{k}{m}\left[\frac{x^{n+2}}{n+2}\right]_0^L \\ &= \frac{kL^{n+2}}{m\left(n+2\right)} \end{aligned}$$

The mass of the rod must then be found. The mass is simply the sum of all the small masses dm. Therefore, the mass is

$$\begin{aligned} m &= \int_0^L \lambda\,dx \\ &= \int_0^L kx^n dx \\ &= k\int_0^L x^n dx \\ &= k\left[\frac{x^{n+1}}{n+1}\right]_0^L \\ &= \frac{kL^{n+1}}{n+1} \end{aligned}$$

Thus, the position of the centre of mass is

$$\begin{aligned} x_{cm} &= \frac{1}{m}\frac{kL^{n+2}}{(n+2)} \\ &= \frac{n+1}{kL^{n+1}}\frac{kL^{n+2}}{(n+2)} \\ &= \frac{n+1}{n+2}L \end{aligned}$$

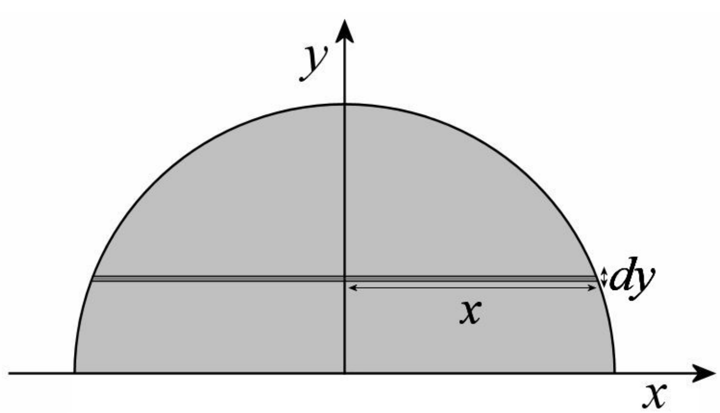
If the centre of mass is at $x = 0.9\ L$, then

$$\begin{gathered} \frac{n+1}{n+2}L = 0.9L \\ \frac{n+1}{n+2} = 0.9 \\ n+1 = 0.9(n+2) \\ n+1 = 0.9n+1.8 \\ 0.1n = 0.8 \\ n = 8 \end{gathered}$$

17. The y-component of the position of the centre of mass is given by

$$y_{cm} = \frac{1}{m}\int y\,dm$$

The plate will be divided into small horizontal slices, as shown in the figure.



The area of this slice is $2xdy$. Thus, the mass of this slice is

$$dm = \sigma 2xdy$$

For a circle with radius a, we have

$$x^2 + y^2 = a^2$$

This means that x is

$$x = \sqrt{a^2 - y^2}$$

The mass of the slice then becomes

$$dm = \sigma 2\sqrt{a^2 - y^2}\,dy$$

Thus, the position of the centre of mass is

$$\begin{aligned} y_{cm} &= \frac{1}{m}\int y\,dm \\ &= \frac{1}{m}\int y\,\sigma 2\sqrt{a^2 - y^2}\,dy \end{aligned}$$

As the slices going from $y = 0$ to $y = a$ are summed *up,* the integral is

$$y_{cm} = \frac{\sigma}{m}\int_o^a 2y\sqrt{a^2 - y^2}\,dy$$

Therefore,

$$\begin{aligned} y_{cm} &= \frac{\sigma}{m}\left[\frac{-\left(a^2 - y^2\right)^{3/2}}{3/2}\right]_0^a \\ &= \frac{\sigma}{m}\left[\frac{-\left(a^2 - a^2\right)^{3/2}}{3/2} - \frac{-\left(a^2 - 0^2\right)^{3/2}}{3/2}\right] \\ &= \frac{\sigma}{m}\frac{a^3}{3/2} \\ &= \frac{2\sigma a^3}{3m} \end{aligned}$$

The mass of the plate is

$$\begin{aligned} m &= \sigma \cdot aire \\ &= \sigma \cdot \frac{1}{2}\pi a^2 \end{aligned}$$

Therefore, the *y*-component of the position of the centre of mass is

$$\begin{aligned} y_{cm} &= \frac{1}{m}\frac{2\sigma a^3}{3} \\ &= \frac{2}{\sigma\pi a^2}\frac{2\sigma a^3}{3} \\ &= \frac{4a}{3\pi} \end{aligned}$$